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UNIVERSAL CENTRAL EXTENSION OF DIRECT LIMITS OF HOM-LIE ALGEBRAS

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Abstract. We prove that the universal central extension of a direct limit of perfect Hom-Lie algebras $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ is (isomorphic to) the direct limit of universal central extensions of $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$. As an application we provide the universal central extensions of some multiplicative Hom-Lie algebras. More precisely, we consider a family of multiplicative Hom-Lie algebras $\{(\mathrm{sl}_k(\mathcal{A}), \alpha_k)\}_{k\in I}$ and describe the universal central extension of its direct limit.

Keywords: Hom-Lie algebra; extension of Hom-Lie algebras and its direct limit

MSC 2010: 17A30, 17B55, 17B60, 17B99

1. Introduction

The notion of a Hom-Lie algebra was initially introduced in [15] as part of a study of deformations of the Witt and the Virasoro algebras. Hom-Lie algebras are $\mathbb{F}$-vector spaces endowed with a bilinear skew-symmetric bracket which the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. If this linear map is the identity map, then the definition of Lie algebra is recovered.

The study of algebraic structure of the Hom-Lie algebras can be found in several papers [4], [20], [21], [26], [29], [29]. In the (co)homology theory for Hom-Lie algebras, which generalizes the Chevalley Eilenberg (co)homology for Lie algebras, it has been the subject of [2], [3], [11], [29], [30].

In the setting of Lie algebras, homology theory is closely related to universal central extensions. Some classical results of the universal central extension of Lie algebras cannot be completely extended to the Hom-Lie algebras setting (see [10]). Centrally extended Hom-Lie algebras have a more interesting representation theory than the original Hom-Lie algebra. That is, the central extension of Hom-Lie algebras is an interesting topic for applications in mathematics and physics. The analysis of

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degeneration, contraction and deformation properties in the Hom-Lie algebra setting have led to dealing with universal central extensions (see [12]).

Universal central extension of the direct limit of Hom-Lie algebras is an important way how to construct infinite dimensional Hom-Lie algebras, which includes various types of Hom-Lie algebras, for example; quadratic Hom-Lie algebras and Hom-Lie algebras with invariant bilinear forms (see [5], [7]). Universal central extension of the direct limit of Hom-Lie algebras has not been investigated yet and we are going to clarify this. On the other hand, universal central extensions of direct limits of Lie algebras, super Lie algebras and root graded Lie algebras have been studied by many others, which can be found in several papers (see [1], [18], [22], [23], [24], [25], [28]).

In this work, we consider the universal central extension of general direct limits of perfect Hom-Lie algebras. More precisely, we will prove in Theorem 3.6 that the universal central extension of a direct limit $\varinjlim(\mathcal{L}_i, a_{\mathcal{L}_i})$ of perfect Hom-Lie algebras $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i})\}_{i\in I}$, is canonically isomorphic to the direct limit of the universal central extension of $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$. As an application, we will describe the direct limit of multiplicative perfect Hom-Lie algebras of Lie-type. It is a well known result that the Stienberg Lie algebra $\mathrm{st}_n(\mathcal{A})$ for $n \geqslant 5$, whenever $\mathcal{A}$ is an unitary associative algebra, is the universal cover of the Lie algebra $\mathrm{sl}_n(\mathcal{A})$. Following this, we introduce a Hom version of Stienberg Lie algebra $(\mathrm{st}_I(\mathcal{A}), \overline{\alpha}_s)$ and prove that the direct limit of multiplicative Hom-Lie algebras $(\mathrm{sl}_I(\mathcal{A}), \alpha_s)$ is isomorphic to the multiplicative Steinberg Hom-Lie algebra $(\mathrm{st}_I(\mathcal{A}), \overline{\alpha}_s)$.

To close this introduction, we briefly outline the contents of the paper. In Section 1, we recall the definition of Hom-Lie algebras and some facts that we will need in the sequel. We also revive an introduction to the theory of universal central extension for perfect Hom-Lie algebras and record some of its properties. In Section 2, we provide a construction for central extension of the direct limit of Hom-Lie algebras and show that the universal central extension of the direct limit of perfect Hom-Lie algebras is canonically isomorphic to the direct limit of the universal central extension of perfect Hom-Lie algebras. Section 3 contains as an example an application of universal central extension of direct limits of Hom-Lie algebras. Namely, we will describe the direct limit of multiplicative perfect Hom-Lie algebras of type A_I.

2. Universal central extensions of Hom-Lie algebras

Throughout this paper we will assume that $\mathbb{K}$ is a field of characteristic zero, unless otherwise mentioned and all vector spaces and all algebras are considered to be over $\mathbb{K}$. In this section we recall necessary concepts on Hom-Lie algebras which will be used in the sequel.

2.1. Hom-Lie algebras.

Definition 2.1 ([20]). A Hom-associative algebra is a triple $(\mathcal{A}, \mu, \alpha)$ consisting of a linear space $\mathcal{A}$, a bilinear map $\mu\colon \mathcal{A} \to \mathcal{A}$ and a linear space homomorphism $\alpha\colon \mathcal{A} \to \mathcal{A}$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)).$$

Definition 2.2. A Hom-Lie algebra is a pair $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ consisting of a $\mathbb{K}$-vector space $\mathcal{L}$ together with a bilinear map $[\cdot,\cdot]\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and an $\mathbb{F}$-linear map $\alpha_{\mathcal{L}}\colon \mathcal{L} \to \mathcal{L}$ satisfying

(i) $[x, y] = -[y, x]$ (skew-symmetry),
(ii) $[\alpha_{\mathcal{L}}(x), [y, z]] + [\alpha_{\mathcal{L}}(y), [z, x]] + [\alpha_{\mathcal{L}}(z), [x, y]] = 0$ (Hom-Jacobi identity)

for all $x, y, z \in \mathcal{L}$. A Hom-Lie algebra $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ is called a multiplicative Hom-Lie algebra if $\alpha_{\mathcal{L}}$ is an algebra homomorphism, that is, $\alpha_{\mathcal{L}}[x, y] = [\alpha_{\mathcal{L}}(x), \alpha_{\mathcal{L}}(y)]$ for all $x, y \in \mathcal{L}$. A multiplicative Hom-Lie algebra is said to be regular if α is invertible.

Proposition 2.3 ([20]). *To any Hom-associative algebra $(\mathcal{A}, \mu, \alpha)$, one may associate a Hom-Lie algebra defined for all $x, y \in \mathcal{A}$ by the bracket $[x, y] = \mu(x, y) - \mu(y, x)$.*

Remark 2.4. (i) Let $(\mathcal{L}, [\cdot,\cdot])$ be a Lie algebra and α a Lie algebra homomorphism. Then $(\mathcal{L}, [\cdot,\cdot]_\alpha, \alpha)$ where $[x, y]_\alpha = \alpha([x, y] = [\alpha(x), \alpha(y)])$ is a multiplicative Hom-Lie algebra.

(ii) Every regular Hom-Lie algebra $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ is the twist of the Lie algebra $(\mathcal{L}, [\cdot,\cdot]_{\alpha^{-1}})$ by $\alpha_{\mathcal{L}}$ (see [30]).

(iii) Consider $\mathrm{sl}_n(\mathbb{C})$, the complex Lie algebra of $n \times n$ matrices with trace 0. Let $E_{i,j}$ denote the matrix with 1 in the (i, j)-entry and 0 everywhere else. Then $\mathrm{sl}_n(\mathbb{C})$ is generated as a Lie algebra by the elements

$$e_i = E_{i,i+1}, \quad f_i = E_{i+1,i}, \quad \text{and} \quad h_i = E_{i,i} - E_{i+1,i+1}$$

for $i = 1, 2, \ldots, n-1$. For any nonzero scalars a_i in $\mathbb{C}$, define the map $\alpha\colon \mathrm{sl}_n(\mathbb{C}) \to \mathrm{sl}_n(\mathbb{C})$ by

$$\alpha(e_i) = a_i e_i, \quad \alpha(f_i) = a_i^{-1} f_i \quad \text{and} \quad \alpha(h_i) = h_i$$

for $i = 1, 2, \ldots, n-1$. Next, define a new bracket $[\cdot,\cdot]_\alpha = \alpha([\cdot,\cdot])$. Then $\mathrm{sl}_n^\alpha(\mathbb{C}) := (\mathrm{sl}_n(\mathbb{C}), [\cdot,\cdot]_\alpha, \alpha)$ is a multiplicative Hom-Lie algebra, which is called a simple Hom-Lie algebra of type A_n. For more details see [30], Example 2.12.

Definition 2.5. A homomorphism of Hom-Lie algebras $\varphi\colon (\mathcal{L},\alpha_{\mathcal{L}}) \to (\mathcal{L}',\alpha_{\mathcal{L}'})$ is an algebra homomorphism $\varphi\colon \mathcal{L} \to \mathcal{L}'$ such that $\varphi \circ \alpha_{\mathcal{L}} = \alpha_{\mathcal{L}'} \circ \varphi$. In other words, the following diagram commutes:

$$
\begin{tikzcd}
(\mathcal{L},\alpha_{\mathcal{L}}) \arrow[r, "\varphi"] \arrow[d, "\alpha_{\mathcal{L}}"] & (\mathcal{L}',\alpha_{\mathcal{L}'}) \arrow[d, "\alpha_{\mathcal{L}'}"] \\
(\mathcal{L},\alpha_{\mathcal{L}}) \arrow[r, "\varphi"] & (\mathcal{L}',\alpha_{\mathcal{L}'}).
\end{tikzcd}
\tag{2.1}
$$

Two Hom-Lie algebras $(\mathcal{L},\alpha_{\mathcal{L}})$ and $(\mathcal{L}',\alpha_{\mathcal{L}'})$ are isomorphic if there is a Hom-Lie algebra homomorphism $\varphi\colon (\mathcal{L},\alpha_{\mathcal{L}}) \to (\mathcal{L}',\alpha_{\mathcal{L}'})$ such that $\varphi\colon \mathcal{L} \to \mathcal{L}'$ is bijective. A homomorphism of multiplicative Hom-Lie algebras is also a homomorphism of the underlying Hom-Lie algebras; using this fact we define the category Hom-Lie (Hom-$\mathrm{Lie}_{\mathrm{mult}}$) that its objects are Hom-Lie algebras (multiplicative Hom-Lie algebras) and that its morphisms are the homomorphism of Hom-Lie algebras (multiplicative Hom-Lie algebras).

2.2. Some generalities on extensions. In this subsection, we review a generalization of the theory of extensions of Hom-Lie algebras which can be found in [15], [21]. In particular, we recall the universal central extension (uce) of Hom-Lie algebras and record some results which we need in the sequel (see [10]).

Definition 2.6. Let $(\mathcal{H},\alpha_{\mathcal{H}})$, $(\mathcal{L},\alpha_{\mathcal{L}})$ and $(\mathrm{e},\alpha_{\mathrm{e}})$ be Hom-Lie algebras. An extension $(\mathrm{e},\alpha_{\mathrm{e}})$ of $(\mathcal{L},\alpha_{\mathcal{L}})$ by $(\mathcal{H},\alpha_{\mathcal{H}})$ is a short exact sequence of the form

$$0 \longrightarrow (\mathcal{H},\alpha_{\mathcal{H}}) \xrightarrow{i} (\mathrm{e},\alpha_{\mathrm{e}}) \xrightarrow{\pi} (\mathcal{L},\alpha_{\mathcal{L}}) \longrightarrow 0. \tag{2.2}$$

Let $(\mathrm{e}_j,\alpha_{\mathrm{e}_j})$, $j=1,2$ be two extensions of $(\mathcal{L},\alpha_{\mathcal{L}})$ by $(\mathcal{H},\alpha_{\mathcal{H}})$. A morphism from an extension $(\mathrm{e}_1,\alpha_{\mathrm{e}_1})$ to another extension $(\mathrm{e}_2,\alpha_{\mathrm{e}_2})$ of $(\mathcal{L},\alpha_{\mathcal{L}})$ by $(\mathcal{H},\alpha_{\mathcal{H}})$ is a Hom-Lie algebra homomorphism $\varphi\colon (\mathrm{e}_1,\alpha_{\mathrm{e}_1}) \to (\mathrm{e}_2,\alpha_{\mathrm{e}_2})$ satisfying $\pi' \circ \varphi = \pi$. Hence we have

$$\ker\varphi \subseteq \varphi^{-1}(\ker\pi') = \ker\pi \quad \text{and} \quad \mathrm{e}_2 = \pi(\mathrm{e}_1) + \ker\pi'. \tag{2.3}$$

Two extensions of Hom-Lie algebras

$$0 \longrightarrow (\mathcal{H},\alpha_{\mathcal{H}}) \xrightarrow{i_j} (\mathrm{e}_j,\alpha_{\mathrm{e}_j}) \xrightarrow{\pi_j} (\mathcal{L},\alpha_{\mathcal{L}}) \longrightarrow 0, \quad j=1,2$$

are equivalent if there is an isomorphism $\varphi\colon (\mathrm{e}_1,\alpha_{\mathrm{e}_1}) \to (\mathrm{e}_2,\alpha_{\mathrm{e}_2})$ such that $\varphi \circ i_1 = i_2$ and $\pi_2 \circ \varphi = \pi_1$. The extension (2.2) is said to be split if there exists a Hom-Lie algebra homomorphism $s\colon (\mathcal{L},\alpha_{\mathcal{L}}) \to (\mathrm{e},\alpha_{\mathrm{e}})$ such that $\pi \circ s = \mathrm{id}_{\mathcal{L}}$, which is called a section.

If the extension (2.2) splits, then

$$\mathrm{e} = \ker \pi \oplus s(\mathcal{L}),$$

where the section $s\colon (\mathcal{L}, \alpha_{\mathcal{L}}) \to (s(\mathcal{L}), \alpha_{\mathrm{e}}|)$ is an isomorphism with the inverse $s^{-1} = \pi\big|_{s(\mathcal{L})}$. Moreover, $\mathrm{e} = \ker \pi \times s(\mathcal{L})$ is a semidirect product. Conversely, any semidirect product $\mathrm{e} = \ker \pi \times \mathcal{L}$ gives rise to the canonical split extension (2.2) by taking $\ker \pi = \mathcal{H}$, where

$$(\mathcal{H}, \alpha_{\mathcal{H}}) \to (\mathrm{e}, \alpha_{\mathrm{e}}), \quad h \mapsto h \oplus 0,$$

and

$$\pi\colon (\mathrm{e}, \alpha_{\mathrm{e}}) \to (\mathcal{L}, \alpha_{\mathcal{L}}), \quad h \oplus l \mapsto l.$$

Then, in this way, the split extension and semidirect products correspond to each other. However, in general an extension need not be split, there need not even exist a module homomorphism $s\colon \mathcal{L} \to \mathrm{e}$.

Definition 2.7. Let $(\mathcal{L}, \alpha_{\mathcal{L}})$ be a Hom-Lie algebra. A central extension (α-central extension) of $(\mathcal{L}, \alpha_{\mathcal{L}})$ by $(\mathcal{H}, \alpha_{\mathcal{H}})$ is an extension of Hom-Lie algebras

$$0 \longrightarrow (\mathcal{H}, \alpha_{\mathcal{H}}) \xrightarrow{i} (\mathcal{K}, \alpha_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \alpha_{\mathcal{L}}) \longrightarrow 0, \tag{2.4}$$

such that $[\mathcal{H}, \mathcal{K}] = 0$ ($[\alpha_{\mathcal{H}}(\mathcal{H}), \mathcal{K}] = 0$), this means $\mathcal{H}$ ($\alpha_{\mathcal{H}}(\mathcal{H})$) is in the center of $\mathcal{K}$. A central extension (α-central extension) $(\mathcal{K}, \alpha_{\mathcal{K}})$ of $(\mathcal{L}, \alpha_{\mathcal{L}})$ by $(\mathcal{H}, \alpha_{\mathcal{H}})$ is called a covering (α-covering) if $(\mathcal{K}, \alpha_{\mathcal{K}})$ is a perfect Hom-Lie algebra ($[\mathcal{K}, \mathcal{K}] = \mathcal{K}$). A central extension ($\alpha$-central extension) $(\mathcal{K}, \alpha_{\mathcal{K}})$ of $(\mathcal{L}, \alpha_{\mathcal{L}})$ by $(\mathcal{H}, \alpha_{\mathcal{H}})$ is called a universal central extension (universal α-central extension) if for any other central extension

$$0 \longrightarrow (\mathcal{H}, \alpha_{\mathcal{H}}) \xrightarrow{i'} (\mathcal{K}', \alpha_{\mathcal{K}}) \xrightarrow{\pi'} (\mathcal{L}, \alpha_{\mathcal{L}}) \longrightarrow 0, \tag{2.5}$$

there exists a unique Hom-Lie algebra homomorphism $\varphi\colon (\mathcal{K}, \alpha_{\mathcal{K}}) \to (\mathcal{K}', \alpha_{\mathcal{K}'})$ such that $\pi' \circ \varphi = \pi$. A perfect Hom-Lie algebra $(\mathcal{L}, \alpha_{\mathcal{L}})$ is said to be centrally closed if its universal central extension is

$$0 \to 0 \to (\mathcal{L}, \alpha_{\mathcal{L}}) \to (\mathcal{L}, \alpha_{\mathcal{L}}) \to 0. \tag{2.6}$$

Remark 2.8. (i) It is obvious that every central extension (or universal α-central extension) is an α-central extension (or universal central extension). Note that both notions coincide when $\alpha_{\mathcal{H}} = \mathrm{id}_{\mathcal{H}}$.

(ii) To verify that a covering is universal, it suffices to show the existence of a Hom-Lie algebra homomorphism from the covering to any central extension of $(\mathcal{L}, \alpha_{\mathcal{L}})$. It is obvious from the universal property that any two universal coverings are isomorphic as central extensions (see [10], Lemma 4.7).

Theorem 2.9 (Chracterization theorem).

(a) *If a central extension* $0 \longrightarrow (\mathcal{H}, \alpha_{\mathcal{H}}) \xrightarrow{i} (\mathcal{K}, \alpha_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \alpha_{\mathcal{L}}) \longrightarrow 0$, *is a universal* α*-central extension, then* $(\mathcal{K}, \alpha_{\mathcal{K}})$ *is a perfect Hom-Lie algebra and every central extension of* $(\mathcal{K}, \alpha_{\mathcal{K}})$ *is split.*

(b) *Let* $0 \longrightarrow (\mathcal{H}, \alpha_{\mathcal{H}}) \xrightarrow{i} (\mathcal{K}, \alpha_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \alpha_{\mathcal{L}}) \longrightarrow 0$ *be a central extension. If* $(\mathcal{K}, \alpha_{\mathcal{K}})$ *is a perfect Hom-Lie algebra and every central extension of* $(\mathcal{K}, \alpha_{\mathcal{K}})$ *is split, then* $0 \longrightarrow (\mathcal{H}, \alpha_{\mathcal{H}}) \xrightarrow{i} (\mathcal{K}, \alpha_{\mathcal{K}}) \xrightarrow{\pi} (\mathcal{L}, \alpha_{\mathcal{L}}) \longrightarrow 0$, *is a universal central extension.*

(c) *A Hom-Lie algebra* $(\mathcal{L}, \alpha_{\mathcal{L}})$ *admits a universal central extension if and only if* $(\mathcal{L}, \alpha_{\mathcal{L}})$ *is perfect.*

(d) *The kernel of the universal central extension is canonically isomorphic to* $H_2^{\alpha}(\mathcal{L})$, *the 2nd homology of* $\mathcal{L}$.

Proof. See [10], Theorem 4.11. $\square$

To describe a model of a universal central extension of a Hom-Lie algebra $(\mathcal{L}, \alpha_{\mathcal{L}})$, one can use the following construction of a Hom-Lie algebra which is valid for any not necessarily perfect Hom-Lie algebra (see [10] for a perfect Hom-Lie algebra and for a perfect Lie algebra [18], [22], [27]).

Consider the $\mathbb{K}$-subspace J of $\mathcal{L} \wedge \mathcal{L}$ spanned by all elements of the form

$$-[x, y] \wedge \alpha_{\mathcal{L}}(z) + [x, z] \wedge \alpha_{\mathcal{L}}(y) - [y, z] \wedge \alpha_{\mathcal{L}}(x)$$

for all $x, y, z \in \mathcal{L}$. Define $\widetilde{\mathcal{L}} = (\mathcal{L} \wedge \mathcal{L})/J$. For all x, y in $\mathcal{L}$, $\langle x, y\rangle$ denotes the image of $x \wedge y$ under the canonical map, i.e. $\langle x, y\rangle = x \wedge y + J \in \widetilde{\mathcal{L}}$. Note that we have the following identities in $\widetilde{\mathcal{L}}$,

$$\langle [x, y] \wedge \alpha_{\mathcal{L}}(z)\rangle + \langle [x, z] \wedge \alpha_{\mathcal{L}}(y)\rangle + \langle [y, z] \wedge \alpha_{\mathcal{L}}(x)\rangle = 0 \tag{2.7}$$

for all x, y and z in $\mathcal{L}$. By a straightforward computation, the $\mathbb{K}$-linear map $\mathcal{L} \wedge \mathcal{L} \to [\mathcal{L}, \mathcal{L}]$ defined by $x \wedge y \mapsto [x, y]$ vanishes on J and hence descends to a $\mathbb{K}$-linear map

$$u_{\mathcal{L}} \colon \widetilde{\mathcal{L}} \to [\mathcal{L}, \mathcal{L}], \quad \langle x, y\rangle \mapsto [x, y] \tag{2.8}$$

for all $x, y \in \mathcal{L}$, with

$$\ker u_{\mathcal{L}} = \left\{\sum_i \langle x_i, y_i\rangle : \sum_i [x_i, y_i] = 0\right\} = H_2^{\alpha}(\mathcal{L}).$$

Now, we define a composition law $[\cdot, \cdot]^{\sim}$ on $\widetilde{\mathcal{L}}$ by

$$[l_1, l_2]^{\sim} = \langle u_{\mathcal{L}}(l_1), u_{\mathcal{L}}(l_2)\rangle \tag{2.9}$$

for all l_1, l_2 in $\widetilde{\mathcal{L}}$. Define $\widetilde{\alpha}\colon \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}}$ by $\widetilde{\alpha}_{\mathcal{L}}(\langle x, y\rangle) = \langle \alpha_{\mathcal{L}}(x), \alpha_{\mathcal{L}}(y)\rangle$. Then $(\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}})$ becomes a Hom-Lie algebra with respect to the product (2.9). Also by definition, $u_{\mathcal{L}}\colon (\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}}) \to (\mathcal{L}, \alpha_{\mathcal{L}})$ is a homomorphism of Hom-Lie algebras. In particular, we have

$$[\langle x, y\rangle, \langle x', y'\rangle]^{\sim} = \langle [x, y], [x', y']\rangle, \quad x, y, x', y' \in \mathcal{L}. \tag{2.10}$$

Therefore, $(\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}})$ is a Hom-Lie algebra with respect to the product $[\cdot,\cdot]^{\sim}$ in (2.10). Then we have the extension

$$0 \longrightarrow (H_2^{\alpha}(\mathcal{L}), \widetilde{\alpha}_{\mathcal{L}}|) \xrightarrow{i} (\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}}) \xrightarrow{u_{\mathcal{L}}} (\mathcal{L}, \alpha_{\mathcal{L}}) \longrightarrow 0, \tag{2.11}$$

which is a central extension, since $\ker(u_{\mathcal{L}}) \subset Z(\widetilde{\mathcal{L}})$. If $(\mathcal{L}, \alpha_{\mathcal{L}})$ is perfect, then the extension (2.11) is a universal central extension, since $u_{\mathcal{L}}\colon (\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}}) \to (\mathcal{L}, \alpha_{\mathcal{L}})$ is a surjective homomorphism of Hom-Lie algebras. A morphism of Hom-Lie algebras $\varphi\colon (\mathcal{L}, \alpha_{\mathcal{L}}) \to (\mathcal{K}, \alpha_{\mathcal{K}})$ gives rise to a morphism of Hom-Lie algebras

$$\widetilde{\varphi}\colon (\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}}) \to (\widetilde{\mathcal{K}}, \widetilde{\alpha}_{\mathcal{K}}), \quad \langle l_1, l_2\rangle = \langle \varphi(l_1), \varphi(l_2)\rangle.$$

The assignments

$$(\mathcal{L}, \alpha_{\mathcal{L}}) \mapsto (\widetilde{\mathcal{L}}, \widetilde{\alpha}_{\mathcal{L}}) \quad \text{and} \quad \varphi \mapsto \widetilde{\varphi} \tag{2.12}$$

define a covariant functor on the category Hom-Lie.

3. Universal central extension of direct limit of Hom-Lie algebras

In this section we give a review of direct limits and a complete description for the universal central extension of a Hom-Lie algebra. We show that the universal central extension of a direct limit $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ of perfect Hom-Lie algebras $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ is canonically isomorphic to the direct limit of the universal central extensions of $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$.

3.1. Direct limit of Hom-Lie algebras. We first recall the definition of a direct limit of Hom-Lie algebras. Let I be a set with the partial order $\leqslant$ satisfying the property that for any $i, j \in I$, there exists a $k \in I$ such that $i \leqslant k$ and $j \leqslant k$. Such a set is called a directed set and is denoted by $(I, \leqslant)$. A directed system of Hom-Lie algebras is a family $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i})\}_{i\in I}$ in Hom-Lie and for each pair $i \leqslant j$ together with the Hom-Lie algebra morphism $\varphi_{ij}\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to (\mathcal{L}_j, \alpha_{\mathcal{L}_j})$ with $\varphi_{ii} = \mathrm{id}_{\mathcal{L}_i}$ for each $i \in I$, and such that whenever $i \leqslant j \leqslant k$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. We denote this directed system by $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$. A direct limit $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ of the directed system

$\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ is a unique up to isomorphism Hom-Lie algebra $(\mathcal{L}, \alpha_{\mathcal{L}})$ satisfying the following universal mapping property: there are Hom-Lie algebra morphisms $f_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to (\mathcal{L}, \alpha_{\mathcal{L}})$ such that $f_i = f_j \circ \varphi_{ij}$ for every pair $i \leqslant j$ (which are called canonical maps) and if there is a Hom-Lie algebra $(\mathcal{M}, \alpha_{\mathcal{M}})$ with Hom-Lie algebra morphisms $g_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to (\mathcal{M}, \alpha_{\mathcal{M}})$ such that $g_i = g_j \circ \varphi_{ij}$ for each pair $i \leqslant j$, then there is a unique Hom-Lie algebra morphism $\psi\colon (\mathcal{L}, \alpha_{\mathcal{L}}) \to (\mathcal{M}, \alpha_{\mathcal{M}})$ such that $g_i = \psi \circ f_i$. That is, the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \arrow[rr, "f_i"] \arrow[dr, "g_i"'] & & (\mathcal{L}, \alpha_{\mathcal{L}}) \arrow[dl, "\psi"] \\
& (\mathcal{M}, \alpha_{\mathcal{M}}). &
\end{tikzcd}$$

Remark 3.1. The usual construction of the direct limit of Lie algebras shows that the direct limit of Hom-Lie algebras exists and is unique up to isomorphism. Moreover:

(i) Every element of $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ can be written in the form $f_i(x)$ for some $x \in \mathcal{L}_i$.
(ii) If $x \in \mathcal{L}_i$ satisfies $f_i(x) = 0$, then there exists a $j \geqslant i$ such that $\varphi_{ij}(x) = 0$.

Definition 3.2. Let $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ and $\{(\mathcal{M}_i, \alpha_{\mathcal{M}_i}), \psi_{ij}\}$ be two directed systems of Hom-Lie algebras, which are indexed by the directed set $(I, \leqslant)$. A morphism from $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ to $\{(\mathcal{M}_i, \alpha_{\mathcal{M}_i}), \psi_{ij}\}$ is a family $\{h_i\}_{i\in I}$ of Hom-Lie algebra homomorphisms $h_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to (\mathcal{M}_i, \alpha_{\mathcal{M}_i})$ such that for all pairs $i \leqslant j$ we have $h_j \circ \varphi_{ij} = \psi_{ij} \circ h_i$. That is, the diagram

$$\begin{tikzcd}
(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \arrow[r, "h_i"] \arrow[d, "\varphi_{ij}"] & (\mathcal{M}_i, \alpha_{\mathcal{M}_i}) \arrow[d, "\psi_{ij}"] \\
(\mathcal{L}_j, \alpha_{\mathcal{L}_j}) \arrow[r, "h_j"] & (\mathcal{M}_j, \alpha_{\mathcal{M}_j})
\end{tikzcd} \tag{3.1}$$

commutes.

Remark 3.3. Here we may look at the direct limit from a categorical point. Let $(I, \leqslant)$ be a directed set and $\mathcal{I}$ its category whose objects are the elements of I and its morphisms are arrows $i \to j$ for each pair $i \leqslant j$. A directed system in Hom-Lie is nothing but a functor from $\mathcal{I}$ to the category of directed systems in Hom-Lie to the category of Hom-Lie algebras. More precisely, let h be a morphism from a directed system $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ to a directed system $\{(\mathcal{M}_i, \alpha_{\mathcal{M}_i}), \psi_{ij}\}$. The study of functor categories shows that h is a natural transformation of the functor, that is, for each $i \in I$ there is a Hom-Lie algebra homomorphism $h_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to (\mathcal{M}_i, \alpha_{\mathcal{M}_i})$, and if $i \leqslant j$, then $h_j \circ \varphi_{ij} = \psi_{ij} \circ h_i$. Now, if $f_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ and $g_i\colon (\mathcal{M}_i, \alpha_{\mathcal{M}_i}) \to \varinjlim(\mathcal{M}_i, \alpha_{\mathcal{M}_i})$ are canonical maps, then for each $i \in I$ we have $g_i \circ h_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to \varinjlim(\mathcal{M}_i, \alpha_{\mathcal{M}_i})$ which is a Hom-Lie algebra homomorphism that has the desired properties to yield a unique Hom-Lie algebra homomorphism

$$\varinjlim h_i\colon \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to \varinjlim(\mathcal{M}_i, \alpha_{\mathcal{M}_i}),$$

with $\varinjlim h_i \circ f_i = g_i \circ h_i$ for all $i \in I$. The assertion $h_i \mapsto \varinjlim h_i$ is then the direct limit functor which operates on Hom-Lie algebra homomorphisms. A straightforward calculation shows that the composition of morphisms is preserved and that $\mathrm{id}_{(\mathcal{L}_i, \alpha_{\mathcal{L}_i})} = \mathrm{id}_{\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})}$. The direct limit does give a functor on maps in the category of directed systems in Hom-Lie. Since the direct limit preserves exact sequences [9], II, 6.2, Proposition 3, it follows that if all h_i are isomorphisms, then $\varinjlim h_i$ is an isomorphism too.

3.2. Construction of the central extension of the direct limit of Hom-Lie algebras. Let $(I, \leqslant)$ be a directed set, and $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ a directed system in Hom-Lie. Let $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ be its direct limit which is a Hom-Lie algebra too. By the universal mapping property there are canonical maps $f_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ such that for all pairs $i \leqslant j$, the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \arrow[rr, "\varphi_{ij}"] \arrow[dr, "f_i"'] & & (\mathcal{L}, \alpha_{\mathcal{L}_j}) \arrow[dl, "f_j"] \\
& \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}). &
\end{tikzcd} \tag{3.2}$$

Since uce is a covariant functor (see (2.12)), it follows that $\{(\mathrm{uce}(\mathcal{L}_i), \mathrm{uce}(\alpha_{\mathcal{L}_i})), \mathrm{uce}(\varphi_{ij})\}$ is a directed system in Hom-Lie, where $\mathrm{uce}(\alpha_{\mathcal{L}_i})$ (which we denote by $\widetilde{\alpha}_{\mathcal{L}_i}$) is defined as in Section 2.2. In the sequel, we denote $\mathrm{uce}(\mathcal{L}_i)$ by $\widetilde{\mathcal{L}}_i$ and $\mathrm{uce}(\varphi_{ij})$ by $\widetilde{\varphi}_{ij}$. Thus $\{(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}), \widetilde{\varphi}_{ij}\}$ is a directed system in Hom-Lie. Also $\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$, the direct limit of $(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$, is a Hom-Lie algebra. If $\hat{f}_i\colon (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ are canonical maps, then we have a commutative diagram

$$\begin{tikzcd}
(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[rr, "\widetilde{\varphi}_{ij}"] \arrow[dr, "\hat{f}_i"'] & & (\widetilde{\mathcal{L}}_j, \widetilde{\alpha}_{\mathcal{L}_j}) \arrow[dl, "\hat{f}_j"] \\
& \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}). &
\end{tikzcd} \tag{3.3}$$

By construction of the central extension of the Hom-Lie algebras, let

$$u_{\mathcal{L}_i}\colon\ (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to (\mathcal{L}_i, \alpha_{\mathcal{L}_i}),$$

be the Hom-Lie algebra homomorphism in the category Hom-Lie.

By Definition 3.2 for $i \leqslant j$, we have a commutative diagram

$$\begin{tikzcd} (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[r, "u_{\mathcal{L}_i}"] \arrow[d, "\widetilde{\varphi}_{ij}"] & (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \arrow[d, "\varphi_{ij}"] \\ (\widetilde{\mathcal{L}}_j, \widetilde{\alpha}_{\mathcal{L}_j}) \arrow[r, "u_{\mathcal{L}_j}"] & (\mathcal{L}_j, \alpha_{\mathcal{L}_j}). \end{tikzcd} \tag{3.4}$$

That is, the family $\{u_{\mathcal{L}_i}\}_{i\in I}$ is a morphism from the directed system $\{(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}), \widetilde{\varphi}_{ij}\}$ to the directed system $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ in Hom-Lie. By Remark 3.3, the direct limit is a covariant functor on maps in the category of directed systems in Hom-Lie, hence for every $i \in I$ the morphism $u_{\mathcal{L}_i}$ gives rise to a unique morphism

$$\varinjlim u_{\mathcal{L}_i}\colon\ \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \tag{3.5}$$

such that $\varinjlim u_{\mathcal{L}_i} \circ \hat{f}_i = f_i \circ u_{\mathcal{L}_i}$. Then we get the commutative diagram

$$\begin{tikzcd} (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[r, "u_{\mathcal{L}_i}"] \arrow[d, "\hat{f}_i"] & (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \arrow[d, "f_i"] \\ \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[r, "\varinjlim u_{\mathcal{L}_i}"] & \varinjlim(\mathcal{L}_j, \alpha_{\mathcal{L}_j}). \end{tikzcd} \tag{3.6}$$

Now, consider the sequence

$$0 \longrightarrow (\ker(\varinjlim u_{\mathcal{L}_i}), \varinjlim \widetilde{\alpha}_{\mathcal{L}_i}|) \xrightarrow{i} \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \xrightarrow{\varinjlim u_{\mathcal{L}_i}} \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \longrightarrow 0. \tag{3.7}$$

The basic object in this note is to prove that the sequence (3.7) is a universal central extension if $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i})\}_{i\in I}$ is a family of perfect Hom-Lie algebras.

Lemma 3.4. *In the setting of* 3.2, *the sequence* (3.7) *is an extension if all the* $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ *are perfect Hom-Lie algebras.*

Proof. By the construction of the universal central extension of Hom-Lie algebras $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$, every Hom-Lie algebra homomorphism $u_{\mathcal{L}_i}\colon\ (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to (\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ is surjective. By Remark 3.3, the direct limit is a functor on maps in the category of directed systems in Hom-Lie, so every $\varinjlim u_{\mathcal{L}_i}\colon\ \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ is an epimorphism. Hence the sequence (3.7) is an extension. $\square$

Proposition 3.5. *In the setting of* 3.2, *the extension* (3.7) *is a central extension if all* $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ *are perfect Hom-Lie algebras.*

Proof. We have to show that $[\ker(\varinjlim u_{\mathcal{L}_i}), \varinjlim \widetilde{\mathcal{L}}_i] = 0$. Let $z \in \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ be such that $\varinjlim u_{\mathcal{L}_i}(z) = 0$. By Remark 3.1 (i), $z = \hat{f}_i(x)$ for some $x \in (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ and $(\varinjlim u_{\mathcal{L}_i} \circ \hat{f}_i)(x) = 0$. So by the commutative diagram (3.6), we have $f_i(u_{\mathcal{L}_i}(x)) = 0 \in \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$. Again by Remark 3.1 (ii), there is a $j \geqslant i$ such that

$$\varphi_{ij}(u_{\mathcal{L}_i}(x)) = 0 \in (\mathcal{L}_j, \alpha_{\mathcal{L}_j}). \tag{3.8}$$

Now for an arbitrary $y \in \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$, we also have $y = \hat{f}_i(y_k)$ for some $y_k \in (\widetilde{\mathcal{L}}_k, \widetilde{\alpha}_{\mathcal{L}_k})$. We must show that $[x, y] = 0$ in $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$. For the above $j, k \in I$, there exists $p \in I$ such that $p \leqslant i \leqslant j$ and $p \leqslant k$. Thus, by equation (3.8), $\varphi_{ip}(u_{\mathcal{L}_i}(x)) = (\varphi_{kp} \circ \varphi_{ik})(u_{\mathcal{L}_i}(x)) = 0 \in (\mathcal{L}_p, a_{\mathcal{L}_p})$. Hence by the commutative diagram (3.4) for $i \leqslant p$ we have $u_{\mathcal{L}_p}(\widetilde{\varphi}_{ip}(x)) = 0$. This means that $\widetilde{\varphi}_{ip}(x) \in \ker(u_{\mathcal{L}_p})$. But $(\widetilde{\mathcal{L}}_p, \widetilde{\alpha}_{\mathcal{L}_p})$ is the central extension of $(\mathcal{L}_p, \alpha_{\mathcal{L}_p})$ by $(\ker(u_{\mathcal{L}_p}), \widetilde{\alpha}_{\mathcal{L}_p}|)$. So $\ker(u_{\mathcal{L}_p}) \subseteq Z(\widetilde{\mathcal{L}}_p)$. Hence we have $\widetilde{\varphi}_{ip}(x) \in Z(\widetilde{\mathcal{L}}_p)$ and also $\widetilde{\varphi}_{pk}(y_k) \in (\widetilde{\mathcal{L}}_p, \widetilde{\alpha}_{\mathcal{L}_p})$. We then get

$$[\widetilde{\varphi}_{ip}(x), \widetilde{\varphi}_{pk}(y_k)] = 0 \in (\mathcal{L}_p, \alpha_{\mathcal{L}_p}). \tag{3.9}$$

Note that for $i \leqslant p$ by the commutative diagram (3.3) we have

$$\hat{f}_i(y_k) = (\hat{f}_p \circ \widetilde{\varphi}_{pk})(y_k). \tag{3.10}$$

Finally, by relations (3.9) and (3.10), we get

$$[z, y] = [\hat{f}_i(x), \hat{f}_i(y_k)] = [(\hat{f}_p \circ \widetilde{\varphi}_{ip})(x), (\hat{f}_p \circ \widetilde{\varphi}_{pk})(y_k)] = \hat{f}_p([\widetilde{\varphi}_{ip}(x), \widetilde{\varphi}_{pk}(y_k)]) = 0.$$

□

Now, we are ready to prove the following main theorem. Notice that its proof follows the same arguments and techniques as [25], Theorem 1.6 for the case of Lie superalgebras.

Theorem 3.6. *In the setting of* 3.2, *if* $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i})\}_{i \in I}$ *is a family of perfect Hom-Lie algebras, then the extension* (3.7) *is a universal central extension in Hom-Lie.*

Proof. Since all $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ are perfect Hom-Lie algebras, $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ is perfect also and by Theorem 2.9 admits a universal central extension

$$0 \longrightarrow (\ker(u), \varinjlim |) \xrightarrow{i} (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \xrightarrow{u} \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \longrightarrow 0. \tag{3.11}$$

Step 1. By the construction of the direct limit $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ of the directed system $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$, the canonical maps $f_i\colon (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \to \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ are homomorphisms of Hom-Lie algebras. We then get a unique Hom-Lie algebra homomorphism $\tilde{f}_i\colon (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})$ such that the following diagram commutes:

(3.12)
$$\begin{tikzcd}
(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[r, "u_{\mathcal{L}_i}"] \arrow[d, "\tilde{f}_i"] & (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \arrow[d, "f_i"] \\
(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \arrow[r, "u"] & \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}).
\end{tikzcd}$$

Now, since uce is a covariant functor, by applying this fact to the commutative diagram (3.2), we have

$$\tilde{f}_i = \mathrm{uce}(f_i) = \mathrm{uce}(f_j \circ \varphi_{ij}) = \mathrm{uce}(f_j) \circ \mathrm{uce}(\varphi_{ij}) = \tilde{f}_j \circ \widetilde{\varphi}_{ij}.$$

We then have the diagram

(3.13)
$$\begin{tikzcd}
(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[rr, "\widetilde{\varphi}_{ij}"] \arrow[dr, "\tilde{f}_i"'] & & (\widetilde{\mathcal{L}}_j, \widetilde{\alpha}_{\mathcal{L}_j}) \arrow[dl, "\tilde{f}_j"] \\
& (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) &
\end{tikzcd}$$

which commutes for all pairs $i \leqslant j$.

Next, consider the direct limit $\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ of the directed system $\{(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}), \widetilde{\varphi}_{ij}\}$ with canonical maps $\hat{f}_i\colon (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ such that $\hat{f}_i = \hat{f}_j \circ \widetilde{\varphi}_{ij}$ for all pairs $i \leqslant j$ (see the commutative diagram (3.3)). By the commutative diagram (3.13), there is a Hom-Lie algebra $(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})$ with Hom-Lie algebra homomorphisms $\tilde{f}_i\colon (\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to (\varinjlim \widetilde{\mathcal{L}}_i, \varinjlim \widetilde{\alpha}_{\mathcal{L}_i}))$ such that $\tilde{f}_i = \tilde{f}_j \circ \widetilde{\varphi}_{ij}$ for all pairs $i \leqslant j$. Then by the universal mapping property, there is a unique Hom-Lie algebra homomorphism $\psi\colon \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})$ such that $\tilde{f}_i = \psi \circ \hat{f}_i$. That is, the following diagram commutes:

(3.14)
$$\begin{tikzcd}
(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[rr, "\hat{f}_i"] \arrow[dr, "\tilde{f}_i"'] & & \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[dl, "\psi"] \\
& (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}). &
\end{tikzcd}$$

Step 2. Consider the universal central extension (3.11) and the central extension (3.7). By the universal property of $(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})$, there is a unique

Hom-Lie algebra homomorphism $\psi'\colon (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \to \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ such that $u = \varinjlim u_{\mathcal{L}_i} \circ \psi'$. In other words, the following diagram commutes:

$$\begin{tikzcd}
(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \arrow[rr, "\psi'"] \arrow[dr, "u"'] & & \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \arrow[dl, "\varinjlim u_{\mathcal{L}_i}"] \\
& \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}). &
\end{tikzcd} \tag{3.15}$$

Our goal is to prove that the central extension (3.7) is universal. For this propose we will prove that

$$(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \cong \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}).$$

Step 3. A standard application of the universal mapping property of $\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ shows that $\psi' \circ \psi = \mathrm{id}_{\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})}$. Indeed, by commutativity of diagrams (3.15), (3.12) and (3.6) we have

$$\varinjlim u_{\mathcal{L}_i} \circ (\psi' \circ \tilde{f}_i) = u \circ \tilde{f}_i = f_i \circ u_{\mathcal{L}_i} = \varinjlim u_{\mathcal{L}_i} \circ \hat{f}_i.$$

Hence, by Definition 3.2 we get

$$\psi' \circ \tilde{f}_i = \hat{f}_i. \tag{3.16}$$

Now, by the commutative diagram (3.14) and the relation (3.16), we have

$$(\psi' \circ \psi) \circ \hat{f}_i = \psi' \circ \tilde{f}_i = \hat{f}_i.$$

Therefore, $\psi' \circ \psi\colon \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \to \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})$ satisfies $\hat{f}_i = (\psi' \circ \psi) \circ \hat{f}_i$. But, $\mathrm{id}_{\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})}$ satisfies $\hat{f}_i = \mathrm{id}_{\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})} \circ \hat{f}_i$. Then by the uniqueness part of the universal mapping property of the direct limit, we conclude that $\psi' \circ \psi = \mathrm{id}_{\varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i})}$.

Step 4. Finally, the universal property of universal central extensions shows that $\psi \circ \psi' = \mathrm{id}_{(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})}$. Indeed, we observe that by commutativity of diagrams (3.14), (3.12) and (3.6) we have

$$u \circ (\psi \circ \hat{f}_i) = u \circ \tilde{f}_i = f_i \circ u_{\mathcal{L}_i} = \varinjlim u_{\mathcal{L}_i} \circ \hat{f}_i,$$

so by Definition 3.2 we get

$$u \circ \psi = \varinjlim u_{\mathcal{L}_i}. \tag{3.17}$$

Now, by using the commutative diagram (3.15) and the relation (3.17) we get

$$u \circ (\psi \circ \psi') = \varinjlim u_{\mathcal{L}_i} \circ \psi' = u.$$

Therefore, $\psi \circ \psi' \colon (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \to (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})$ satisfies $u \circ (\psi \circ \psi') = u$. However, $\mathrm{id}_{(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})}$ satisfies $u \circ \mathrm{id}_{(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})} = u$. Then by the uniqueness part of the universal property of universal central extension, we conclude that $\psi \circ \psi' = \mathrm{id}_{(\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}})}$. □

Corollary 3.7. *In the setting of* 3.2, *if* $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i})\}_{i \in I}$ *is a family of perfect Hom-Lie algebras, then*

$$\varinjlim H_2^{\alpha}(\widetilde{\mathcal{L}}_i) \cong H_2^{\alpha}(\varinjlim \widetilde{\mathcal{L}}_i).$$

Proof. Since all $(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ are perfect Hom-Lie algebras, $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ is also perfect and by Theorem 2.9 admits a universal central extension

$$0 \longrightarrow (\ker(u), \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}|) \xrightarrow{i} (\widetilde{\varinjlim \mathcal{L}_i}, \widetilde{\varinjlim \alpha_{\mathcal{L}_i}}) \xrightarrow{u} \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \longrightarrow 0. \tag{3.18}$$

Note that $\ker(u) \cong H_2^{\alpha}(\varinjlim \mathcal{L}_i)$ by Theorem 4.11 in [10]. Consider the universal central extension

$$0 \longrightarrow (H_2^{\alpha}(\mathcal{L}_i), \alpha_{\mathcal{L}_i}|) \longrightarrow (\widetilde{\mathcal{L}}_i, \alpha_{\widetilde{\mathcal{L}}_i}) \xrightarrow{u_{\mathcal{L}_i}} (\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \longrightarrow 0, \tag{3.19}$$

we see that $\{(H_2^{\alpha}(\mathcal{L}_i), \alpha_{\mathcal{L}_i}|), \widetilde{\varphi}_{ij}|\}$ is a directed system in Hom-Lie. By Remark 3.3 the direct limit preserves exact sequences, so from 3.19 we derive that

$$0 \longrightarrow (\varinjlim H_2^{\alpha}(\widetilde{\mathcal{L}}_i), \varinjlim \widetilde{\alpha}_{\mathcal{L}_i}|) \longrightarrow \varinjlim(\widetilde{\mathcal{L}}_i, \widetilde{\alpha}_{\mathcal{L}_i}) \xrightarrow{\varinjlim u_{\mathcal{L}_i}} \varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i}) \longrightarrow 0 \tag{3.20}$$

is an extension which is a universal central extension. By Theorem 3.6 the two extensions (3.18) and (3.20) are equivalent and we get the result. □

Corollary 3.8. *In the setting of* 3.2, *if* $\{(\mathcal{L}_i, \alpha_{\mathcal{L}_i}), \varphi_{ij}\}$ *is a directed system of perfect and centrally closed Hom-Lie algebras, then* $\varinjlim(\mathcal{L}_i, \alpha_{\mathcal{L}_i})$ *is perfect and centrally closed Hom-Lie algebra.*

Proof. By Remark 3.3 and Theorem 3.6, we get the results. □

4. Example: Universal central extension of the direct limit of an A-type Hom-Lie algebra

In this section, we will provide an example of the universal central extension of the direct limit of regular Hom-Lie algebras. More precisely, we will consider a family of multiplicative Hom-Lie algebras $\{(\mathrm{sl}_k(\mathcal{A}), \alpha_k)\}_{k\in I}$ with α_k invertible where $(\mathcal{A}, \alpha_{\mathcal{A}})$ is a unital multiplicative Hom-associative algebra (see Definition 2.1), and I is a not necessarily finite set of cardinality $|I| \geqslant 5$. We will describe the universal central extension of the direct limit of $(\mathrm{sl}_k(\mathcal{A}), \alpha_k)$.

Definition 4.1 ([19]). Let $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ be a Hom-Lie algebra. If there exists a Lie algebra $(\mathcal{L}, [\cdot,\cdot]')$ such that $[x,y] = \alpha([x,y]') = [\alpha(x), \alpha(y)]'$ for all $x, y \in \mathcal{L}$, then $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ is said to be of Lie-type, and $(\mathcal{L}, [\cdot,\cdot]')$ is called the compatible Lie algebra of $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$. Furthermore, if the compatible Lie algebra of $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ is a classical type Lie algebra, one calls $(\mathcal{L}, [\cdot,\cdot], \alpha_{\mathcal{L}})$ a classical type Hom-Lie algebra.

Let $(\mathcal{A}, \alpha_{\mathcal{A}})$ be a unital Hom-associative algebra where $\alpha_{\mathcal{A}}$ is an automorphism of $\mathcal{A}$. Let I be a not necessarily finite set of cardinality $|I| \geqslant 3$. We denote by $\mathrm{Mat}_I(\mathcal{A})$ the set of all finitary $I \times I$-matrices with entries from $\mathcal{A}$ (only finitely many nonzero entries). By definition, an element of $\mathrm{Mat}_I(\mathcal{A})$ is a square matrix $x = (x_{ij})_{i,j\in I}$ with all $x_{ij} \in \mathcal{A}$ and $x_{ij} \neq 0$ for only finitely many indices $(i,j) \in I \times I$. We can view $(\mathrm{Mat}_I(\mathcal{A}), \alpha_M)$ as a multiplicative Hom-associative algebra (see [16]). We also denote by $(\mathrm{gl}_I(\mathcal{A}), \alpha_g)$ the multiplicative Hom-Lie algebra associated to the multiplicative Hom-associative algebra $(\mathrm{Mat}_I(\mathcal{A}), \alpha_M)$ (see [13]). Let $\mathrm{sl}_I(\mathcal{A}) = [\mathrm{gl}_I(\mathcal{A}), \mathrm{gl}_I(\mathcal{A})]$ be its derived algebra which can be described as the subalgebra of $(\mathrm{gl}_I(\mathcal{A}), \alpha_g)$ generated by $e_{ij}(a)$ for $i \neq j$, $a \in \mathcal{A}$, where e_{ij} is the standard matrix unit. The elements $e_{ij}(a)$ satisfy certain canonical relations. Clearly, for any $a, b \in \mathcal{A}$ for $i \neq j$, we have $[e_{ij}(a), e_{jk}(b)] = e_{ik}([a,b])$, and if i, j, k are distinct we have $[e_{ij}(a), e_{kl}] = 0$. If $|I| = n \in \mathbb{N}$, then $\mathrm{sl}_I(\mathcal{A}) = \mathrm{sl}_n(\mathcal{A})$. Note that $(\mathrm{sl}_I(\mathcal{A}), \alpha_s)$ is a regular Hom-Lie algebra of type A_I, where α_s is the invertible restriction linear map of α_g. One can check that $(\mathrm{sl}_I(\mathcal{A}), \alpha_s)$ is perfect.

Remark 4.2. One can consider the regular Hom-Lie algebra $(\mathrm{sl}_I(\mathcal{A}), \alpha_s)$ defined by

$$\mathrm{sl}_I(\mathcal{A}) = \{X \in \mathrm{Mat}_I(\mathcal{A})\colon\ \mathrm{tr}(X) \in [\mathcal{A}, \mathcal{A}]\}.$$

Here the trace of a matrix $X = X_{ij} \in \mathrm{Mat}_I(\mathcal{A})$ is given by $\mathrm{tr}(X) = \sum_{i\in I} X_{ii}$ and $[\mathcal{A}, \mathcal{A}]$ is the span of all commutators $[a,b]$, $a, b \in \mathcal{A}$. We also define $\alpha_s(X) = A^{-1}XA$ where A is an arbitrary invertible $I \times I$-matrix.

Now we will describe the universal central extension of $(\mathrm{sl}_I(\mathcal{A}), \alpha_s)$ which is the direct limit of $\{(\mathrm{sl}_n(\mathcal{A}), \alpha_n)\}_{n\in\mathbb{N}}$ (see Proposition 4.1 in [6]). To do this, let $\mathcal{A} \wedge \mathcal{A} =$

$(\mathcal{A}\otimes\mathcal{A})/J(\mathcal{A},\alpha_{\mathcal{A}})$, where

$$
\begin{aligned}
J(\mathcal{A},\alpha_{\mathcal{A}}) := \operatorname{span}_{\mathbb{K}}(\{&a\otimes b - b\otimes a\colon\ a,b\in\mathcal{A}\}\\
&\cup\{ab\otimes\alpha_{\mathcal{A}}(c)+bc\otimes\alpha_{\mathcal{A}}(a)+ca\otimes\alpha_{\mathcal{A}}(b)\colon\ a,b,c\in\mathcal{A}\}).
\end{aligned}
$$

It follows that $\mathcal{A}\wedge\mathcal{A}$ is spanned by elements $a\wedge b = a\otimes b + J(\mathcal{A},\alpha_{\mathcal{A}})$, $a,b\in\mathcal{A}$, and that the map $\nu\colon\ \mathcal{A}\wedge\mathcal{A}\to[\mathcal{A},\mathcal{A}] : a\wedge b\mapsto[a,b]$ is well-defined and also

$$HC_1^{\alpha}(\mathcal{A}) = \ker\nu = \Big\{\sum_i a_i\wedge b_i\colon\ \sum_i [a_i,b_i]=0\Big\}.$$

Now, we introduce the multiplicative Steinberg Hom-Lie algebra $(\mathrm{st}_I(\mathcal{A}),\overline{\alpha}_s)$ associated to $\mathcal{A}$ (see [14] for Steinberg Lie algebra), where $\overline{\alpha}_s$ is the induced map of α_s. It is the multiplicative Hom-Lie algebra presented by generators $X_{ij}(a)$, where $i,j\in I$, $i\neq j$, $a\in\mathcal{A}$, subject to the relations (i)–(iii):

(i) The map

$$\varphi\colon\ (\mathcal{A},\alpha_{\mathcal{A}})\to(\mathrm{st}_I(\mathcal{A}),\overline{\alpha}_s),\quad a\mapsto X_{ij}(a),$$

is $\mathbb{K}$-linear such that $\varphi\circ\alpha_{\mathcal{A}} = \overline{\alpha}_s\circ\varphi$,

(ii) for distinct i,j,k we have $[X_{ij}(a),X_{jk}(b)] = X_{ik}(ab)$,

(iii) for $i\neq l$ and $j\neq k$, we have $[X_{ij}(a),X_{kl}]=0$, where $a,b\in\mathcal{A}$ and $i,j,k\in I$.

Both Hom-Lie algebras $(\mathrm{sl}_I(\mathcal{A}),\alpha_s)$ and $(\mathrm{st}_I(\mathcal{A}),\overline{\alpha}_s)$ are perfect. Consider the canonical multiplicative Hom-Lie algebra epimorphism

$$\pi_I\colon\ \mathrm{st}_I(\mathcal{A})\to\mathrm{sl}_I(\mathcal{A}),\quad X_{ij}(a)\mapsto e_{ij}(a).$$

The following result is well known for Lie algebra setting, see [8], [17].

Proposition 4.3. *If $n\geqslant 5$, then*

$$0\longrightarrow HC_1(\mathcal{A})\twoheadrightarrow\mathrm{st}_n(\mathcal{A})\xrightarrow{\pi_n}\mathrm{sl}_n(\mathcal{A})\longrightarrow 0,$$

is the universal covering with $\ker\pi_I = HC_1(\mathcal{A})$ and $H_2(\mathrm{st}_n(\mathcal{A}))=0$.

Now, let $\mathcal{K}$ be the set of finite subsets of I ordered by inclusion. We know that the multiplicative Hom-Lie algebra $(\mathrm{sl}_I(\mathcal{A}),\alpha_s)$ is a direct union of multiplicative Hom-Lie algebras of type $A_k (k\in\mathcal{K})$, i.e., $(\mathrm{sl}_I(\mathcal{A}),\alpha_s) = \bigcup_{k\in\mathcal{K}}(\mathrm{sl}_k(\mathcal{A}),\alpha_k)$, so

$$(\mathrm{sl}_I(\mathcal{A}),\alpha_s)\cong\varinjlim_{k\in\mathcal{K}}(\mathrm{sl}_k(\mathcal{A}),\alpha_k).$$

By applying uce functor and Proposition 4.3, we have

$$(\widetilde{\mathrm{sl}_I(\mathcal{A})}, \widetilde{\alpha}_s) \cong \varinjlim_{k\in\mathcal{K}} (\mathrm{sl}_k(\mathcal{A}), \alpha_k).$$

Thus, it is enough to show that

$$(\mathrm{st}_I(\mathcal{A}), \widetilde{\alpha}_s) \cong \varinjlim_{k\in\mathcal{K}} (\mathrm{sl}_k(\mathcal{A}), \alpha_k).$$

Using an argument similar to that in the proof of Theorem 3.6, in step 1, we see that the diagram

$$\begin{tikzcd} (\mathrm{st}_k(\mathcal{A}), \overline{\alpha}_k) \arrow[rr, "\widetilde{\varphi}_{km}"] \arrow[dr, "\tilde{f}_k"'] & & (\mathrm{st}_m(\mathcal{A}), \overline{\alpha}_m) \arrow[dl, "\tilde{f}_m"] \\ & \varinjlim_{k\in\mathcal{K}} (\mathrm{st}_k(\mathcal{A}), \widetilde{\alpha}_k) & \end{tikzcd} \tag{4.1}$$

commutes for all pairs $k \leqslant m$. Now considering the direct limit $(\mathrm{st}_I(\mathcal{A}), \overline{\alpha}_s)$ of the directed system $\{(\mathrm{st}_k(\mathcal{A}), \overline{\alpha}_k), \widetilde{\varphi}_{km}\}$ with canonical maps $\hat{f}_k\colon (\mathrm{st}_k(\mathcal{A}), \overline{\alpha}_k) \to (\mathrm{st}_I(\mathcal{A}), \overline{\alpha}_s)$ by $X_{ij} \mapsto X_{ij}$, on generators and extend linearity, we have by the universal mapping property that there exists a multiplicative Hom-Lie algebra homomorphism

$$\psi\colon (\mathrm{st}_I(\mathcal{A}), \widetilde{\alpha}_s) \to \varinjlim_{k\in\mathcal{K}} (\mathrm{st}_k(\mathcal{A}), \widetilde{\alpha}_k).$$

The families $\{X_{ij}(a) \in \mathrm{st}_k(\mathcal{A})\colon k \in \mathcal{K}\}$ give rise to elements $X_{ij}(a) \in \varinjlim_{k\in\mathcal{K}} \mathrm{st}_k(\mathcal{A})$ satisfying the relations (i)–(iii). Then there is a multiplicative Hom-Lie algebra homomorphism

$$\psi'\colon \varinjlim_{k\in\mathcal{K}} (\mathrm{st}_k(\mathcal{A}), \widetilde{\alpha}_k) \to (\mathrm{st}_I(\mathcal{A}), \widetilde{\alpha}_s),$$

which is defined by sending $X_{ij}(a) \in \varinjlim_{k\in\mathcal{K}} \mathrm{st}_k(\mathcal{A})$ to $X_{ij}(a) \in \mathrm{st}_k(\mathcal{A})\colon k \in \mathcal{K}$ such that the diagram

$$\begin{tikzcd} (\mathrm{st}_k(\mathcal{A}), \overline{\alpha}_k) \arrow[rr, "\widetilde{\varphi}_{km}"] \arrow[dr, "\hat{f}_k"'] & & (\mathrm{st}_m(\mathcal{A}), \overline{\alpha}_m) \arrow[dl, "\hat{f}_m"] \\ & (\mathrm{st}_I(\mathcal{A}), \widetilde{\alpha}_s) & \end{tikzcd} \tag{4.2}$$

commutes. Again, by an argument similar to that in the proof of Theorem 3.6, one can prove that ψ and ψ' are inverses of each other, and that $\ker(\pi_I) = HC_1^{\alpha}(\mathcal{A})$, where $HC_1(\mathcal{A})$, is the first cyclic homology group of $\mathcal{A}$.

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