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Qiang Zhou

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ON WEAKLY-SUPPLEMENTED SUBGROUPS AND THE SOLVABILITY OF FINITE GROUPS

Qiang Zhou, Nanjing

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The paper is dedicated to Professor J. C. Beidleman for his 80th birthday

Abstract. A subgroup H of a finite group G is weakly-supplemented in G if there exists a proper subgroup K of G such that $G = HK$. In this paper, some interesting results with weakly-supplemented minimal subgroups or Sylow subgroups of G are obtained.

Keywords: weakly-supplemented subgroup; complemented subgroup; solvable group

MSC 2010: 20D10, 20D20

1. Introduction

Only finite groups are considered in this paper. The terminology and notions employed agree with standard usage, as in Doerk and Hawkes [2]. In addition, the set of distinct primes dividing the order of the group G will be denoted by $\pi(G)$.

A subgroup H of G is complemented in G if there exists a subgroup K of G such that $G = HK$ and $H \cap K = 1$. In 1937, Hall proved that a finite group is solvable if and only if every Sylow subgroup of G is complemented (see [4]). Arad and Ward in [1] proved that a finite group is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup are complemented. In particular, Hall in [5] proved that a finite group G is supersolvable with elementary abelian Sylow subgroups if and only if every subgroup of G is complemented in G. In a recent paper [7], Kong and Liu studied finite groups for which every minimal subgroup is weakly-supplemented. A subgroup H of G is weakly-supplemented in G if there exists a proper subgroup K

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of G such that $G = HK$. One can easily see that being weakly-supplemented is a generalization of being complemented. Kong and Liu proved that every minimal subgroup of G is weakly-supplemented in G if and only if G is a supersolvable group and all Sylow subgroups of G are elementary abelian.

The purpose of this paper is to take the above mentioned studies further. More precisely, we improve and generalize the result of [4] and [7] as follows.

Theorem 1.1. *If every subgroup of a group G of prime odd order is weakly-supplemented in G, then G is solvable.*

Theorem 1.2. *Let G be a group. If every Sylow subgroup of G of odd order is weakly-supplemented in G, then G is solvable.*

2. Preliminary results

In this section, we give one result that will be needed later in this paper.

Lemma 2.1 ([7], Lemma 2.2). *Let G be a group and N be a normal subgroup of G.*

(1) *If $H \leqslant K \leqslant G$ and H is weakly-supplemented in G, then H is weakly-supplemented in K.*
(2) *If N is contained in H and H is weakly-supplemented in G, then H/N is weakly-supplemented in G/N.*
(3) *Let π be a set of primes. Let N be a π'-subgroup and A be a π-subgroup of G. If A is weakly-supplemented in G, then AN/N is weakly-supplemented in G/N.*

3. The proof of the main result

P r o o f of Theorem 1.1. Suppose that the theorem is false and let G be a counterexample of minimal order. Then we prove the theorem following these steps:

Step 1. $|\pi(G)| \geqslant 3$. Assume that $1 \leqslant |\pi(G)| \leqslant 2$. Then G is solvable by Burnside's theorem (see [2], page 21), a contradiction.

Step 2. Every subgroup of G is solvable. Let M be any subgroup of G. Hence, if M is of prime power order, M is solvable. So, assume that M is of composite order. Then, by Lemma 2.1. (1), every subgroup of prime odd order of M is weakly-supplemented in M. Then M is solvable by our choice of G.

Step 3. For each odd prime p dividing the order of G there exists a non-normal subgroup L of G of order p. Assume that there exists an odd prime, say p, such that each subgroup L of G of order p is normal in G. Then $G' \leqslant C_G(L)$, where G' is the commutator subgroup of G. Hence, $\Omega_1(G' \cap P) \leqslant Z(G')$ a Sylow p-subgroup of G and so G' is p-nilpotent by [6], page 435, Satz 5.5. This, together with Step 2, imply that G is solvable, a contradiction.

Step 4. There exist two subgroups H and K of G such that $|G : H| = p$ and $|G : K| = q$, where p and q are distinct odd primes. By Step 1, there exist two distinct odd primes p and q with $p < q$. By Step 3, there exist two non-normal subgroups L_1 and L_2 such that $|L_1| = p$ and $|L_2| = q$. By the hypothesis, L_1 and L_2 are weakly-supplemented in G and since L_1 and L_2 are non-normal subgroups in G, we have that L_1 and L_2 are complemented in G. Then there exist two subgroups H and K such that $G = L_1H = L_2K$, $L_1 \cap H = 1 = L_2 \cap K$, that is, $|G : H| = p$ and $|G : K| = q$.

Step 5. $H_G = K_G = 1$. Assume that $H_G \neq 1$. Clearly, if $H \trianglelefteq G$, then G is solvable, a contradiction. Thus H is non-normal in G. Since H is solvable by Step 2, it follows that H_G is solvable. Hence, if H_G is not contained in K, $G = H_GK$ and so G is solvable, a contradiction. Thus $H_G \leqslant K$ and so $H_G \leqslant K_G$. Clearly $K_G \leqslant H$, otherwise G is solvable, and we have a contradiction. So $K_G \leqslant H_G$, and since $H_G \leqslant K_G$, it follows that $H_G = K_G = 1$. Clearly, $H_G \cap L_i = 1$, $i = 1, 2$ and $(H/H_G)_G = (K/K_G)_G = (K/H_G)_G = 1$. Now it follows easily that $|(G/H_G) : (H/H_G)| = |G : H| = p$ and $|(G/H_G) : (K/H_G)| = |G : K| = q = |(G/K_G) : (K/K_G)|$. Then G/H_G is isomorphic to a subgroup of S_p, where S_p is a symmetric group of degree p and $G/H_G = G/K_G$ is isomorphic to a subgroup of S_q. Hence q divides $|S_p| = p!$ and since $p < q$, we have a contradiction. Thus $H_G = 1$. Similarly, $K_G = 1$.

Step 6. Finishing the proof. By step 5, H has no nontrivial normal subgroup of G and $|G : H| = p$. Then G is isomorphic to a subgroup of S_p. Similarly, G is isomorphic to a subgroup of S_q, where $p < q$. Hence q divides $|S_p|$, a final contradiction. The proof of the theorem is complete. □

P r o o f of Theorem 1.2. If G is solvable, then by Hall's theorem [4], every Sylow subgroup of G is complement in G, and so every Sylow subgroup of G is weakly-supplemented in G, in particular, every Sylow subgroup of G of odd order is weakly-supplemented in G.

Conversely, assume that every Sylow subgroup of G of odd order is weakly-supplemented in G. We argue that every Sylow subgroup of G of odd order is weakly-supplemented in G. If not, there exists a Sylow p-subgroup P of G, where $p > 2$, such that P is not complement in G. By hypothesis, P is weakly-supplemented in G. Then $P_G \neq 1$. Consider the group G/P_G. Hence, if $P_G = P$, then by the Schur-Zassenhaus theorem in [2], Theorem 11.3, page 38, P is complement in G, a contradiction. Thus $1 \neq P_G < P$. By Lemma 2.1, G/P_G satisfies the hypothesis of the theorem. By induction on the order of G, G/P_G is solvable. Hence G is solvable, and this implies that P is complemented in G, a contradiction. Thus, every Sylow subgroup of G of odd order is complemented in G.

By Burnside's Theorem, we may assume that $|\pi(G)| \geqslant 3$. Thus, there exist two different odd primes p and q dividing the order of G with $p < q$. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. Since Sylow subgroups of odd order are complemented in G, G possesses two subgroups H and K such that $|G:H| = |P|$ and $|G:K| = |Q|$. Hence if G is simple, $G \cong \mathrm{PSL}(2,7)$ by Guralnick [3], page 304. Therefore $|G:H| = 3$, and consequently, G is not simple, a contradiction. Thus we may assume that G is not simple.

Now, let L be a minimal normal subgroup of G. Then by [2], Lemma 4.20, page 15, either L is an elementary abelian p-group for some prime p or L is the direct product of isomorphic non-abelian simple groups. Assume first that L is an elementary abelian p-group. Then G/L satisfies the hypothesis of the theorem by Lemma 2.1. So, by induction on the order of G, G/L is solvable, and hence G is solvable as desired. So, assume that L is the direct product of isomorphic nonabelian simple groups. Consider PL, where P is any Sylow subgroup of G of odd order. Since P is complemented in G, it follows that $P \cap L$ is complemented in L (note that $P \cap L$ is a Sylow p-subgroup of L). This means that every Sylow subgroup of L of odd order is complemented in L. Since complemented subgroups are weakly-supplemented, it follows that every Sylow subgroup of L of odd order is weakly-supplemented in L. By induction on the order of G, L is solvable, a contradiction. This completes the proof of the theorem. $\square$

Based on Theorem 1.1 and Theorem 1.2, the following conjecture arise.

Conjecture. *Let G be a finite group such that every noncyclic Sylow subgroup P of G of odd order has a subgroup such that $1 < |D| \leqslant |P|$ and all subgroups H of P with $|H| = |D|$ are weakly-supplemented in G. Is G solvable?*

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Author's address: Qiang Zhou, School of Automation, Southeast University, Nanjing 210096, P. R. China and College of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, P. R. China, e-mail: zq5240_2008@163.com.