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On minimal spectrum of multiplication lattice modules


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ON MINIMAL SPECTRUM OF MULTIPLICATION LATTICE MODULES

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Abstract. We study the minimal prime elements of multiplication lattice module $M$ over a $C$-lattice $L$. Moreover, we topologize the spectrum $\pi(M)$ of minimal prime elements of $M$ and study several properties of it. The compactness of $\pi(M)$ is characterized in several ways. Also, we investigate the interplay between the topological properties of $\pi(M)$ and algebraic properties of $M$.

Keywords: prime element; minimal prime element; Zariski topology

MSC 2010: 06D10, 06E10, 06E99, 06F99

1. Introduction

The notion of minimal prime elements of a lattice module is a generalization of minimal prime elements of a multiplicative lattice. The prime and minimal prime elements of multiplicative lattice were introduced and studied by Thakare, Manjarekar and Maeda [12], Thakare and Manjarekar [11], and the minimal prime ideals of 0-distributive lattices by Pawar and Thakare [9]. Keimel [7] unified the study of minimal prime ideals for various structures, e.g. commutative rings, distributive lattices, lattice ordered groups, $f$-rings. In this paper, we have carried out investigations leading to the study of generalizations of notions in commutative rings and multiplicative lattices along the lines of Dilworth (see [6]).

A complete lattice $L$ with the least element 0 and the greatest element 1 is said to be a multiplicative lattice if a binary operation “$\cdot$” called multiplication on $L$ satisfying the following conditions is defined:

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implies either a < q

with a minimal prime element of prime

over an element radical

et al. [1], if L

principal elements of

the set of all compact elements of a multiplicative lattice L

maximal elements exist in L

lattice

elements is a compact element, then prime elements and minimal primes over

in a compactly generated multiplicative lattice, if every finite product of compact

lattice

An element

a multiplicatively closed subset

which is compact as well as multiplicative identity, that is, generated under joins by

a

identity

is a multiplication between elements of

N

∈

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is said to be a

(2)

a

(1)

a

(3)

a

(4)

a

1 = a for all a ∈ L.

Henceforth, a · b will be simply denoted by ab.

An element p ≠ 1 of a multiplicative lattice L is said to be prime if ab ≤ p
imply either a ≤ p or b ≤ p. A prime element p ∈ L is said to be a minimal
prime over an element a ∈ L if a ≤ p and whenever there is a prime element q ∈ L
with a < q ≤ p, then q = p. In L, a minimal prime element over 0 will be called a
minimal prime element of L. For a ∈ L, its radical is denoted by √a and defined as

√a = \{x ∈ L: x^n ≤ a for some n ∈ \mathbb{Z}^+\}. An element a ∈ L is called semiprime
or radical if √a = a.

An element a ∈ L is said to be compact if a ≤ \bigvee X, X ⊆ L implies that there
exists a finite number of elements x_1, x_2, . . . , x_n ∈ X such that a ≤ \bigvee_{i=1}^n x_i. We denote
the set of all compact elements of a multiplicative lattice L by L∗. In a multiplicative
lattice L, an element a ∈ L is said to be nilpotent if a^n = 0 for some n ∈ \mathbb{Z}^+ and is
said to be reduced if the only nilpotent element of L is 0.

An element e ∈ L is said to be meet principal or join principal if it satisfies the
identity a ∧ be = ((a : e) ∧ b)e or (ae ∨ b) : e = (b : e) ∨ a, respectively, for a, b ∈ L.
Also, e is said to be principal if it is both join and meet principal. A multiplicative
lattice L is said to be principally generated (PG) if every element of L is a join of
principal elements of L. A multiplicative lattice L is said to be compactly generated (CG) if every element of L is the join of compact elements of L. According to Alarcon
et al. [1], if L is a compactly generated multiplicative lattice with 1 compact, then
maximal elements exist in L and every maximal element is a prime element. Further,
in a compactly generated multiplicative lattice, if every finite product of compact
elements is a compact element, then prime elements and minimal primes over a ∈ L
exist (see [1]).

By a C-lattice we mean a multiplicative lattice L with the greatest element 1,
which is compact as well as multiplicative identity, that is, generated under joins by
a multiplicatively closed subset C of compact elements of L.

A complete lattice M with the smallest element 0_M and the greatest element 1_M
is said to be a lattice module over the multiplicative lattice L or L-module if there
is a multiplication between elements of M and L, denoted by aN for a ∈ L and
N ∈ M, which satisfies the following properties:

(1) (ab)N = a(bN);
(2) \bigvee_{\alpha} a_{\alpha} \bigvee_{\beta} N_{\beta} = \bigvee_{\alpha,\beta} (a_{\alpha}N_{\beta});
(3) $1_L N = N$;
(4) $0_L N = 0_M$ for $a, b, a_\alpha \in L$ and for $N, N_\beta \in M$.

Let $M$ be a lattice module over a multiplicative lattice $L$. For $N \in M$ and $b \in L$, denote $(N : b) = \bigvee \{X \in M : aX \leq N\}$. If $a, b \in L$, we write $(a : b) = \bigvee \{x \in L : bx \leq a\}$. If $A, B \in M$, then $(A : B) = \bigvee \{x \in L : xB \leq A\}$.

An element $A \in M$ is called weak meet principal if $(B : A)A = B \land A$ for all $B \in M$; $A$ is called weak join principal if $bA = b \lor (0 : A)$ for all $b \in L$; and $A$ is weak principal if $A$ is both weak meet principal and weak join principal. Lattice module $M$ over a multiplicative lattice $L$ is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$. An element $N \neq 1_M$ in $M$ is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$, i.e. $a \leq (N : 1_M)$ for every $a \in L$ and $X \in M$. An element $N \neq 1_M$ of $M$ is called a maximal element if for every element $B$ of $M$ such that $N \leq B$, either $N = B$ or $B = 1_M$. Let $M$ be an $L$-module. An element $N$ in $M$ is called compact if $N \leq \bigvee A_\alpha$ ($I$ is an indexed set) implies $N \leq A_{\alpha_1} \lor A_{\alpha_2} \lor \ldots \lor A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $I$.

In this paper, a lattice module $M$ will be a multiplication lattice module, which is compactly generated with the largest element $1_M$ being compact and $L$ will be a $C$-lattice.

For general background and terminology of multiplicative lattice and multiplication lattice module, the reader may consult [1], [2], [4]–[6], [12], [11].

2. The Zariski topology

In [3], the Zariski topology over the prime spectrum Spec($M$) of a lattice module $M$ over a $C$-lattice $L$ has been studied by Ballal and Kharat. In [10], Phadatare et al. introduced and studied the concept of quasi-prime elements as a generalization of prime elements and also the Zariski topology on the quasi-prime spectrum of a lattice module $M$ over a $C$-lattice $L$.

In this paper most of the results in [12] and [11] are generalized.

**Definition 2.1.** Let $M$ be a lattice module over a multiplicative lattice $L$. An element $P \in M$ is called a minimal prime over an element $N \in M$ if $N \leq P$ and there is no other prime element $Q$ of $M$ such that $N \leq Q < P$.

**Lemma 2.2.** Let $M$ be a multiplication lattice module over a $C$-lattice $L$ and $(0_M : 1_M)$ be a radical element. Then for $x \in L$, $(0_M : x) = (0_M : x^n)$ for every integer $n \geq 1$.
Proof. Note that \((0_M : x) = \sqrt{\{N \in M : xN \leq 0_M\}}\) and as \(x^n \leq x\), we have \((0_M : x) \leq (0_M : x^n)\) for every integer \(n \geq 1\). Let \(N_1 = (0_M : x^n)\). Since \(M\) is a multiplication lattice module, \(N_1 = a_1M\) for some \(a \in L\). So \(x^na^1M \leq x^naM = 0_M\). Hence \(xa \leq \sqrt{(0_M : 1M)} = (0_M : 1M)\). So \(xa1M \leq 0_M\), i.e. \(N_1 \leq (0_M : x)\) and consequently \((0_M : x) = (0_M : x^n)\) for each integer \(n \geq 1\). □

**Theorem 2.3 ([8]).** Let \(M\) be a multiplication lattice module over a \(C\)-lattice \(L\) and \(a \in L\) be proper. A prime element \(P \in M\) with \(a_1M \leq P\) is minimal if and only if for \(x \in L_*\) with \(x1M \leq P\) there is an element \(y \in L_*\) such that \(y1M \not\leq P\) and \(x^ny1M \leq a_1M = N\) for some positive integer \(n\).

The following result characterizes a prime element to be a minimal prime.

**Theorem 2.4.** Let \(M\) be a multiplication lattice module over a \(C\)-lattice \(L\) and \((0_M : 1M)\) be a radical element. A prime element \(P \in M\) is a minimal prime if and only if for \(x \in L_*\), \(P\) contains precisely one of \(x1M\) and \((0_M : x)\).

Proof. Suppose that the condition is true for prime element \(P \in M\). Let \(x \in L_*\) be such that \(x1M \leq P\) and \((0_M : x) \not\leq P\). Then there exists \(y \in L_*\) such that \(y1M \leq (0_M : x)\) but \(y1M \not\leq P\). Thus, \(xy1M \leq 0_M\) and hence \(x^n y1M \leq 0_M\) for every integer \(n \geq 1\). This shows that for each \(x \in L_*\) with \(x1M \leq P\) there exists an element \(y \in L_*\) such that \(y1M \not\leq P\) and \(x^n y1M \leq 0_M\). By Theorem 2.3, it follows that \(P\) is minimal.

Conversely, suppose that a prime element \(P \in M\) is minimal and also that \(x1M \leq P\) for \(x \in L_*\). Then by Theorem 2.3, there exists \(y \in L_*\) such that \(y1M \not\leq P\) and \(x^n y1M = 0_M\) for some positive integer \(n\). Consequently, \(y1M \leq (0_M : x^n)\). By Lemma 2.2, we have \((0_M : x^n) = (0_M : x)\) and hence \(y1M \leq (0_M : x)\). This implies that \((0_M : x) \not\leq P\).

Now, if \(x1M \not\leq P\) and \((0_M : x) \not\leq P\), then there exists \(y \in L_*\) such that \(y1M \leq (0_M : x)\) but \(y1M \not\leq P\). Hence, we have \(xy1M \leq 0_M\) and so \(xy1M \leq P\). But \(x1M \not\leq P\) and \(y1M \not\leq P\) together contradict the fact that \(P\) is a prime. This shows that \(P\) contains precisely one of \(x1M\) and \((0_M : x)\). □

Let \(\sigma(M)\) be the set of prime elements of a lattice module \(M\). For an element \(N \in M\) we set \(V(N) = \{P \in \sigma(M) : N \leq P\}\). Taking the sets \(\{V(N) : N \in M\}\) as a base for closed sets, \(\sigma(M)\) becomes a topological space and this topology is called the Zariski topology (see [3]).

The restriction of the Zariski topology to the set of minimal prime elements \(\pi(M)\) makes it a topological space and it is called the minimal prime spectrum of \(M\).

The following results about a minimal prime spectrum are immediate.

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Corollary 2.5. Let $M$ be a multiplication lattice module over a reduced $C$-lattice $L$. For $a \in L$, $V(0_M : a) = \pi(M) - V(a1_M)$. In particular, $V(a1_M)$ and $V(0_M : a)$ are disjoint open and closed sets.

Corollary 2.6. Let $M$ be a multiplication lattice module over a reduced $C$-lattice $L$ with $1_M$ being compact. Then $\pi(M)$ is a Hausdorff space with a base of open and closed sets.

Definition 2.7 ([11]). A subset $S$ of a multiplicative lattice $L$ is said to be multiplicatively closed if $x, y \in S$ implies $xy \in S$, and is said to be sub-multiplicatively closed if $x, y \in X$ implies $a \leq xy$ for some $a \in S$.

In order to characterize prime elements of lattice modules in terms of multiplicatively closed subset of $L$, we need the following lemma.

Lemma 2.8 ([4]). Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$ and $N \in M$ with $N < 1_M$. Then the following conditions are equivalent.

1. $N$ is a prime element in $M$.
2. $(N : 1_M)$ is a prime element in $L$.
3. There exists a prime element $p$ in $L$ with $(0_M : 1_M) \leq p$ such that $N = p1_M$.

For $N \in M$ we define $C(N) = \{x \in L : x \notin (N : 1_M)\}$.

Lemma 2.9. Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$. An element $P \in M$ is a prime if and only if $C(P)$ is a multiplicatively closed subset of $L$.

Proof. Suppose that $P \in M$ is a prime and $x, y \in C(P)$. Then $x \notin (P : 1_M)$ and $y \notin (P : 1_M)$. Since $P \in M$ is a prime, by Lemma 2.8 we have that $(P : 1_M) \subseteq L$ is a prime. As $x \notin (P : 1_M)$, $y \notin (P : 1_M)$ and $(P : 1_M)$ is a prime, $xy \notin (P : 1_M)$, i.e. $xy \in C(P)$ and hence $C(P)$ is multiplicatively closed.

Conversely, suppose that $C(P)$ is a multiplicatively closed subset of $L$ and $xy1_M \leq P$ for $x, y \in L$. Then $xy \leq (P : 1_M)$ and so $xy \notin C(P)$. If $x \notin (P : 1_M)$ and $y \notin (P : 1_M)$, then $x \in C(P)$, $y \in C(P)$ and this contradicts the fact that $C(P)$ is multiplicatively closed. Therefore $x \leq (P : 1_M)$ or $y \leq (P : 1_M)$, i.e. $x1_M \leq P$ or $y1_M \leq p$. Consequently, $P$ is a prime. \qed

Lemma 2.10 ([11]). Let $a$ be an element of a $C$-lattice $L$ and $S$ be a multiplicatively closed subset of $L$ satisfying the property $s \notin a$ for all $s \in S$. Then there is a multiplicatively closed subset $S'$ of $L$ containing $S$ which is maximal with respect to the property $s' \notin a$ for all $s' \in S'$.
Lemma 2.11 ([11]). (Separation lemma) Let $S$ be a sub-multiplicatively closed subset of a $C$-lattice $L$. Suppose that $a \in L$ and $t \not\leq a$ for every $t \in S$. Then there exists a prime element $p \in L$ such that $a \leq p$ and it is maximal with respect to $t \not\leq p$ for each $t \in S$.

An element $a$ in a complete lattice $L$ is said to be completely join prime if $a \leq \bigvee S$, $S \subseteq L$ implies $a \leq s$ for some $s \in S$.

Lemma 2.12. Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$ and suppose every element of $L$ is a completely join prime. A prime element $P \in M$ with $a1_M \leq P$ is minimal if and only if $C(P)$ is a maximal multiplicatively closed subset of $L$ with $x \not\leq a$ for all $x \in C(P)$ and $a \in L$.

Proof. Suppose that $C(P)$ is a maximal multiplicatively closed subset of $L$ with $x \not\leq a$ for all $x \in C(P)$. By Lemma 2.11 there is a prime element $(Q : 1_M) \geq a$ that is maximal with respect to the property that $x \not\leq (Q : 1_M)$ for all $x \in C(P)$. Hence, by Lemma 2.9, $C(Q)$ is a multiplicatively closed subset of $L$. As $a \leq (Q : 1_M)$, we have $x \not\leq a$ for any $x \in C(Q)$. But $C(P)$ is a maximal multiplicatively closed subset of $L$ with the property that $x \not\leq a$ for all $x \in C(P)$, hence we must have $C(Q) \subseteq C(P)$. Now, if $y \in C(P)$, then $y \not\leq (Q : 1_M)$ and hence $y \in C(Q)$. Consequently, we have $C(P) = C(Q)$. Now, let $z \leq (P : 1_M)$, i.e. $z \in C(P)$. Then $z \not\in C(Q)$ and it implies that $z \leq (Q : 1_M)$ and it further implies $(P : 1_M) \leq (Q : 1_M)$. Similarly, we have $(Q : 1_M) \leq (P : 1_M)$ and hence $(P : 1_M) = (Q : 1_M)$. It follows that $P = Q$.

Now we show that $P$ is a minimal prime. Suppose that $P' \in M$ is a prime with $a \leq (P' : 1_M) < (P : 1_M)$. Then by Lemma 2.9, $C(P')$ is a multiplicatively closed subset of $L$ with $x \not\leq a$ for all $x \in C(P')$ and $C(P) \subseteq C(P')$. This contradicts the maximality of $C(P)$. Hence, $P$ is a minimal prime element of $M$ with $a1_M \leq P$.

Conversely, suppose that $P \in M$ is a minimal prime with $a1_M \leq P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of $L$ with $x \not\leq a$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed subset $S$ which contains $C(P)$ and $x \not\leq a$ for all $x \in S$. We show that $S = C(P')$, where $P' = p1_M$ and $p = \bigvee (L - S)$. Let $y \in C(P') = \{ z \in L: z \not\leq \bigvee (L - S) \}$. This gives $y \not\leq \bigvee (L - S)$, i.e. $y \in S$ and $C(P') \subseteq S$. On the other hand, if $s \in S$, then $s \not\in L - S$ and $s \not\in \bigvee (L - S)$. As each element of $L$ is a completely join prime, we have $s \in C(P')$ and therefore $C(P) = C(P')$.

By the first part, as $S$ is a maximal multiplicatively closed subset of $L$ with respect to $x \not\leq a$ for all $x \in S$, we conclude that $P'$ is a minimal prime with $a1_M \leq P'$. Clearly, $C(P) \subseteq S = C(P')$ gives that $P' \leq P$ and since $P$ is minimal, we must have $P = P'$. Hence, $C(P) = S = C(P')$ is the required maximal multiplicatively closed subset of $L$ with $x \not\leq a$ for all $x \in M$ and $a \in L$. \qed
For $N \in M$ define $\sqrt[N]{N} = \sqrt[{\text{radical}}]{\{x \in L: x^n1_M \leq N\}1_M}.$

**Theorem 2.13.** Let $L$ be a PG $C$-lattice in which every element is completely join prime and let $M$ be a multiplication lattice module over $L$. For $N \in M$, the radical $\sqrt[N]{N}$ is a minimal prime element of $M$ with $N \leq P$.

**Proof.** Observe that for a prime element $P \in M$ with $N \leq P$ we have $\sqrt[N]{N} \leq P$. Therefore $\sqrt[N]{N} \leq \bigwedge \{P: P$ is a minimal prime element of $M$ with $N \leq P\}$.

Now, let $x \in L_s$ be such that $x1_M \notin \sqrt[N]{N}$ and let $S = \{x^i: x^i \notin (N : 1_M)$ and $i$ is an integer}. Observe that $S$ is a multiplicatively closed subset of $L$. By Lemma 2.10, there is a maximal multiplicatively closed set $S'$ such that $y \notin (N : 1_M)$ for $y \in S'$. Let $p' = \sqrt[(L - S')]{M}$ then $S' = C(p'1_M) = C(P')$. By Lemma 2.12, $P'$ is a minimal prime element of $M$ with $N \leq P'$. Clearly, $x \in C(P')$ and as such $x \notin (P : 1_M)$. This gives that $\bigwedge \{P: P$ is a minimal prime element of $M$ with $N \leq P\} \leq \sqrt[N]{N}$. Consequently, $\sqrt[N]{N} = \bigwedge \{P: P$ is a minimal prime element of $M$ with $N \leq P\}$. □

**Corollary 2.14.** Let $M$ be a lattice module over a reduced PG $C$-lattice $L$ and $N \in M$. Then for a prime element $P \in M$ with $N \leq P$ there exists a minimal prime element $Q \in M$ such that $N \leq Q \leq P$.

**Proof.** Suppose $P \in M$ is a prime element with $N \leq P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of $L$ with $x \notin (N : 1_M)$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed set $S$ such that $y \notin (N : 1_M)$ for all $y \in S$. Also, $C(Q) = S$, where $Q = p1_M = \sqrt[(L - S)]{M}$ is a minimal prime element of $M$ with $N \leq Q$ and $C(P) \subseteq C(Q) = S$ implies that $Q \leq P$. □

**Lemma 2.15 ([12]).** Let $L$ be a $C$-lattice. Then each nonzero element of $L$ is contained in a maximal multiplicatively closed subset of $L$ not containing zero.

For $N \in M$ we set $U(N) = \{P \in \pi(M): N \notin P\}$.

**Theorem 2.16.** Let $L$ be a PG $C$-lattice in which every element is completely join prime and let $M$ be a multiplication lattice module over $L$. Then $(0_M : a) = \bigwedge U(a1_M) = \{P \in \pi(M): a1_M \notin P\}$, $a \in L$.

**Proof.** Suppose $P \in M$ is a minimal prime. Then by Theorem 2.4 we have $(0_M : a) \leq P$ when $a1_M \notin P$ and therefore $(0_M : a) \leq \bigwedge \{P \in \pi(M): a1_M \notin P\} = Q$. If $(0_M : a) < Q$, then there exists $x \in L_s$ such that $x1_M \subseteq Q$ and $x1_M \notin (0_M : a)$. Clearly, $ax1_M \notin 0_M$ and so $ax \neq 0$. By Lemma 2.15, $ax$ is contained in some maximal multiplicatively closed subset $S$ of $L$ not containing 0. As proved in Lemma 2.12, $S = C(P)$, where $P = p1_M$ and $p = \sqrt[(L - S)]{M}$ is a minimal prime element of $L$. Now $ax \in S$ implies $ax \notin (P : 1_M)$ and hence $ax1_M \notin P$. 91
Since \( P \) is a minimal prime and \( a1_M \not\in P \), we have \( x1_M \not\in P \). Therefore \( x1_M \not\in Q \), a contradiction and consequently, \((0_M : a) = \bigwedge\{P \in \pi(M) : a1_M \not\in P\}\).

\[\Box\]

**Theorem 2.17.** Let \( L \) be a PG C-lattice in which every element is a completely join prime and let \( M \) be a multiplication lattice module over \( L \). Then \( a1_M = (0_M : (0_M : a1_M)) \) if and only if \( a1_M = \bigwedge\{P \in \pi(M) : a1_M \not\in P, a \in L\} \).

**Proof.** Suppose \( a1_M = (0_M : (0_M : a1_M)), a \in L \). By Theorem 2.4 we have \( \bigwedge\{P \in \pi(M) : (0_M : a) \not\in P\} = \bigwedge\{P \in \pi(M) : a1_M \not\in P\} \). But \( (0_M : (0_M : a1_M)) = \bigwedge\{P \in \pi(M) : (0_M : a) \not\in P\} \) gives that \( a1_M = \bigwedge\{P \in \pi(M) : a1_M \not\in P\} \).

Conversely, suppose that \( a1_M = \bigwedge\{P \in \pi(M) : a1_M \not\in P\} \). By Theorem 2.16 we have \( (0_M : (0_M : a1_M)) = \bigwedge\{P \in \pi(M) : (0_M : a) \not\in P\} \). Now, by Theorem 2.4 we have \( \bigwedge\{P \in \pi(M) : (0_M : a) \not\in P\} = \bigwedge\{P \in \pi(M) : a1_M \not\in P\} \) and by assumption, \( a1_M = (0_M : (0_M : a1_M)) \).

\[\Box\]

**Theorem 2.18.** Let \( M \) be a multiplication lattice module over a PG C-lattice \( L \). Then \( (0_M : a) = \bigwedge\{V(0_M : a)\}, a \in L \).

**Proof.** Note that \((0_M : a) \not\in \bigwedge\{V(0_M : a)\}, a \in L \) follows immediately. Now, let \( x \in L_* \) be such that \( x1_M \not\in (0_M : a) \). Then \( ax1_M \not\in 0_M \) and so \( ax \neq 0 \). Therefore \( ax \) is contained in some maximal multiplicatively closed subset \( S \) of \( L \). Then \( S = V(P) = V(p1_M) \), where \( p = \sqrt{L - S} \) and \( p \) is a minimal prime element of \( L \). Now \( ax \in C(P) \) implies \( ax \not\in (P : 1_M) \) and hence \( ax1_M \not\in P \). Since \( P \) is a minimal prime, we have \( x1_M \not\in P \) and \( a1_M \not\in P \). By Theorem 2.4 we have \( (0_M : a) \not\in P \) and hence \( P \in V(0_M : a) \). As \( x1_M \not\in P \), we have \( x1_M \not\in \bigwedge\{V(0_M : a)\} \). Thus, \( x1_M \not\in (0_M : a) \) implies \( x1_M \not\in \bigwedge\{V(0_M : a)\} \), i.e. \( \bigwedge\{V(0_M : a)\} \subseteq (0_M : a) \).

\[\Box\]

We now show that the minimal prime spectrum \( \pi(M) \) is a completely regular Hausdorff space, i.e. a Tychonoff space.

**Theorem 2.19.** Let \( M \) be a multiplication lattice module over a PG C-lattice \( L \). Then the topology on \( \pi(M) \) for which the collection \( \{U(a1_M) : a \in L\} \) is a base for open sets is Tychonoff.

**Proof.** Suppose that \( P_1, P_2 \in \pi(M) \) with \( P_1 \not\in P_2 \). Clearly \( P_1 \not\in P_2 \) and \( P_2 \not\in P_1 \). Let \( x \in L_* \) with \( x1_M \leq P_1 \) be such that \( x1_M \not\in P_2 \). By Theorem 2.3, there is \( y \in L_* \) with \( y1_M \leq P_1 \) and \( x^n y1_M = 0_M \) for some integer \( n \). If \( y1_M \not\in P_2 \), then this together with \( x1_M \not\in P_2 \) gives \( x^n y1_M \not\in P_2 \), which is a contradiction to the fact that \( 0_M \leq P_2 \). Therefore \( y1_M \leq P_2 \). Clearly, \( P_1 \in U(y1_M), P_2 \in U(x1_M) \) and
$U(x_1 M) \cap U(y_1 M) = \{ P \in \pi(M) : x_1 M \not\in P, y_1 M \not\in P \} = U(xy_1 M) = U(x^n y_1 M) = U(0_M) = \varphi$. Consequently, $\pi(M)$ is a Hausdorff space and hence singletons are closed.

Now, let $Q \in \pi(M)$ and $F$ be a closed subset of $\pi(M)$ such that $Q \not\in F$. Then $Q \in \pi(M) - F$ and $\pi(M) - F$ is open in $\pi(M)$. Then there is an open set $U(s_1 M)$ for some $s \in L$ such that $Q \in U(s_1 M) \subseteq \pi(M) - F$. Define a function $f$ on $\pi(M)$ as $f(Q) = 0_M$ if $Q \in U(s_1 M)$ and $f(Q) = 1_M$ otherwise. Then $f(Q) = 0_M$ and $f(F) = 1$. Note that $f$ is continuous and hence $\pi(M)$ is completely regular. Consequently, $\pi(M)$ is a completely regular Hausdorff space, i.e. a Tychonoff space. 

Corollary 2.20. $\pi(M)$ is totally disconnected and zero dimensional space.

**Theorem 2.21.** Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$. Let $x, y \in L$. Then $U(x_1 M) \subseteq U(y_1 M)$ if and only if $0_M : (0_M : x_1 M) \leq 0_M : (0_M : y_1 M)$. In addition, $U(x_1 M) = U(y_1 M)$ if and only if $(0_M : x) \leq (0_M : y)$.

**Proof.** Suppose that $U(x_1 M) \subseteq U(y_1 M)$ for $x, y \in L$. By Theorem 2.16 we have $(0_M : y) \leq (0_M : x)$ and hence $(0_M : y) \not\in P$ which implies $(0_M : x) \not\in P$ and so $\{P \in \pi(M) : (0_M : y) \not\in P\} \subseteq \{P \in \pi(M) : (0_M : x) \not\in P\}$. By Theorem 2.4 we have $0_M : (0_M : x_1 M) \leq 0_M : (0_M : y_1 M)$.

Conversely, suppose that $0_M : (0_M : x_1 M) \leq 0_M : (0_M : y_1 M)$. Therefore $\{P \in \pi(M) : (0_M : y) \not\in P\} \subseteq \{P \in \pi(M) : (0_M : x) \not\in P\}$ and so $\{P \in \pi(M) : y_1 M \not\in P\} \subseteq \{P \in \pi(M) : x_1 M \not\in P\}$ by Theorem 2.4. This gives $\{P \in \pi(M) : x_1 M \not\in P\} \subseteq \{P \in \pi(M) : y_1 M \not\in P\}$ and therefore $U(x_1 M) \subseteq U(y_1 M)$.

For the second part, suppose that $U(x_1 M) = U(y_1 M)$. Then $U(x_1 M) \subseteq U(y_1 M)$ implies $0_M : (0_M : x_1 M) \leq 0_M : (0_M : y_1 M)$ and $U(y_1 M) \subseteq U(x_1 M)$ implies $0_M : (0_M : y_1 M) \leq 0_M : (0_M : x_1 M)$. Hence, $0_M : (0_M : y_1 M) = 0_M : (0_M : x_1 M)$ and $0_M : (0_M : (0_M : y_1 M)) \leq 0_M : (0_M : (0_M : x_1 M))$. Consequently, $(0_M : x) = (0_M : y)$.

Conversely, suppose that $(0_M : x) = (0_M : y)$. Then $0_M : (0_M : x_1 M) = 0_M : (0_M : y_1 M)$, i.e. $0_M : (0_M : x_1 M) \leq 0_M : (0_M : y_1 M)$ and $0_M : (0_M : y_1 M) \leq 0_M : (0_M : x_1 M)$ and the result follows by the first part.

**Theorem 2.22.** Let $M$ be a multiplication lattice module over a PG $C$-lattice $L$. Let $I$ be an indexing set and $S = \{x_r : r \in I\}$ be a set of points in $L$ such that the collection of sets $\{U(x_r 1_M) : r \in I\}$ has the finite intersection property. Then the intersection of all $\{U(x_r 1_M) : r \in I\}$ is nonempty.

**Proof.** We have $\bigcap_{i=1}^n U(x_i) = U(y_1 M)$, where $y = x_1 x_2 \ldots x_n$. Note that the multiplication of finite number of nonzero $x_r$, $r \in I$ is nonzero. The collection of
all nonzero \( x_r, r \in I \) together with finite multiplication of \( x_r \in S \) is multiplicatively closed subset of \( L \) not containing 0. By Lemma 2.10, there is a maximal multiplicatively closed subset \( S' \) of \( L \) containing \( S \) and not containing 0. We have \( S' = C(P) = C(p_1 M) \), where \( p = \sqrt{L - S'} \) and \( p \) is a minimal prime element of \( L \). Clearly, \( P \in U(x_r M) \) for all \( x_r \in S' \). As \( S \subseteq S' \), we have \( P \in U(x_r M) \) for all \( x_r \in S \). Thus, \( P \in \bigcap_{r \in I} U(x_r M) \), which implies that \( \bigcap_{r \in I} U(x_r M) \neq \varnothing \). □

If the family \( \{ V(x_M) : x \in L \} \) is considered as an open basis for \( \pi(M) \), the resulting topology is called the dual topology and denoted by \( \tau^d \). We denote the topology for which \( \{ U(x_M) : x \in L \} \) is an open basis by \( \tau \).

**Theorem 2.23.** Let \( M \) be a multiplication lattice module over a PG \( C \)-lattice \( L \). The topology \( \tau \) on \( \pi(M) \) for which \( \{ U(x_M) : x \in L \} \) is a basis for open sets is finer than the topology \( \tau^d \) on \( \pi(M) \) for which \( \{ V(x_M) : x \in L \} \) is a basis for open sets and moreover \( \tau = \tau^d \).

**Proof.** We know that \( \{ V(x_M) : x \in L \} \) is a basis for open sets for the topology on \( \pi(M) \) denoted by \( \tau^d \). Clearly, \( V(x_M) = \pi(M) - U(x_M) \) for all \( x \in L \). Note that for \( x \in L, U(x_M) \) is closed in \( \pi(M) \). Hence, \( V(x_M) \) is open in the topology \( \tau \) for \( \pi(M) \), i.e. \( \tau \) is finer than \( \tau^d \).

Now, for any \( x \in L \) we have \( U(x_M) = V(0_M : x) \). Thus, every basic open set in \( \tau \) is open in \( \tau^d \) and so we conclude that \( \tau = \tau^d \). □

**Theorem 2.24.** Let \( M \) be a multiplication lattice module over a PG \( C \)-lattice \( L \). The following statements are equivalent in \( M \).

1. \( \pi(M) \) is compact.
2. The poset \( \{ U(x_M) : x \in L \} \), under set inclusion, is a Boolean lattice.
3. For \( x \in L \) there exist \( N_1 = y_1 M, N_2 = y_2 M, \ldots, N_n = y_n M \in M \) with
   \( y_i M = N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \) and \( (0_M : x) \wedge \bigwedge_{i=1}^n (0_M : y_i) = 0_M \).
4. For \( x \in L \) there exist \( N_1 = y_1 M, N_2 = y_2 M, \ldots, N_n = y_n M \in M \) such that
   \( 0_M : (0_M : x_M) = \bigwedge_{i=1}^n (0_M : y_i) \).
5. \( \tau = \tau^d \).
6. \( \{ U(x_M) : x \in L \} \) is a subbasis for open sets of \( \pi(M) \) with respect to the topology \( \tau \).
7. \( \{ V(x_M) : x \in L \} \) is a subbasis for open sets of \( \pi(M) \) with respect to the topology \( \tau^d \).

**Proof.** (1) \( \Rightarrow \) (2): Clearly the set \( \{ U(x_M) : x \in L \} \) is partially ordered under set inclusion.
Now, we first show that
(i) \( U(x_1M) \cup U(y_1M) = U(x_1M \lor y_1M) \);
(ii) \( U(x_1M) \cap U(y_1M) = U(xy_1M) \).

Let \( P \in U(x_1M) \cup U(y_1M) \), then \( P \in U(x_1M) \) or \( P \in U(y_1M) \) and so \( x_1M \not\in P \) or \( y_1M \not\in P \). Therefore \( x_1M \lor y_1M \not\in P \) and this implies \( P \in U(x_1M \lor y_1M) \). Now, let \( Q \in U(x_1M \lor y_1M) \), then \( x_1M \lor y_1M \not\in Q \) and this implies that \( x_1M \not\in Q \) or \( y_1M \not\in Q \). Therefore \( Q \in U(x_1M) \cup U(y_1M) \). Consequently, \( U(x_1M) \cup U(y_1M) = U(x_1M \lor y_1M) \). Similarly, \( U(x_1M) \cap U(y_1M) = U(xy_1M) \).

From this we conclude that \( \{U(x_1M): x \in L\}, \cup, \cap, \setminus \) is a lattice.

Now, \( U(0_1M) = U(0_M) = \varphi \) and \( U(1_1M) = U(1_M) = \pi(M) \). This shows that \( \{U(x_1M): x \in L\}, \cup, \cap, \setminus \) is a bounded lattice. Again, observe that \( U(x_1M) \cup (y_1M) \cap U(z_1M) = \cap (U(x_1M) \cup U(y_1M) \cap U(z_1M)) \) and \( U(x_1M) \cap (y_1M) \cap U(z_1M) = \cap (U(x_1M) \cap U(y_1M) \cap U(z_1M)) \). This shows that \( \{U(x_1M): x \in L\}, \cup, \cap, \setminus \) is a distributive lattice.

Finally, we show that \( \{U(x_1M): x \in L\}, \cup, \cap, \setminus \) is complemented. Note that for \( x \in L \) we have \( V(x_1M) \cap V(0_M : x) = \varphi \). Then \( V(x_1M) \cap \{V(N_i): N_i \leq (0_M : x)\} = \varphi \).

Since \( \pi(M) \) is compact, there exist \( N_1, N_2, \ldots, N_n \leq (0_M : x) \) such that \( V(x_1M) \cap \{V(N_i): N_i \leq (0_M : x), i = 1, 2, \ldots, n\} = \varphi \). By taking complements in \( \pi(M) \), we get \( \pi(M) = U(x_1M) \cup U(N_1) \cup \cdots \cup U(N_n) \). Since each \( N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \), we have \( U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) = \varphi \). For, if \( P \in U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) \), then \( x_1M \not\in P \), which implies \( (0_M : x) \not\in P \). Therefore \( N_i \not\in P \) for \( i = 1, 2, \ldots, n \), a contradiction as \( P \in \bigcup_{i=1}^{n} U(N_i) \) and so \( N_k \not\in P \) for some \( k, 1 \leq k \leq n \). Thus, we have \( \pi(M) = U(x_1M) \cup \bigcup_{i=1}^{n} U(N_i) \) and \( U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) = \varphi \). Consequently, \( \{U(x_1M): x \in L\}, \cup, \cap, \setminus \) is a Boolean lattice.

(2) \( \Rightarrow \) (3): Suppose that the finite union of \( \{U(x_1M): x \in L\} \) forms a Boolean lattice and suppose that the complement of \( U(x_1M) \) is \( \bigcup_{i=1}^{n} U(N_i) \). As \( U(x_1M) \cap \bigcup_{i=1}^{n} U(N_i) = \varphi \), we get \( U(x_1M) \cap U(N_i) = \varphi \), \( i = 1, 2, \ldots, n \). Therefore \( \{P \in \pi(M): xN_i \not\in P\} = \varphi \), \( i = 1, 2, \ldots, n \), i.e. \( U(xN_i) = \varphi \) for \( i = 1, 2, \ldots, n \), which implies \( xN_i = 0_M \) for \( i = 1, 2, \ldots, n \). Thus \( N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \). Also, \( \pi(M) = U(x_1M) \cup \bigcup_{i=1}^{n} U(N_i) \) gives \( \bigwedge(\pi(M)) = \bigwedge\left(U(x_1M) \cup \bigcup_{i=1}^{n} U(N_i)\right) \), i.e. \( 0_M = \bigwedge(\pi(M)) = \bigwedge\left(U(x_1M) \lor \bigcup_{i=1}^{n} N_i\right) \). Note that \( \bigwedge\left(U(x_1M) \lor \bigcup_{i=1}^{n} N_i\right) = \bigwedge(U(x_1M)) \land \bigwedge_{i=1}^{n} (U(N_i)) \). Then by Theorem 2.16 we have \( (0_M : x) \land \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M \).

(3) \( \Rightarrow \) (4): Suppose that (3) holds. Then for any \( x \in L \) there exist \( N_1 = y_11_M, N_2 = y_21_M, \ldots, N_n = y_n1_M \in M \) with \( y_i1_M = N_i \leq (0_M : x) \) for \( i = 1, 2, \ldots, n \)
and \((0_M : x) \land \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M\). This implies \((0_M : x1_M) \land (0_M : y_i) = 0_M\), i.e. \(\bigwedge_{i=1}^{n} (0_M : y_i) \leq (0_M : (0_M : x1_M))\). Also note that \((0_M : (0_M : x1_M)) \leq (0_M : y_i)\) for \(i = 1, 2, \ldots, n\). Hence \((0_M : (0_M : x1_M)) \leq \bigwedge_{i=1}^{n} (0_M : y_i)\) and consequently, \((0_M : (0_M : x1_M)) = \bigwedge_{i=1}^{n} (0_M : y_i)\).

(4) \implies (5): Let \(x\) be an element of \(L\). By (4), there exist \(N_1 = y_11_M, N_2 = y_21_M, \ldots, N_n = y_n1_M \in M\) such that \((0_M : (0_M : x1_M)) = \bigwedge_{i=1}^{n} (0_M : y_i)\). Hence we have

\[
V(0_M : (0_M : x1_M)) = V\left(\bigwedge_{i=1}^{n} (0_M : y_i)\right) = \bigcup_{i=1}^{n} V(0_M : y_i) = \bigcup_{i=1}^{n} U(y_i1_M) = V(x1_M).
\]

Taking complements in \(\pi(M)\), we have \(\pi(M) - V(x1_M) = \pi(M) - \bigcup_{i=1}^{n} U(y_i1_M)\), i.e. \(U(x1_M) = \bigcap_{i=1}^{n} V(y_i1_M)\). It follows that \(U(x1_M)\) is a finite intersection of open sets in dual topology \(\tau^d\). Hence, \(U(x1_M)\) is open in \(\tau^d\), which implies \(\tau^d\) is finer than \(\tau\), and \(\tau\) is finer than \(\tau^d\) follows by Theorem 2.23.

(5) \implies (1): Suppose that \(\tau = \tau^d\). Then \(\{U(x1_M) : x \in L\}\) is also a base for closed sets in \(\pi(M)\). Let \(\{U(y1_M) : y \in K\}\) be a family of closed sets with finite intersection property in \(\pi(M)\), where \(K \subseteq L\). Then \(\bigcap_{i=1}^{n} U(y_i1_M) = U(y_1y_2\ldots y_n1_M) \neq \emptyset\) and so \(y_1y_2\ldots y_n1_M \neq 0_M\) for any finite number of elements \(y_1, y_2, \ldots, y_n \in K\). All the nonzero elements in \(K\) together with the finite multiplication of elements in \(K\) form a multiplicatively closed set not containing 0. This multiplicatively closed set is again contained in some maximal multiplicatively closed set \(S\) not containing 0. As proved in Lemma 2.12, \(S = C(P) = C(p1_M)\), where \(p = \sqrt{L - S}\) is a minimal prime element of \(L\). Note that \(K \subseteq C(P)\) and therefore \(P \in U(y1_M)\) for all \(y \in K\). Thus, \(p \in \bigcap\{U(y1_M) : y \in K\} \neq \emptyset\) and so \(\pi(L)\) is compact.

(5) \implies (6): The implication follows immediately as \(\{V(x1_M) : x \in L\}\) is a basis for open sets in \(\tau^d\).

(6) \implies (5): Let \(\{U(x1_M) : x \in L\}\) be any basis for open sets in \(\tau\). Then we have \(U(x1_M) = \bigcap_{i=1}^{n} V(x_i)\) as \(\{V(x1_M) : x \in L\}\) is a subbasis for open sets in \(\pi(M)\) with respect to \(\tau\). This implies that \(\{U(x1_M) : x \in L\}\) is open in \(\tau^d\) and hence \(\tau \subseteq \tau^d\) and the result follows by Theorem 2.23.

(6) \implies (7): If \(\{V(x1_M) : x \in L\}\) is a subbasis for open sets in \(\tau\), then \(\{\pi(M) - V(x1_M) : x \in L\}\) = \(\{U(x1_M) : x \in L\}\) forms a subbasis for open sets in \(\tau^d\) and conversely. \(\square\)
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References


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