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GENERALIZATION OF THE WEAK AMENABILITY ON VARIOUS
BANACH ALGEBRAS

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Abstract. The generalized notion of weak amenability, namely (φ, ψ) -weak amenability, where φ, ψ are continuous homomorphisms on a Banach algebra \mathcal{A} , was introduced by Bodaghi, Eshaghi Gordji and Medghalchi (2009). In this paper, the (φ, ψ) -weak amenability on the measure algebra $M(G)$, the group algebra $L^1(G)$ and the Segal algebra $S^1(G)$, where G is a locally compact group, are studied. As a typical example, the (φ, ψ) -weak amenability of a special semigroup algebra is shown as well.

Keywords: Banach algebra; (φ, ψ) -derivation; group algebra; locally compact group; measure algebra; Segal algebra; weak amenability

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1. INTRODUCTION

Let \mathcal{A} be a Banach algebra and X be a Banach \mathcal{A} -bimodule. Let the products of $a \in \mathcal{A}$ and $x \in X$ be denoted by $a \cdot x$ and $x \cdot a$. A derivation $D: \mathcal{A} \rightarrow X$ is a linear map which satisfies $D(ab) = a \cdot D(b) + D(a) \cdot b$ for all $a, b \in \mathcal{A}$. The derivation δ is said to be *inner* if there exists $x \in X$ such that $\delta(a) = \delta_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. The linear space of bounded derivations from \mathcal{A} into X is denoted by $Z^1(\mathcal{A}, X)$ and the linear subspace of inner derivations is denoted by $N^1(\mathcal{A}, X)$. We consider the quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$, called the *first Hochschild cohomology group* of \mathcal{A} with coefficients in X . A Banach algebra \mathcal{A} is amenable if every continuous derivation from \mathcal{A} into every dual Banach \mathcal{A} -module is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach \mathcal{A} -module X . This definition was introduced by Johnson in [12]. One of the important results that Johnson obtained was that the group algebra $L^1(G)$ is amenable if and only if the

locally compact group G is amenable. Also, a Banach algebra \mathcal{A} is called *weakly amenable* if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Weak amenability of the group algebra $L^1(G)$ was proved by Johnson in [13] for the first time. After that, Despić and Ghahramani in [8] gave a different and shorter proof for it; see also [1] and [10]. Dales and Pandey studied the weak amenability of special case of Segal algebras in [7], and then the weak amenability of Segal algebras and the Lebesgue-Fourier algebra of a locally compact group were proved by Ghahramani and Lau in [9]. For more details about Segal algebras refer to [16].

Let X be an \mathcal{A} -bimodule and let φ, ψ be continuous homomorphisms of \mathcal{A} into itself. A bounded linear mapping $d: \mathcal{A} \rightarrow X$ is called a (φ, ψ) -*derivation* if

$$d(ab) = d(a) \cdot \varphi(b) + \psi(a) \cdot d(b), \quad a, b \in \mathcal{A}.$$

A bounded linear mapping $d: \mathcal{A} \rightarrow X$ is called a (φ, ψ) -*inner derivation* if there exists $x \in X$ such that

$$d(a) = x \cdot \varphi(a) - \psi(a) \cdot x, \quad a \in \mathcal{A}.$$

Let \mathcal{A} be Banach algebra and let φ and ψ be as in the above. We consider the following module actions on \mathcal{A} :

$$a \cdot x := \varphi(a)x, \quad x \cdot a := x\psi(a), \quad a, x \in \mathcal{A}.$$

We denote the above \mathcal{A} -bimodule by $\mathcal{A}_{(\varphi, \psi)}$. Then \mathcal{A} is called (φ, ψ) -*weakly amenable* if $H^1(\mathcal{A}, (\mathcal{A}_{(\varphi, \psi)})^*) = \{0\}$. These concepts are introduced and investigated in [4] and [15] (for the generalization of n -weak amenability refer to [5]). It is also proved in [4], Example 4.2 that for any locally compact group algebra G , $L^1(G)$ is $(\varphi, 0)$ and $(0, \psi)$ -weakly amenable. For the module versions of these notions refer to [3] and [2].

Let \mathcal{A} and \mathcal{B} be Banach algebras. Similarly to [4], we denote by $\text{Hom}(\mathcal{A}, \mathcal{B})$ the space of all bounded homomorphisms from \mathcal{A} into \mathcal{B} and denote $\text{Hom}(\mathcal{A}, \mathcal{A})$ by $\text{Hom}(\mathcal{A})$. Suppose that \mathcal{A} is a Banach algebra, X is a Banach \mathcal{A} -module and $\varphi, \psi \in \text{Hom}(\mathcal{A})$. A derivation $D: \mathcal{A} \rightarrow X$ is called *approximately (φ, ψ) -inner* if there exists a net (x_α) in X such that for all $a \in \mathcal{A}$, $D(a) = \lim_{\alpha} (x_\alpha \cdot \varphi(a) - \psi(a) \cdot x_\alpha)$ in norm. A Banach algebra \mathcal{A} is *approximately (φ, ψ) -weakly amenable* if every derivation $D: \mathcal{A} \rightarrow (\mathcal{A}_{(\varphi, \psi)})^*$ is approximately (φ, ψ) -inner.

In this work, we prove that if the group algebra $L^1(G)$ is a two-sided G -module, then it is (φ, ψ) -amenable. We also show that (φ, ψ) -weak amenability of the measure algebra $M(G)$ necessitates G being discrete and amenable locally compact group. Finally, we investigate (approximate) (φ, ψ) -weak amenability of Segal algebras.

2. RESULTS FOR GROUP ALGEBRAS

In this section, by providing some new results on (φ, ψ) -weak amenability of Banach algebras, we study the (φ, ψ) -weak amenability of group algebra $L^1(G)$, where $\varphi, \psi \in \text{Hom}(L^1(G))$ and G is a locally compact group. From now on, the notation $\langle f, x \rangle$ means the value of the mapping f at x .

Proposition 2.1. *Let \mathcal{A} and \mathcal{B} be Banach algebras such that \mathcal{B} is a Banach \mathcal{A} -bimodule and $\varphi, \psi \in \text{Hom}(\mathcal{A})$. Suppose that $\Phi: \mathcal{A} \rightarrow \mathcal{B}$ is an \mathcal{A} -bimodule morphism with dense range, and $d: \mathcal{A} \rightarrow (\mathcal{B}_{(\varphi, \psi)})^*$ is a nonzero (φ, ψ) -derivation. Then $D := \Phi^* \circ d$ is a nonzero (φ, ψ) -derivation.*

Proof. For every $x, y, z \in \mathcal{A}$ we have

$$\begin{aligned} \langle D(xy), z \rangle &= \langle \Phi^* \circ d(xy), z \rangle = \langle \Phi^*(d(x) \cdot \varphi(y) + \psi(x) \cdot d(y)), z \rangle \\ &= \langle d(x) \cdot \varphi(y) + \psi(x) \cdot d(y), \Phi(z) \rangle \\ &= \langle d(x), \varphi(y) \cdot \Phi(z) \rangle + \langle d(y), \Phi(z) \cdot \psi(x) \rangle \\ &= \langle d(x), \Phi(\varphi(y)z) \rangle + \langle d(y), \Phi(z\psi(x)) \rangle \\ &= \langle \Phi^*(d(x)), \varphi(y)z \rangle + \langle \Phi^*(d(y)), z\psi(x) \rangle \\ &= \langle D(x) \cdot \varphi(y) + \psi(x) \cdot D(y), z \rangle. \end{aligned}$$

Therefore D is a (φ, ψ) -derivation. If $D = 0$, then for every $x, y \in \mathcal{A}$ we have $\langle D(x), y \rangle = 0$. Thus, $\langle \Phi^* \circ d(x), y \rangle = \langle d(x), \Phi(y) \rangle = 0$. This means that $d(\mathcal{A}) = 0$, and so $d = 0$. \square

Theorem 2.2. *Let \mathcal{A} be a Banach algebra, \mathcal{B} be a closed subalgebra of \mathcal{A} , and \mathcal{I} be a closed ideal of \mathcal{A} such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{I}$. If \mathcal{A} is (φ, ψ) -weakly amenable, where $\varphi, \psi \in \text{Hom}(\mathcal{A})$ and $\varphi(\mathcal{B}), \psi(\mathcal{B}) \subseteq \mathcal{B}$, then \mathcal{B} is (φ, ψ) -weakly amenable.*

Proof. Let $\pi: \mathcal{A} \rightarrow \mathcal{B}$ be the natural projection from \mathcal{A} onto \mathcal{B} . For every $a, b \in \mathcal{A}$ there are $x, y \in \mathcal{B}$ such that $a = x + I$ and $b = y + I$. Then $\pi(ab) = xy = x\pi(b) = \pi(a)y$. Suppose that $d: \mathcal{B} \rightarrow (\mathcal{B}_{(\varphi, \psi)})^*$ is an arbitrary (φ, ψ) -derivation. From Proposition 2.1, $D = \pi^* \circ d: \mathcal{A} \rightarrow (\mathcal{A}_{(\varphi, \psi)})^*$ is a (φ, ψ) -derivation. Since \mathcal{A} is (φ, ψ) -weakly amenable, then there exists an element $\xi \in (\mathcal{A}_{(\varphi, \psi)})^*$ such that

$$D(x) = \xi \cdot \varphi(x) - \psi(x) \cdot \xi, \quad x \in \mathcal{A}.$$

Set $\eta = \xi|_{\mathcal{B}}$. Then

$$\begin{aligned} \langle d(x), y \rangle &= \langle d(x), \pi(y) \rangle = \langle \pi^* \circ d(x), y \rangle = \langle D(x), y \rangle \\ &= \langle \xi \cdot \varphi(x) - \psi(x) \cdot \xi, y \rangle = \langle \xi, \varphi(x)y \rangle - \langle \xi, y\psi(x) \rangle \\ &= \langle \eta, \varphi(x)y \rangle - \langle \eta, y\psi(x) \rangle = \langle \eta \cdot \varphi(x) - \psi(x) \cdot \eta, y \rangle \end{aligned}$$

for all $x, y \in \mathcal{B}$ (note that \mathcal{B} is a closed subalgebra of \mathcal{A}). Hence, d is a (φ, ψ) -inner derivation. \square

Let G be a non-discrete locally compact group. Then $M(G) = M_d(G) \oplus M_c(G)$, where $M_d(G)$ and $M_c(G)$ are discrete measures and continuous measures on G , respectively (see Theorem 19.20 of [11]).

Corollary 2.3. *Let $M(G)$ be (φ, ψ) -weakly amenable, where $\varphi, \psi \in \text{Hom}(M(G))$. If $\varphi(M_d(G)) \subseteq M_d(G)$ and $\psi(M_d(G)) \subseteq M_d(G)$, then $M_d(G)$ is (φ, ψ) -weakly amenable.*

Remark 2.4. In [6], Theorem 1.2, Dales, Ghahramani and Helemskii showed that $M(G)$ is weakly amenable if and only if G is discrete if and only if there is no nonzero continuous point derivation at a character of $M(G)$. Similarly, suppose that $\varphi \in \text{Hom}(M(G))$ which is onto. If $M(G)$ is (φ, φ) -weakly amenable and ψ is a nonzero character on $M(G)$, then by Theorem 2.10 of [4], there are no nonzero point derivations at $\psi \circ \varphi$. This means that G is discrete and $M(G)$ is weakly amenable. Converse of this assertion is true when φ and ψ are identity homomorphisms. This shows that the class of (φ, ψ) -weakly amenable Banach algebras is wider than that of weakly amenable Banach algebras.

Let X be a Banach space, a net $(m_\alpha) \subset X^*$ is said to converge *weak \sim* to $m \in X^*$ if $m_\alpha \xrightarrow{w^*} m$ and $\|m_\alpha\| \rightarrow \|m\|$. This notion was introduced by Lau and Loy in [14]. For a particular case, if $\mu \in M(G)$, suppose that $\nu \in L^\infty(G)^*$ is a norm preserving extension of μ . Then there exists a net $(f_\gamma) \subset L^1(G)$ with $\|f_\gamma\| \leq \|\mu\|$, and $f_\gamma \xrightarrow{w^*} \nu$. Passing to a suitable subnet, we can write $\|f_\gamma\| \rightarrow \|\mu\|$, so we have $f_\gamma \rightarrow \mu$ in weak \sim . Let \mathcal{A} be a Banach algebra. We say that an operator φ satisfies the weak \sim condition if for every net $(m_\alpha) \subset \mathcal{A}^*$ which converges weak \sim to $m \in \mathcal{A}^*$, $m_\alpha \xrightarrow{w^*} \varphi(m)$ holds.

Theorem 2.5. *Let G be a locally compact group and X be a $M(G)$ -bimodule with the module actions $\mu \cdot x = \tilde{\varphi}(\mu) \cdot x$ and $x \cdot \mu = x \cdot \tilde{\psi}(\mu)$, where $\varphi, \psi \in \text{Hom}(L^1(G))$ and $\tilde{\varphi}, \tilde{\psi}$ are extensions of φ and ψ , respectively. Then every (φ, ψ) -derivation $D: L^1(G) \rightarrow X^*$ extends to a unique $(\tilde{\varphi}, \tilde{\psi})$ -derivation from $M(G)$ into X^* .*

Proof. Let $\mu \in M(G)$. Take a net $(f_\gamma) \subset L^1(G)$ such that $f_\gamma \rightarrow \mu$ weak \sim . Then the net (Df_γ) converges in w^* -topology. Define $\overline{D}\mu = w^*\text{-}\lim_{\gamma} Df_\gamma$. Thus, \overline{D} is a bounded linear operator which extends D . For every $x \in X$, $g_1, g_2 \in L^1(G)$ and $\mu \in M(G)$ we have

$$\begin{aligned} (2.1) \quad \langle \overline{D}(\mu), g_1 \cdot x \cdot g_2 \rangle &= w^*\text{-}\lim_{\gamma} \langle D(f_\gamma), g_1 \cdot x \cdot g_2 \rangle = w^*\text{-}\lim_{\gamma} \langle D(f_\gamma) \cdot g_1, x \cdot g_2 \rangle \\ &= w^*\text{-}\lim_{\gamma} \langle D(f_\gamma) \cdot \varphi(g_1), x \cdot g_2 \rangle \end{aligned}$$

$$\begin{aligned}
&= w^*-\lim_{\gamma} \langle D(f_{\gamma} \cdot g_1) - \psi(f_{\gamma}) \cdot D(g_1), x \cdot g_2 \rangle \\
&= w^*-\lim_{\gamma} \langle D(f_{\gamma} \cdot g_1), x \cdot g_2 \rangle - w^*-\lim_{\gamma} \langle \psi(f_{\gamma}) \cdot D(g_1), x \cdot g_2 \rangle \\
&= \langle \overline{D}(\mu \cdot g_1), x \cdot g_2 \rangle - \langle \tilde{\psi}(\mu) \cdot D(g_1), x \cdot g_2 \rangle \\
&= \langle \overline{D}(\mu \cdot g_1), x \cdot g_2 \rangle - \langle D(g_1), x \cdot g_2 \cdot \tilde{\psi}(\mu) \rangle.
\end{aligned}$$

Now, we show that \overline{D} is a $(\tilde{\varphi}, \tilde{\psi})$ -derivation. As above, assume that $f_{\gamma} \rightarrow \mu$ weak \sim . By (2.1) we get

$$\begin{aligned}
\langle \overline{D}(\eta\mu), g \cdot x \rangle &= \lim_{\gamma} \langle \overline{D}(\eta \cdot f_{\gamma}), g \cdot x \rangle \\
&= \lim_{\gamma} \langle \overline{D}(\eta), f_{\gamma} g \cdot x \rangle + \lim_{\gamma} \langle D(f_{\gamma}), g \cdot x \cdot \tilde{\psi}(\eta) \rangle \\
&= \lim_{\gamma} \langle \overline{D}(\eta) \cdot \varphi(f_{\gamma}), g \cdot x \rangle + \lim_{\gamma} \langle D(f_{\gamma}), g \cdot x \cdot \tilde{\psi}(\eta) \rangle \\
&= \lim_{\gamma} \langle \overline{D}(\eta) \cdot \varphi(f_{\gamma}), g \cdot x \rangle + \lim_{\gamma} \langle \tilde{\psi}(\eta) \cdot D(f_{\gamma}), g \cdot x \rangle \\
&= \langle \overline{D}(\eta) \cdot \tilde{\varphi}(\mu) + \tilde{\psi}(\eta) \cdot \overline{D}(\mu), g \cdot x \rangle
\end{aligned}$$

for every $\eta \in M(G)$, $g \in L^1(G)$ and $x \in X$ (note that $\|f_{\gamma} * g - \mu * g\| \rightarrow 0$ and $L^1(G) \cdot X = X$). For the uniqueness of \overline{D} , let D' be another $(\tilde{\varphi}, \tilde{\psi})$ -derivation which extends D to $M(G)$. Let f_{γ} and μ be as above, then $D'(\mu) = \lim_{\gamma} D(f_{\gamma}) = \overline{D}(\mu)$. \square

Let G be a locally compact group. A Banach space X is called a *left Banach G -module* if the following statements hold:

- (1) There is $k \geq 0$ such that $\|g \cdot x\| \leq k\|x\|$ for every $g \in G, x \in X$;
- (2) For $x \in X$, the map $G \rightarrow X: g \mapsto g \cdot x$ is continuous.

Similarly for the right Banach G -modules and two-sided Banach G -modules, where in the latter case we require the map $G \times G \rightarrow X: (g_1, g_2) \mapsto g_1 \cdot x \cdot g_2$ to be continuous. If X^* is the dual of X , then X^* is a two-sided G -module with the actions defined as follows:

$$\langle f \cdot \theta, x \rangle = \langle f, \theta \cdot x \rangle \quad \text{and} \quad \langle \theta \cdot f, x \rangle = \langle f, x \cdot \theta \rangle$$

for every $\theta \in G$, $x \in X$ and $f \in X^*$. Here we use the technique of the proof in [8] to show that $L^1(G)$ is (φ, ψ) -weakly amenable. In fact, we generalize the result of [4], Example 4.2 which asserts that for any locally compact group G , $L^1(G)$ is (φ, ψ) -weakly amenable in which either φ or ψ is zero homomorphism.

Theorem 2.6. *Let $L^1(G)$ be a two-sided Banach G -module and let $\varphi, \psi \in \text{Hom}(L^1(G))$. Then $L^1(G)$ is (φ, ψ) -weak amenable.*

Proof. Let $D: L^1(G) \rightarrow L^\infty(G)$ be a (φ, ψ) -derivation. By Theorem 2.5, it suffices to show that the extension of D to $M(G)$ is inner. For $t \in G$, by δ_t we mean the point mass at t . Then

$$\begin{aligned}
(2.2) \quad \psi(\delta_{t-1}) \cdot D(\delta_t) &= \psi(\delta_{t-1}) \cdot D(\delta_{tx^{-1}} * \delta_x) \\
&= \psi(\delta_{t-1})\psi(\delta_{tx^{-1}}) \cdot D(\delta_x) + \psi(\delta_{t-1}) \cdot D(\delta_{tx}) \cdot \varphi(\delta_x) \\
&= \psi(\delta_{x-1}) \cdot D(\delta_x) + \psi(\delta_{x-1}) \cdot \psi(\delta_{(tx^{-1})^{-1}}) \cdot D(\delta_{tx}) \cdot \varphi(\delta_x).
\end{aligned}$$

For every $\lambda \in L^\infty(G)$ let $\operatorname{Re}(\lambda)$ denote the real part of λ and let

$$S = \{\operatorname{Re}(\psi(\delta_{t-1}) \cdot D(\delta_t)) : t \in G\}.$$

Take $\xi = \sup(S)$. Since $L^1(G)$ is two-sided G -module, then we obtain

$$\begin{aligned}
(2.3) \quad \sup(\psi(\delta_{x-1}) \cdot S \cdot \varphi(\delta_x)) &= \psi(\delta_{x-1}) \cdot \sup(S) \cdot \varphi(\delta_x) \quad \text{and} \\
\sup(\lambda + S) &= \lambda + \sup(S), \quad x \in G, \lambda \in L^\infty(G).
\end{aligned}$$

Now, by (2.2) and (2.3) we have

$$\xi = \psi(\delta_{x-1}) \cdot \operatorname{Re}(D(\delta_x)) + \psi(\delta_{x-1}) \cdot \xi \cdot \varphi(\delta_x),$$

then

$$\operatorname{Re}(D(\delta_x)) = \psi(\delta_x) \cdot \xi - \xi \cdot \varphi(\delta_x)$$

for every $x \in G$. Similarly, for imaginary part there exists an element $\zeta \in L^\infty(G)$ such that

$$\operatorname{Im}(D(\delta_x)) = \psi(\delta_x) \cdot \zeta - \zeta \cdot \varphi(\delta_x)$$

for every $x \in G$. Therefore by taking $\varsigma = \xi + i\zeta$, we find

$$D(\delta_x) = \psi(\delta_x) \cdot \varsigma - \varsigma \cdot \varphi(\delta_x)$$

for every $x \in G$. Let $\mu \in M(G)$, then there exists a net (μ_α) with each μ_α a linear combination of point masses such that $\mu_\alpha \rightarrow \mu$ in strong topology. Note that point masses are extreme points of $M(G)$. On the other hand, we can take this net to be the one from the proof of Theorem 2.5. Then

$$\overline{D}(\mu) = \tilde{\psi}(\mu) \cdot \varsigma - \varsigma \cdot \tilde{\varphi}(\mu)$$

for every $\mu \in M(G)$. This means that $L^1(G)$ is (φ, ψ) -weak amenable. \square

Example 2.7. Suppose that S is a discrete infinite semigroup and s_0 is a fixed element in S . Define an algebra product in $l^1(S)$ via $st := s(s_0)t$, $s, t \in l^1(S)$. This Banach algebra has been introduced by Yong Zang in [17]. For every $\varphi, \psi \in \text{Hom}(l^1(S))$ we show that $l^1(S)$ is (φ, ψ) -weakly amenable. This Banach algebra has a left identity e_0 defined by

$$e_0(s) = \begin{cases} 1 & \text{if } s = s_0, \\ 0 & \text{if } s \neq s_0. \end{cases}$$

The $l^1(S)$ -bimodule actions on the dual module $l^1(S)^* = l^\infty(S)$ are in fact defined as follows:

$$f \cdot s = s(s_0)f, \quad s \cdot f = f(s)e_0^*, \quad s \in l^1(S), \quad f \in l^\infty(S),$$

where e_0^* is the element of $l^\infty(S)$ satisfying $e_0^*(s_0) = 1$ and $e_0^*(s) = 0$ for $s \neq s_0$. Let $\varphi: l^\infty(S) \rightarrow l^\infty(S)$ be a nonzero homomorphism. Then

$$s(s_0)\varphi(t) = \varphi(s(s_0)t) = \varphi(st) = \varphi(s)\varphi(t) = \varphi(s)(s_0)\varphi(t).$$

Hence, $\varphi(t)(\varphi(s)(s_0) - s(s_0))$ for all $s, t \in l^1(S)$. Since φ is nonzero,

$$(2.4) \quad \varphi(s)(s_0) = s(s_0), \quad s \in l^1(S).$$

Now, suppose that $\varphi, \psi \in \text{Hom}(l^1(S))$ and $D: l^1(S) \rightarrow l^\infty(S)$ is a bounded (φ, ψ) -derivation. For each $s, t \in l^1(S)$ we have

$$\begin{aligned} s(s_0)D(t) &= D(s(s_0)t) = D(st) \\ &= D(s) \cdot \varphi(t) + \psi(s) \cdot D(t) = \varphi(t)(s_0)D(s) + \psi(s) \cdot D(t). \end{aligned}$$

Replacing t by s in the last equalities and using (2.4), we get $\psi(s) \cdot D(s) = 0$ for all $s \in l^1(S)$. The last equality implies that $\psi(s) \cdot D(t) = -\psi(t) \cdot D(s)$ for all $s, t \in l^1(S)$. Thus

$$\begin{aligned} D(s) &= D(s_0s) = D(s_0) \cdot \varphi(s) + \psi(s_0) \cdot D(s) \\ &= D(s_0) \cdot \varphi(s) - \psi(s) \cdot D(s_0) \end{aligned}$$

for all $s \in l^1(S)$. Therefore $l^1(S)$ is (φ, ψ) -weakly amenable. Note that $l^1(S)$ cannot be amenable because this Banach algebra does not have a bounded right approximate identity (see [17]).

3. RESULTS FOR SEGAL ALGEBRAS

Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Then $(\mathfrak{B}, \|\cdot\|')$ is an abstract Segal algebra with respect to $(\mathcal{A}, \|\cdot\|)$ if

- (1) \mathfrak{B} is a dense left ideal in \mathcal{A} and \mathfrak{B} is a Banach algebra with respect to $\|\cdot\|'$;
- (2) There exists $M > 0$ such that $\|b\| \leq M\|b\|'$ for each $b \in \mathfrak{B}$;
- (3) There exists $C > 0$ such that $\|ab\|' \leq C\|a\|'\|b\|'$ for each $a, b \in \mathfrak{B}$.

Let G be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra if it satisfies the following conditions:

- (i) $S^1(G)$ is dense in $L^1(G)$;
- (ii) If $f \in S^1(G)$, then $L_x f \in S^1(G)$, where $L_x f(a) = f(xa)$, i.e. $S^1(G)$ is left translation invariant;
- (iii) $S^1(G)$ is a Banach space under some norm $\|\cdot\|_S$ and $\|L_x f\|_S = \|f\|_S$ for all $f \in S^1(G)$ and $x \in G$;
- (iv) Map $x \mapsto L_x f$ from G into $S^1(G)$ is continuous.

In the upcoming result we study approximate (φ, ψ) -weak amenability of Segal algebras.

Theorem 3.1. *Let G be an amenable group and let $S^1(G)$ be a symmetric Segal algebra with approximate identity $(e_\alpha)_\alpha$ contained in the center of $L^1(G)$. Given $\varphi, \psi \in \text{Hom}(S^1(G))$ such that extensions of φ, ψ to $L^1(G)$ are continuous, and ψ is onto, every bounded (φ, ψ) -derivation from $S^1(G)$ into $S^1(G)^*$ is approximately (φ, ψ) -inner.*

Proof. Let $(e_\alpha)_\alpha$ be an approximate identity of $S^1(G)$ contained in the center of $L^1(G)$, and let $\tilde{\varphi}$ and $\tilde{\psi}$ be the extensions of φ and ψ to $L^1(G)$, respectively. For each α define $D_\alpha: L^1(G) \rightarrow S^1(G)^*$ by $D_\alpha(f) = D(e_\alpha * f) - D(e_\alpha) \cdot \tilde{\varphi}(f)$ for every $f \in L^1(G)$. At first, we prove that D_α is a bounded (φ, ψ) -derivation. The boundedness of D_α comes from that $L^1(G)$ acts continuously on $S^1(G)$ on the right, and so $f \mapsto D(e_\alpha * f)$ is continuous from $L^1(G)$ into $S^1(G)^*$. Similarly $L^1(G)$ acts continuously on $S^1(G)$ on the left, and so $f \mapsto D(e_\alpha) \cdot \tilde{\varphi}(f)$ is continuous, which implies that D_α is continuous. Let $f_1, f_2 \in L^1(G)$. Then

$$\begin{aligned}
 D_\alpha(f_1 * f_2) &= D(e_\alpha * f_1 * f_2) - D(e_\alpha) \cdot \tilde{\varphi}(f_1 * f_2) \\
 &= \text{norm-}\lim_{\beta} (D(e_\alpha * f_1 * e_\beta * f_2) - D(e_\alpha) \cdot \tilde{\varphi}(f_1 * f_2)) \\
 &= \text{norm-}\lim_{\beta} (D(e_\alpha * f_1) \cdot \varphi(e_\beta * f_2) + \psi(e_\alpha * f_1) \cdot D(e_\beta * f_2) \\
 &\quad - D(e_\alpha) \cdot \tilde{\varphi}(f_1 * f_2))
 \end{aligned}$$

$$\begin{aligned}
&= D(e_\alpha * f_1) \cdot \tilde{\varphi}(f_2) - D(e_\alpha) \cdot \tilde{\varphi}(f_1 * f_2) \\
&\quad + w^*\text{-}\lim_{\beta} \psi(f_1 * e_\alpha) \cdot D(e_\beta * f_2) \\
&= (D(e_\alpha * f_1) - D(e_\alpha) \cdot \tilde{\varphi}(f_1)) \cdot \tilde{\varphi}(f_2) \\
&\quad + w^*\text{-}\lim_{\beta} (\tilde{\psi}(f_1) \psi(e_\alpha) \cdot D(e_\beta * f_2)) \\
&= (D(e_\alpha * f_1) - D(e_\alpha) \cdot \tilde{\varphi}(f_1)) \cdot \tilde{\varphi}(f_2) \\
&\quad + w^*\text{-}\lim_{\beta} (\tilde{\psi}(f_1) \cdot D(e_\alpha * e_\beta * f_2) - \tilde{\psi}(f_1) \cdot D(e_\alpha) \cdot \varphi(e_\beta * f_2)) \\
&= (D(e_\alpha * f_1) - D(e_\alpha) \cdot \tilde{\varphi}(f_1)) \cdot \tilde{\varphi}(f_2) + \tilde{\psi}(f_1) \cdot D(e_\alpha * f_2) \\
&\quad - \tilde{\psi}(f_1) \cdot D(e_\alpha) \cdot \tilde{\varphi}(f_2) \\
&= D_\alpha(f_1) \cdot \tilde{\varphi}(f_2) + \tilde{\psi}(f_1) \cdot D(f_2).
\end{aligned}$$

Therefore D_α is a (φ, ψ) -derivation. By Johnson's theorem there exists ξ_α in $S^1(G)^*$ such that

$$D_\alpha(f) = \xi_\alpha \cdot \tilde{\varphi}(f) - \tilde{\psi}(f) \cdot \xi_\alpha = D(e_\alpha * f) - D(e_\alpha) \cdot \tilde{\varphi}(f)$$

for all $f \in L^1(G)$. So we get

$$D(e_\alpha) \cdot \tilde{\varphi}(f) = D(e_\alpha * f) - \psi(e_\alpha) \cdot D(f) \xrightarrow{w^*} 0$$

for every $f \in S^1(G)$. Then

$$D(f) = w^*\text{-}\lim_{\alpha} \xi_\alpha \cdot \tilde{\varphi}(f) - \tilde{\psi}(f) \cdot \xi_\alpha, \quad f \in S^1(G).$$

Take $f = e_\alpha$. Since (e_α) is in the center of $L^1(G)$, then $D(e_\alpha) = 0$. Therefore for every $f \in S^1(G)$ we have

$$D(f) = \text{norm-}\lim_{\alpha} \xi_\alpha \cdot \tilde{\varphi}(f) - \tilde{\psi}(f) \cdot \xi_\alpha,$$

and so the proof is complete. \square

In the upcoming result, we show that the concept of generalized weak amenability on Banach algebras can be induced on Segal algebras.

Theorem 3.2. *Let \mathcal{A} be a commutative Banach algebra and let \mathfrak{B} be an abstract Segal algebra of \mathcal{A} with approximate identity (e_α) . Suppose that $\varphi \in \text{Hom}(\mathfrak{B})$ has a continuous extension $\tilde{\varphi}$ to \mathcal{A} . If \mathcal{A} is $(\tilde{\varphi}, \tilde{\varphi})$ -weakly amenable, then \mathfrak{B} is (φ, φ) -weakly amenable.*

Proof. Let $D: \mathfrak{B} \rightarrow \mathfrak{B}^*$ be a continuous (φ, φ) -derivation. Similarly as in the proof of Theorem 3.1, define $D_\alpha: \mathcal{A} \rightarrow \mathfrak{B}^*$ by $D_\alpha(f) = D(e_\alpha * f) - D(e_\alpha) \cdot \tilde{\varphi}(f)$ for every $f \in \mathcal{A}$. According to the proof of Theorem 3.1, D_α is a continuous (φ, φ) -derivation. Since \mathcal{A} is $(\tilde{\varphi}, \tilde{\varphi})$ -weakly amenable and commutative, \mathfrak{B}^* is a symmetric \mathcal{A} -bimodule. This means that $D_\alpha = 0$, and so $D = 0$. \square

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