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A NOTE ON SOME RESULTS OF LI AND LI

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Received December 18, 2016. Published online October 19, 2017.
Communicated by Stanislava Kanas

Abstract. The purpose of the paper is to study the uniqueness problems of linear differential polynomials of entire functions sharing a small function and obtain some results which improve and generalize the related results due to J. T. Li and P. Li (2015). Basically we pay our attention to the condition \( \lambda(f) \neq 1 \) in Theorems 1.3, 1.4 from J. T. Li and P. Li (2015). Some examples have been exhibited to show that conditions used in the paper are sharp.

Keywords: entire function; linear differential polynomial; uniqueness

MSC 2010: 30D35

1. Introduction, definitions and results

In this paper, by a meromorphic or entire function we shall always mean meromorphic or entire, respectively, function in the complex plane \( \mathbb{C} \). We denote by \( n(r, \infty; f) \) the number of poles of \( f \) lying in \( |z| < r \); the poles are counted according to their multiplicities. The quantity

\[
N(r, \infty; f) = \int_0^r \frac{n(t, \infty; f) - n(0, \infty; f)}{t} \, dt + n(0, \infty; f) \log r
\]

is called the integrated counting function or simply the counting function of poles of \( f \).

Also \( m(r, \infty; f) = \frac{1}{2} \pi^{-1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \) is called the proximity function of poles of \( f \), where \( \log^+ x = \log x \) for \( x \geq 1 \) and \( \log^+ x = 0 \) for \( 0 \leq x < 1 \).

The sum \( T(r, f) = m(r, \infty; f) + N(r, \infty; f) \) is called the Nevanlinna characteristic function of \( f \). We denote by \( S(r, f) \) any quantity satisfying \( S(r, f) = o\{T(r, f)\} \) as \( r \to \infty \) except possibly a set of finite linear measure. We denote by \( T(r) \) the maximum of \( T(r, f) \) and \( T(r, g) \). The notation \( S(r) \) denotes any quantity satisfying

DOI: 10.21136/MB.2017.0106-16
\( S(r) = o(T(r)) \) as \( r \to \infty \), outside of a possible exceptional set of finite linear measure.

For \( a \in \mathbb{C} \), we put \( N(r, a; f) = N(r, \infty; (f - a)^{-1}) \) and \( m(r, a; f) = m(r, \infty; (f - a)^{-1}) \).

Let us denote by \( \pi(r, a; f) \) the number of distinct \( a \)-points of \( f \) lying in \( |z| < r \), where \( a \in \mathbb{C} \cup \{ \infty \} \). The quantity
\[
N(r, a; f) = \int_0^r \frac{\pi(t, a; f) - \pi(0, a; f)}{t} \, dt + \pi(0, a; f) \log r
\]
denotes the reduced counting function of \( a \)-points of \( f \) (see, e.g. [2], [13]).

The order of \( f \) is defined by
\[
\lambda(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

Let \( k \) be a positive integer and \( a \in \mathbb{C} \cup \{ \infty \} \). We use \( \overline{N}_k(r, a; f) \) to denote the counting function of \( a \)-points of \( f \) with multiplicity not greater than \( k \); \( \overline{N}_{k+1}(r, a; f) \) to denote the counting function of \( a \)-points of \( f \) with multiplicity greater than \( k \). Similarly, \( \overline{N}_k(r, a; f) \) and \( \overline{N}_{k+1}(r, a; f) \) are their reduced functions, respectively.

For \( a \in \mathbb{C} \cup \{ \infty \} \) and a positive integer \( p \) we denote by \( \overline{N}_p(r, a; f) \) the sum
\[
\overline{N}_1(r, a; f) + \overline{N}_2(r, a; f) + \ldots + \overline{N}_p(r, a; f).
\]

For \( a \in \mathbb{C} \cup \{ \infty \} \) and \( p \in \mathbb{N} \) we put
\[
\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)}.
\]

Clearly,
\[
0 \leq \delta(a; f) \leq \delta_p(a; f) \leq \delta_{p-1}(a; f) \leq \ldots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f).
\]

A meromorphic function \( a \) is said to be a small function of \( f \) provided that \( T(r, a) = S(r, f) \), that is \( T(r, a) = o(T(r, f)) \) as \( r \to \infty \) except possibly a set of finite linear measure.

Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions. Let \( a(z) \) be a small function with respect to \( f(z) \) and \( g(z) \). We say that \( f(z) \) and \( g(z) \) share \( a(z) \) CM (counting multiplicities) if \( f(z) - a(z) \) and \( g(z) - a(z) \) have the same zeros with the same multiplicities and we say that \( f(z) \), \( g(z) \) share \( a(z) \) IM (ignoring multiplicities) if we do not consider the multiplicities.

In 1976, Yang [11] posed the following question:
What can be said about the relationship between two non-constant entire functions \( f \) and \( g \) if \( f \) and \( g \) share the value 0 CM and \( f' \) and \( g' \) share the value 1 CM?

The above problem has been studied by Shibazaki [10], Yi [15], [14], Yang and Yi [12], Hua [4], Muse-Reinders [9] and Lahiri [5]. And Yi [14] proved the following theorem.

**Theorem A** ([14]). Let \( f \) and \( g \) be two non-constant entire functions and let \( k \) be a non-negative integer. If \( f \) and \( g \) share the value 0 CM, \( f^{(k)} \) and \( g^{(k)} \) share the value 1 CM and \( \delta(0; f) > \frac{1}{2} \), then \( f \equiv g \) unless \( f^{(k)}g^{(k)} \equiv 1 \).

**Remark 1.1.** The following example shows that in Theorem A the condition \( \delta(0; f) > \frac{1}{2} \) is sharp.

**Example 1.2 ([14]).** Let

\[
f(z) = \frac{1}{2^k}e^{2z} + \frac{(-1)^{k+1}}{2^k}e^z \quad \text{and} \quad g(z) = \frac{(-1)^{k+1}}{2^k}e^{-2z} - \frac{1}{2^k}e^{-z},
\]

where \( k \) is a non-negative integer. Then \( f \) and \( g \) share the value 0 CM, \( f^{(k)} \) and \( g^{(k)} \) share the value 1 CM and \( \delta(0; f) = \frac{1}{2} \), but \( f \not\equiv g \) and \( f^{(k)}g^{(k)} \not\equiv 1 \).

Let \( h \) be a non-constant meromorphic function. We denote by

\[
P(h) = h^{(k)} + a_1h^{(k-1)} + a_2h^{(k-2)} + \ldots + a_{k-1}h' + a_kh
\]

the differential polynomial of \( h \), where \( a_1, a_2, \ldots, a_k \) are finite complex numbers and \( k \) is a positive integer.

**Remark 1.3.** The following example shows that in Theorem A the functions \( f^{(k)} \) and \( g^{(k)} \) cannot be replaced by \( P(f) \) and \( P(g) \).

**Example 1.4 ([8]).** Let \( f(z) = \frac{1}{2}e^{-2z} \) and \( g(z) = e^{-2z} \). Then \( f \) and \( g \) share the value 0 CM, \( f'' + 2f' \) and \( g'' + 2g' \) share the value 1 CM and \( \delta(0; f) > \frac{1}{2} \), but \( f \not\equiv g \) and \( (f'' + 2f')(g'' + 2g') \not\equiv 1 \).

In 2015, Li and Li proved the following results.

**Theorem B** ([8]). Let \( f \) and \( g \) be two non-constant entire functions. Suppose that \( f \) and \( g \) share the value 0 CM, \( P(f) \) and \( P(g) \) share the value 1 CM and \( \delta(0; f) > \frac{1}{2} \). If \( \lambda(f) \neq 1 \), then \( f \equiv g \) unless \( P(f)P(g) \equiv 1 \).
**Theorem C ([8])**. Let $f$ and $g$ be two non-constant entire functions. Suppose that $f$ and $g$ share the value 0 CM, $P(f)$ and $P(g)$ share the value 1 IM and $\delta(0; f) > \frac{1}{2}$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $P(f)P(g) \equiv 1$.

Now observing the above results the following questions are inevitable.

**Question 1.5.** Is the condition “$\lambda(f) \neq 1$” sharp in Theorems B, C?

**Question 1.6.** Is the condition “$\delta(0; f) > \frac{1}{2}$” sharp in Theorem B?

**Question 1.7.** What can be said if the sharing value in Theorems B, C is replaced by a small function of $f$ and $g$?

**Question 1.8.** Is it really possible in any way to relax the nature of sharing the 1-point in Theorem B (Theorem C)?

In this paper we pay our attention to the nature of the differential polynomial $P(h)$ of $h$ defined as in (1.1). Actually, we want to show that when $a_k \neq 0$ in (1.1), the condition $\lambda(f) \neq 1$ is not necessary. On the other hand, when $a_k = 0$ in (1.1), the condition $\lambda(f) \neq 1$ is necessary.

We now explain the notation of weighted sharing as introduced in [6].

**Definition 1.9 ([6])**. Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f, g$ share the value $a$ with the weight $k$.

We write $f, g$ share $(a, k)$ to mean that $f, g$ share the value $a$ with the weight $k$. Clearly, if $f, g$ share $(a, k)$, then $f, g$ share $(a, p)$ for any integer $p, 0 \leq p < k$. Also we note that $f, g$ share a value $a$ IM or CM if and only if $f, g$ share $(a, 0)$ or $(a, \infty)$, respectively.

Let $h$ be a non-constant meromorphic function. We denote by

\[ P_1(h) = h^{(k)} + a_1h^{(k-1)} + a_2h^{(k-2)} + \ldots + a_k h' + a_k h \]

and

\[ P_2(h) = h^{(k)} + b_1h^{(k-1)} + b_2h^{(k-2)} + \ldots + b_{k-1} h' \]

the differential polynomials of $h$, where $a_1, a_2, \ldots, a_k (\neq 0), b_1, b_2, \ldots, b_{k-1}$ are finite complex numbers with $(b_1, b_2, \ldots, b_{k-1}) \neq (0, 0, \ldots, 0)$ and $k$ is a positive integer.

Now taking the possible answers of the above questions into background we obtain the following results.

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Theorem 1.10. Let $f$ and $g$ be two non-constant entire functions and let $\alpha(z) (\neq 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $f$ and $g$ share $(0, \infty)$, $P_1(f) - \alpha$ and $P_1(g) - \alpha$ share $(0, 2)$. If $\delta_{k+2}(0; f) > \frac{1}{2}$, then $f \equiv g$ unless $P_1(f)P_1(g) \equiv \alpha^2$.

Theorem 1.11. Let $f$ and $g$ be two non-constant entire functions and let $\alpha(z) (\neq 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $f$ and $g$ share $(0, \infty)$, $P_1(f) - \alpha$ and $P_1(g) - \alpha$ share $(0, 1)$. If $\delta_{k+2}(0; f) > \frac{2}{5}$, then $f \equiv g$ unless $P_1(f)P_1(g) \equiv \alpha^2$.

Theorem 1.12. Let $f$ and $g$ be two non-constant entire functions and let $\alpha(z) (\neq 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $f$ and $g$ share $(0, \infty)$, $P_2(f) - \alpha$ and $P_2(g) - \alpha$ share $(0, 0)$. If $\lambda(f) \neq 1$ and $\delta_{k+2}(0; f) > \frac{1}{2}$, then $f \equiv g$ unless $P_2(f)P_2(g) \equiv \alpha^2$.

Theorem 1.13. Let $f$ and $g$ be two non-constant entire functions and let $\alpha(z) (\neq 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $f$ and $g$ share $(0, \infty)$, $P_2(f) - \alpha$ and $P_2(g) - \alpha$ share $(0, 1)$. If $\lambda(f) \neq 1$ and $\delta_{k+2}(0; f) > \frac{2}{5}$, then $f \equiv g$ unless $P_2(f)P_2(g) \equiv \alpha^2$.

Theorem 1.14. Let $f$ and $g$ be two non-constant entire functions and let $\alpha(z) (\neq 0, \infty)$ be a small function with respect to $f$ and $g$. Suppose that $f$ and $g$ share $(0, \infty)$, $P_2(f) - \alpha$ and $P_2(g) - \alpha$ share $(0, 0)$. If $\lambda(f) \neq 1$ and $\delta_{k+2}(0; f) > \frac{3}{5}$, then $f \equiv g$ unless $P_2(f)P_2(g) \equiv \alpha^2$.

Remark 1.16. From the following example it is easy to see that the condition

$$\delta_{k+2}(0; f) > \frac{1}{2}$$

in Theorem 1.10 is sharp.

Example 1.17. Let

$$f(z) = e^z \left(1 - \frac{1}{2}e^z\right) \quad \text{and} \quad g(z) = e^{-z} \left(\frac{1}{2} - e^{-z}\right).$$
Then
\[ P_1(f) = -\frac{3}{8} \left( f^{(iv)} + \frac{2}{3} f''' - 5f'' - 2f' + 8f \right) = e^z(1 - e^z) \]
and
\[ P_1(g) = -\frac{3}{8} \left( g^{(iv)} + \frac{2}{3} g''' - 5g'' - 2g' + 8f \right) = e^{-z}(1 - e^{-z}). \]
Clearly \( P_1(f) \) and \( P_1(g) \) share \((1, \infty)\), \( f, g \) share \((0, \infty)\) and \( \delta_{k+2}(0; f) = \frac{1}{2} \), but neither \( f \equiv g \) nor \( P_1(f)P_1(g) \equiv 1 \).

Remark 1.18. From the following example it is easy to see that the conditions
\[ \delta_{k+2}(0; f) > \frac{1}{2} \quad \text{and} \quad \lambda(f) \neq 1 \]
in Theorem 1.13 are sharp.

Example 1.19. Let
\[ f(z) = e^z \left( 1 - \frac{1}{2} e^z \right) \quad \text{and} \quad g(z) = e^{-z} \left( \frac{1}{2} - e^{-z} \right). \]
Then
\[ P_2(f) = -\frac{3}{8} \left( f^{(iv)} - \frac{2}{3} f''' + 5f'' + 2f' + 8f \right) = e^z(1 - e^z) \]
and
\[ P_2(g) = -\frac{3}{8} \left( g^{(iv)} - \frac{2}{3} g''' + 5g'' + 2g' \right) = e^{-z}(1 - e^{-z}). \]
Clearly \( P_2(f) \) and \( P_2(g) \) share \((1, \infty)\), \( f, g \) share \((0, \infty)\), \( \delta_{k+2}(0; f) = \frac{1}{2} \) and \( \lambda(f) = 1 \), but neither \( f \equiv g \) nor \( P_2(f)P_2(g) \equiv 1 \).

Remark 1.20. From the following example it is easy to see that the condition
\[ \text{“} f \text{ and } g \text{ share } (0, \infty)\text{”} \]
in Theorem 1.10 is necessary.

Example 1.21. Let
\[ f(z) = e^{3z} - e^{2z} \quad \text{and} \quad g(z) = e^z - e^{-2z}. \]
Then
\[ P_1(f) = \frac{1}{24} \left( f^{(iv)} + 6f''' + 23f'' + 42f' + 48f \right) = e^{3z} - e^{2z} \]
and
\[ P_1(g) = \frac{1}{24} \left( g^{(iv)} + 6g''' + 23g'' + 42g' + 48g \right) = e^z - e^{-2z}. \]
Clearly \( P_1(f) \) and \( P_1(g) \) share \((1, \infty)\), \( f, g \) do not share \((0, \infty)\) and \( \delta_{k+2}(0; f) = \frac{2}{3} > \frac{1}{2} \), but neither \( f \equiv g \) nor \( P_1(f)P_1(g) \equiv 1 \).
2. Lemmas

Let \( F, G \) be two non-constant meromorphic functions. Henceforth we shall denote by \( H \) the following function:

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right).
\]

**Lemma 2.1** ([17]). Let \( f \) be a non-constant meromorphic function, \( P(f) \) be defined by (1.1) and \( p, k \) be positive integers. If \( P(f) \not\equiv 0 \), we have

\[
N_p(r, 0; P(f)) \leq T(r, P(f)) - T(r, f) + S(r, f),
\]

\[
N_p(r, 0; P(f)) \leq kN(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).
\]

**Lemma 2.2.** Let \( f \) and \( g \) be two non-constant entire functions. Suppose \( P_2(f) \equiv P_2(g) \), where \( P_2(f) \) is defined by (1.3). If \( \lambda(f) \neq 1 \), then \( f \equiv g \).

**Proof.** Proof of the lemma follows from the proof of Theorem 1.4 in [8]. □

**Lemma 2.3** ([13]). Suppose \( f_j, j = 1, 2, \ldots, m + 1 \) and \( g_j, j = 1, 2, \ldots, m \) are entire functions satisfying the following conditions:

(i) \( \sum_{j=1}^{m} f_j(z)e^{g_j(z)} \equiv f_{m+1} \);

(ii) The order of \( f_j(z) \) is less than the order of \( e^{g_k(z)} \) for \( 1 \leq j \leq m + 1, 1 \leq k \leq m \); and furthermore, the order of \( f_j(z) \) is less than the order of \( e^{g_j-g_k} \) for \( m \leq 2 \) and \( 1 \leq j \leq m + 1, 1 \leq l, k \leq m, l \neq k \).

Then \( f_j \equiv 0, j = 1, 2, \ldots, m + 1 \).

**Lemma 2.4.** Let us consider the linear differential equations

\[
a_n(z)f^{(n)}(z) + a_{n-1}(z)f^{(n-1)}(z) + \ldots + a_0(z)f(z) = 0
\]

with entire coefficients \( a_0(z) (\neq 0), a_1(z), \ldots, a_n(z) (\neq 0) \). Then all solutions of (2.2) are entire functions of finite order if and only if the coefficients \( a_0, a_1, \ldots, a_n \) of (2.2) are polynomials.

**Proof.** Proof of the lemma follows from the proof of Theorem 4.1 (see [7]) and Remark 1 (see [7], page 58). □
Lemma 2.5 ([6]). Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \((1, 2)\). Then one of the following holds:

(i) \( T(r, f) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r, f) + S(r, g) \),

(ii) \( fg \equiv 1 \),

(iii) \( f \equiv g \). 

Lemma 2.6 ([1]). Let \( F \) and \( G \) be two non-constant meromorphic functions sharing \((1, 1)\) and \( H \not\equiv 0 \). Then

\[
T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2} \overline{N}(r, 0; F) \\
+ \frac{1}{2} \overline{N}(r, \infty; F) + S(r, F) + S(r, G).
\]

Lemma 2.7 ([1]). Let \( F \) and \( G \) be two non-constant meromorphic functions sharing \((1, 0)\) and \( H \not\equiv 0 \). Then

\[
T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2 \overline{N}(r, 0; F) \\
+ \overline{N}(r, 0; G) + 2 \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G).
\]

Lemma 2.8 ([16]). Let \( H \) be defined as in (2.1). If \( H \equiv 0 \) and

\[
\limsup_{r \to \infty} \frac{\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)}{T(r)} < 1, \quad r \in I,
\]

where \( I \) is a set of infinite linear measures. Then \( F \equiv G \) or \( FG \equiv 1 \).

3. Proofs of the theorems

Proof of Theorem 1.10. Let \( F(z) = P_1(f)/\alpha(z) \) and \( G(z) = P_1(g)/\alpha(z) \). Then \( F \) and \( G \) share \((1, 2)\) except for the zeros and poles of \( \alpha(z) \). Now applying Lemma 2.5 we see that one of the following three cases holds.

Case 1. Suppose

\[
T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + S(r, F) + S(r, G).
\]
Now applying Lemma 2.1 we have

\[ T(r, f) \leq T(r, F) + N_{k+2}(r, 0; f) - N_2(r, 0; F) + S(r, f) + S(r, g) \]
\[ \leq N_{k+2}(r, 0; f) + N_2(r, 0; G) + S(r, f) + S(r, g) \]
\[ \leq N_{k+2}(r, 0; f) + N_{k+2}(r, 0; g) + S(r, f) + S(r, g) \]
\[ \leq 2N_{k+2}(r, 0; f) + S(r, f) + S(r, g) \]
\[ \leq (2 - 2\delta_{k+2}(0; f) + \varepsilon)T(r, f) + S(r, f) + S(r, g) \]
\[ \leq (2 - 2\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r), \]

i.e.

(3.1) \[ T(r, f) \leq (2 - 2\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r). \]

Similarly we have

(3.2) \[ T(r, g) \leq (2 - 2\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r). \]

Combining (3.1) and (3.2) we get

(3.3) \[ (-1 + 2\delta_{k+2}(0; f) - \varepsilon)T(r) \leq S(r). \]

Since \( \varepsilon > 0 \) is arbitrary, we see that (3.3) leads to a contradiction.

**Case 2.** \( F \equiv G \). Then we have

(3.4) \[ P_1(f) \equiv P_1(g). \]

Let

(3.5) \[ \frac{f}{g} = h = e^\alpha, \]

where \( \alpha \) is an entire function.

We now consider the following subcases.

**Subcase 2.1.** Suppose \( \alpha \) is a constant. Let \( e^\alpha = c_0 \), where \( c_0 \) is a finite complex constant. We obtain \( f \equiv c_0g \) and so \( P_1(f) \equiv c_0P_1(g) \). Now by (3.4) we find that \( c_0 = 1 \) and so \( f \equiv g \).

**Subcase 2.2.** Suppose \( \alpha \) is a non-constant entire function.

Now from (3.4) we have \( P_1(f - g) \equiv 0 \). Solving this equation (see [3], [7]) we get

(3.6) \[ f(z) - g(z) = \sum_{j=1}^{m} p_j(z)e^{\beta_j z}, \]
where \( m (\leq k) \) is a positive integer, \( \beta_j, j = 1, 2, \ldots, m \) are distinct complex constants and \( p_j(z), j = 1, 2, \ldots, m \) are polynomials.

We deduce from (3.5) that
\[
\begin{align*}
 f' &= (g' + \alpha'g)e^\alpha \\
 f'' &= (g'' + 2\alpha'g' + (\alpha'' + (\alpha')^2)g)e^\alpha \\
 f''' &= (g''' + 3\alpha'g'' + 3(\alpha'' + (\alpha')^2)g' + n^3(\alpha')^3 + (\alpha'' + 3\alpha'\alpha'' + (\alpha')^3)g)e^\alpha \\
 & \vdots \\
 f^{(k)} &= (g^{(k)} + Q_{k-1}^k g^{(k-1)} + Q_{k-2}^k g^{(k-2)} + \ldots + Q_0^k g)e^\alpha,
\end{align*}
\]
where \( Q_i^k(\alpha', \alpha'', \ldots, \alpha^{(k)}), i = 0, 1, 2, \ldots, k - 1 \) are differential polynomials in \( \alpha', \alpha'', \ldots, \alpha^{(k)} \). Next we suppose
\[
P_i(f) = f^{(k)} + a_1 f^{(k-1)} + a_2 f^{(k-2)} + \ldots + a_{k-1} f' + a_k f
= (g^{(k)} + Q_{k-1}^k g^{(k-1)} + \ldots + Q_1^k g' + Q_0^k g)e^\alpha,
\]
where \( Q_i(\alpha', \alpha'', \ldots, \alpha^{(k)}), i = 0, 1, 2, \ldots, k - 1 \) are differential polynomials in \( \alpha', \alpha'', \ldots, \alpha^{(k)} \). Since \( \alpha \) is an entire function, we obtain \( T(r, \alpha^{(j)}) = S(r, h) \) for \( j = 1, 2, \ldots, k \). Hence \( T(r, Q_i) = S(r, h) \) for \( i = 0, 1, 2, \ldots, k - 1 \). Now from (3.4) we have
\[
(e^\alpha - 1)g^{(k)} + (e^\alpha Q_{k-1} - a_1)g^{(k-1)} + \ldots + (e^\alpha Q_1 - a_{k-1})g' + (e^\alpha Q_0 - a_k)g \equiv 0.
\]
Clearly \( e^\alpha - 1 \neq 0 \) and \( e^\alpha Q_0 - a_k \neq 0 \). Now by Lemma 2.4 one can easily conclude that both \( f \) and \( g \) are of infinite order. By the Weierstrass’s factorization theorem we have
\[
f(z) = \gamma(z)e^{\alpha_1(z)}, \quad g(z) = \gamma(z)e^{\alpha_2(z)},
\]
where \( \gamma(z) \) is canonical product formed with common zeros of \( f \) and \( g \) and \( \alpha_1(z), \alpha_2(z) \) are non-constant entire functions.

Clearly \( \alpha_1(z) \neq \alpha_2(z) \). Since \( \alpha(z) \) is a non-constant entire function, from (3.5) it follows that \( \alpha_1(z) - \alpha_2(z) \) is a non-constant entire function. Since \( \lambda(\gamma) \) is equal to \( \tau(f) \) which is the exponent of convergence of zeros of \( f(z) \) and \( \tau(f) \leq \tau(f-g) \leq \lambda(f-g) \), by (3.6) we have
\[
\lambda(\gamma) \leq \lambda(f-g) = \lambda \left( \sum_{j=1}^{m} p_j(z)e^{\beta_j z} \right) \leq 1.
\]
Note that \( \lambda(e^{\alpha_1}) = \lambda(f/\gamma) \) and \( \lambda(e^{\alpha_2}) = \lambda(g/\gamma) \). Since \( \lambda(f) > 1 \), \( \lambda(g) > 1 \) and \( \lambda(\gamma) \leq 1 \), it follows that \( \lambda(e^{\alpha_1}) > 1 \) and \( \lambda(e^{\alpha_2}) > 1 \). Also we see that
\[
f - g = (e^{\alpha_1 - \alpha_2} - 1)g.
\]
Clearly,
\[
\lambda(e^{\alpha_1 - \alpha_2}) = \lambda(e^{\alpha_1 - \alpha_2} - 1) = \lambda\left(\frac{f - g}{g}\right).
\]
Since \(\lambda(g) > 1\) and \(\lambda(f - g) \leq 1\), it follows that \(\lambda(e^{\alpha_1 - \alpha_2}) > 1\). From (3.6) we see that
\[
\gamma(z)e^{\alpha_1(z) - \alpha_2(z)} + \sum_{j=1}^{m}(-p_j(z))e^{\beta_j z - \alpha_2(z)} = \gamma(z),
\]
where \(\lambda(e^{\beta_j z - \alpha_2(z)}) > 1\) for \(j = 1, 2, \ldots, m\). Now by Lemma 2.3, we see that \(\gamma(z) = 0\). Therefore \(f(z) \equiv 0\), which is a contradiction.

**Case 3.** \(FG \equiv 1\). Then we have \(P_1(f)P_1(g) \equiv \alpha^2(z)\). This completes the proof. 

\[\square\]

**Proof of Theorem 1.11.** Let \(F(z) = P_1(f)/\alpha(z)\) and \(G(z) = P_1(g)/\alpha(z)\). Then \(F\) and \(G\) share \((1, 1)\) except for the zeros and poles of \(\alpha(z)\). We now consider the following two cases.

**Case 1.** \(H \not\equiv 0\). Applying Lemmas 2.1 and 2.6 we have
\[
T(r, f) \leq T(r, F) + N_{k+2}(r, 0; f) - N_{2}(r, 0; F) + S(r, f) + S(r, g)
\]
\[
\leq N_{2}(r, 0; F) + N_{2}(r, 0; G) + \frac{1}{2}N(r, 0; F)
\]
\[
+ N_{k+2}(r, 0; f) - N_{2}(r, 0; F) + S(r, f) + S(r, g)
\]
\[
\leq N_{k+2}(r, 0; g) + \frac{1}{2}N_{k+1}(r, 0; f) + N_{k+2}(r, 0; f) + S(r, f) + S(r, g)
\]
\[
\leq \left(\frac{5}{2} - \frac{5}{2}\delta_{k+2}(0; f) + \varepsilon\right)T(r) + S(r),
\]
i.e.
\[
(3.7) \quad T(r, f) \leq \left(\frac{5}{2} - \frac{5}{2}\delta_{k+2}(0; f) + \varepsilon\right)T(r) + S(r).
\]
Similarly we have
\[
(3.8) \quad T(r, g) \leq \left(\frac{5}{2} - \frac{5}{2}\delta_{k+2}(0; f) + \varepsilon\right)T(r) + S(r).
\]
Combining (3.7) and (3.8) we get
\[
(3.9) \quad \left(-\frac{3}{2} + \frac{5}{2}\delta_{k+2}(0; f) - \varepsilon\right)T(r) \leq S(r).
\]
Since \(\varepsilon > 0\) is arbitrary, we see that (3.9) leads to a contradiction.
Case 2. \( H \equiv 0 \). In view of Lemma 2.4 we get

\[
\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)
\leq N_{k+2}(r, 0; f) + N_{k+2}(r, 0; g) + S(r, f) + S(r, g)
\leq 2N_{k+2}(r, 0; f) + S(r, f) + S(r, g)
\leq (2 - 2\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r).
\]

Since \( \varepsilon > 0 \) is arbitrary and \( \delta_{k+2}(0; f) > \frac{3}{5} \), we must have

\[
\limsup_{r \to \infty} \frac{\overline{N}(r, 0; F) + \overline{N}(r, 0; G) + \overline{N}(r, \infty; F) + \overline{N}(r, \infty; G)}{T(r)} < 1
\]

and so by Lemma 2.8 we have either \( F \equiv G \) or \( FG \equiv 1 \). So the theorem follows from the proof of Theorem 1.10. This completes the proof. \( \square \)

Proof of Theorem 1.12. Let \( F(z) = P_1(f)/\alpha(z) \) and \( G(z) = P_1(g)/\alpha(z) \). Then \( F \) and \( G \) share \((1, 0)\) except for the zeros and poles of \( \alpha(z) \). We now consider the following two cases.

Case 1. \( H \not\equiv 0 \). Applying Lemmas 2.1 and 2.7 we have

\[
T(r, f) \leq T(r, F) + N_{k+2}(r, 0; f) - N_2(r, 0; F) + S(r, f) + S(r, g)
\leq N_2(r, 0; F) + N_2(r, 0; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G)
+ N_{k+2}(r, 0; f) - N_2(r, 0; F) + S(r, f) + S(r, g)
\leq 3N_{k+2}(r, 0; f) + 2N_{k+2}(r, 0; g) + S(r, f) + S(r, g)
\leq 5N_{k+2}(r, 0; f) + S(r, f) + S(r, g)
\leq (5 - 5\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r),
\]

i.e.

(3.10) \[ T(r, f) \leq (5 - 5\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r). \]

Similarly we have

(3.11) \[ T(r, g) \leq (5 - 5\delta_{k+2}(0; f) + \varepsilon)T(r) + S(r). \]

Combining (3.10) and (3.11) we get

(3.12) \[ (-4 + 5\delta_{k+2}(0; f) - \varepsilon)T(r) \leq S(r). \]

Since \( \varepsilon > 0 \) is arbitrary, we see that (3.12) leads to a contradiction.

Case 2. \( H \equiv 0 \). The remaining part of the theorem follows from the proof of Theorem 1.10. This completes the proof. \( \square \)
Proof of Theorems 1.13–1.15. The proofs of theorems follow from the proof of Theorem 1.10, Theorem 1.11, Theorem 1.12, respectively, and Lemma 2.2. So we omit the detailed proofs. □

References


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