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EXTENSIONS OF COVARIANTLY FINITE SUBCATEGORIES REVISITED

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Abstract. Extriangulated categories were introduced by Nakaoka and Palu by extracting the similarities between exact categories and triangulated categories. A notion of homotopy cartesian square in an extriangulated category is defined in this article. We prove that in an extriangulated category with enough projective objects, the extension subcategory of two covariantly finite subcategories is covariantly finite. As an application, we give a simultaneous generalization of a result of X. W. Chen (2009) and of a result of R. Gentle, G. Todorov (1996).

Keywords: extriangulated category; covariantly finite subcategory

MSC 2010: 18E30, 18E10

1. Introduction

Let $\mathscr{C}$ be an additive category. By a subcategory $\mathscr{X}$ of $\mathscr{C}$ we always mean a full additive subcategory. Recall that a subcategory $\mathscr{X}$ of $\mathscr{C}$ is said to be *covariantly finite* in $\mathscr{C}$ if for every object M of $\mathscr{C}$, there exists some X in $\mathscr{X}$ and a morphism $f\colon M \to X$ such that for every X' in $\mathscr{X}$ the sequence

$$\operatorname{Hom}_{\mathscr{C}}(X, X') \xrightarrow{\circ f} \operatorname{Hom}_{\mathscr{C}}(M, X') \longrightarrow 0$$

is exact. In this case such an f is called a left $\mathscr{X}$-approximation of M. For details, see [1].

Let $\mathscr{C}$ be an abelian category and let $\mathscr{X}$ and $\mathscr{Y}$ be two subcategories. We denote by $\mathscr{X} * \mathscr{Y}$ the subcategory which consists of objects B in $\mathscr{C}$ admitting a short exact sequence $0 \to A \to B \to C \to 0$ where $A \in \mathscr{X}$ and $C \in \mathscr{Y}$. It is called the extension subcategory of $\mathscr{Y}$ by $\mathscr{X}$. Recall that an abelian category $\mathscr{C}$ has *enough projective objects*, if for each object M there exists an epimorphism $P \to M$ where P is a projective object. The following result is due to Gentle and Todorov.

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Theorem 1.1 ([6], Theorem 1.1). *Let $\mathscr{C}$ be an abelian category with enough projective objects. If $\mathscr{X}$ and $\mathscr{Y}$ are covariantly finite subcategories of $\mathscr{C}$, then the extension subcategory $\mathscr{X} * \mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$.*

Let $\mathscr{C}$ be a triangulated category with the shift functor [1], and let $\mathscr{X}$ and $\mathscr{Y}$ be two subcategories. Let $\mathscr{X} * \mathscr{Y}$ be the extension subcategory, that is, the subcategory consisting of objects B such that there exists a triangle $A \to B \to C \to A[1]$ where $A \in \mathscr{X}$ and $C \in \mathscr{Y}$. Chen proved the following result.

Theorem 1.2 ([5], Theorem 1.3). *Let $\mathscr{C}$ be a triangulated category. If $\mathscr{X}$ and $\mathscr{Y}$ are covariantly finite subcategories of $\mathscr{C}$, then the extension subcategory $\mathscr{X} * \mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$.*

Applications of the results of Gentle-Todorov and Chen are presented, see [5], [7], [8], [10].

Recently, the notion of an extriangulated category was introduced in [11], which is a simultaneous generalization of an exact category (which is itself a generalization of the concept of an abelian category) and of a triangulated category. The main result of this paper is the following, which is a simultaneous generalization of a result of Chen [5], Theorem 1.3 and of a result of Gentle-Todorov [6], Theorem 1.1.

Theorem 1.3. *Let $\mathscr{C}$ be an extriangulated category with enough projective objects. If $\mathscr{X}$ and $\mathscr{Y}$ are covariantly finite subcategories of $\mathscr{C}$, then the extension subcategory $\mathscr{X} * \mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$.*

Moreover, under suitable conditions, we can prove that a partial converse of Theorem 1.3 holds.

2. Preliminaries

We recall the definition and basic properties of extriangulated categories from [11].

Let $\mathscr{C}$ be an additive category. Suppose that $\mathscr{C}$ is equipped with a biadditive functor

$$\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab},$$

where Ab is the category of abelian groups. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*. Thus formally, an $\mathbb{E}$-extension is a triplet (A, δ, C). Let (A, δ, C) be an $\mathbb{E}$-extension. Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C', C)$, we have $\mathbb{E}$-extensions

$$\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).$$

We briefly denote them by $a_*\delta$ and $c^*\delta$. For any $A, C \in \mathscr{C}$, the zero element $0 \in \mathbb{E}(C, A)$ is called the *split* $\mathbb{E}$*-extension*.

Definition 2.1 ([11], Definition 2.3)**.** Let (A, δ, C), (A', δ', C') be any pair of $\mathbb{E}$-extensions. A *morphism*

$$(a, c)\colon\ (A, \delta, C) \to (A', \delta', C')$$

of $\mathbb{E}$-extensions is a pair of morphisms $a \in \mathscr{C}(A, A')$ and $c \in \mathscr{C}(C, C')$ in $\mathscr{C}$, satisfying the equality $a_*\delta = c^*\delta'$. Simply we denote it as $(a, c)\colon\ \delta \to \delta'$.

Definition 2.2 ([11], Definition 2.6)**.** Let $\delta = (A, \delta, C)$, $\delta' = (A', \delta', C')$ be any pair of $\mathbb{E}$-extensions. Let

$$C \xrightarrow{\iota_C} C \oplus C' \xleftarrow{\iota_{C'}} C'$$

and

$$A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$$

be the coproduct and the product in $\mathscr{B}$, respectively. Since $\mathbb{E}$ is biadditive, we have a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through the above isomorphism. This is the unique element which satisfies

$$\mathbb{E}(\iota_C, p_A)(\delta \oplus \delta') = \delta, \quad \mathbb{E}(\iota_C, p_{A'})(\delta \oplus \delta') = 0,$$
$$\mathbb{E}(\iota_{C'}, p_A)(\delta \oplus \delta') = 0, \quad \mathbb{E}(\iota_{C'}, p_{A'})(\delta \oplus \delta') = \delta'.$$

Let $A, C \in \mathscr{C}$ be any pair of objects. Sequences of morphisms in $\mathscr{C}$

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \simeq\downarrow b & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A, C \in \mathscr{C}$, we denote $0 = \left[A \xrightarrow{\binom{1}{0}} A \oplus C \xrightarrow{(0,1)} C\right]$.

For any two equivalence classes, we denote $[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C']$.

Definition 2.3 ([11], Definition 2.9). Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$, if it satisfies the following condition:

▷ Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{x} C], \qquad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then, for any morphism $(a, c)\colon \delta \to \delta'$, there exists $b \in \mathscr{C}(B, B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \tag{2.1}$$

In the above situation, we say that the triplet (a, b, c) realizes (a, c).

Definition 2.4 ([11], Definition 2.10). A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions.

(1) For any $A, C \in \mathscr{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.

(2) For any pair of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$,

$$\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$$

holds.

Definition 2.5 ([11], Definition 2.12). A triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is called an *externally triangulated category* (or *extriangulated category* for short) if it satisfies the following conditions:

(ET1) $\mathbb{E}\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Ab}$ is a biadditive functor.

(ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$-extensions, realized as

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in $\mathscr{C}$, there exists a morphism $(a, c)\colon \delta \to \delta'$ satisfying $cy = y'b$.

(ET3)$^{\mathrm{op}}$ Dual of (3).

(ET4) Let (A,δ,D) and (B,δ',F) be $\mathbb{E}$-extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} D \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F,$$

respectively. Then there exist an object $E \in \mathscr{C}$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathscr{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E,A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_*\delta'$,

(ii) $d^*\delta'' = \delta$,

(iii) $f_*\delta'' = e^*\delta'$.

$(\mathrm{ET4})^{\mathrm{op}}$ Dual of (4).

We use the following terminology.

Definition 2.6 ([11]). Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be a triplet satisfying (ET1) and (ET2).

(1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$. In this article, we write the conflation as $A \overset{x}{\rightarrowtail} B \overset{y}{\twoheadrightarrow} C$.

(2) A morphism $f \in \mathscr{C}(A,B)$ is called an *inflation* if it admits some conflation $A \overset{f}{\rightarrowtail} B \twoheadrightarrow C$.

(3) A morphism $f \in \mathscr{C}(A,B)$ is called a *deflation* if it admits some conflation $K \rightarrowtail A \overset{f}{\twoheadrightarrow} B$.

(4) If a conflation $A \overset{x}{\rightarrowtail} B \overset{y}{\twoheadrightarrow} C$ realizes $\delta \in \mathbb{E}(C,A)$, we call the pair $(A \overset{x}{\rightarrowtail} B \overset{y}{\twoheadrightarrow} C, \delta)$ an $\mathbb{E}$-*triangle*, and write it in the following way:

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

(5) Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathbb{E}$-triangles. If a triplet (a,b,c) realizes $(a,c)\colon \delta \to \delta'$ as in (2.1), then we write it as

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \overset{\delta'}{\dashrightarrow} & \end{array}$$

and call (a,b,c) a *morphism of* $\mathbb{E}$-*triangles*.

(6) An object $P \in \mathscr{C}$ is called *projective* if for any $\mathbb{E}$-triangle $A \stackrel{x}{\longrightarrow} B \stackrel{y}{\longrightarrow} C \stackrel{\delta}{\dashrightarrow}$ and any morphism $c \in \mathscr{C}(P, C)$, there exists $b \in \mathscr{C}(P, B)$ satisfying $yb = c$. We denote the subcategory of projective objects by $\mathcal{P} \subseteq \mathscr{C}$. Dually, the subcategory of injective objects is denoted by $\mathcal{I} \subseteq \mathscr{C}$.

(7) We say that $\mathscr{C}$ *has enough projective objects* if for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $A \stackrel{x}{\longrightarrow} P \stackrel{y}{\longrightarrow} C \stackrel{\delta}{\dashrightarrow}$ satisfying $P \in \mathcal{P}$. We can define the notion of having enough injectives dually.

(8) An extriangulated category $\mathscr{C}$ is said to be *Frobenius* if it has enough projectives and enough injectives and the projectives coincide with the injectives.

We now give some examples of extriangulated categories.

Example 2.7. (1) An exact category $\mathcal{B}$ can be viewed as an extriangulated category. For the definition and basic properties of an exact category, see [4]. In fact, a biadditive functor $\mathbb{E} := \mathrm{Ext}^1_{\mathcal{B}}\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Ab}$ and the realization $\mathfrak{s}$ is defined by associating the equivalence classes of short exact sequences to themselves. For more details, see [11], Example 2.13.

(2) Let $\mathscr{C}$ be a triangulated category with the shift functor [1]. Put $\mathbb{E} := \mathscr{C}(-, -[1])$. For any $\delta \in \mathbb{E}(C, A) = \mathscr{C}(C, A[1])$, take a triangle

$$A \stackrel{x}{\longrightarrow} B \stackrel{y}{\longrightarrow} C \stackrel{\delta}{\longrightarrow} A[1]$$

and define $\mathfrak{s}(\delta) = [A \stackrel{x}{\longrightarrow} B \stackrel{y}{\longrightarrow} C]$. Then $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. It is easy to see that extension closed subcategories of triangulated categories are also extriangulated categories. For more details, see [11], Proposition 3.22.

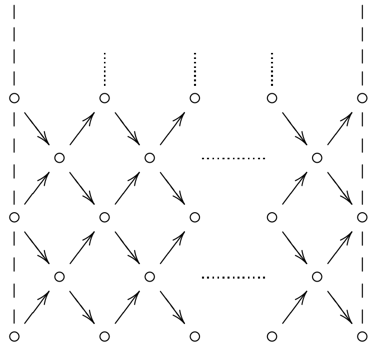
(3) Let $\mathscr{C}$ be an extriangulated category, and $\mathcal{J}$ a subcategory of $\mathscr{C}$. If $\mathcal{J} \subseteq \mathcal{P} \cap \mathcal{I}$, then $\mathscr{C}/\mathcal{J}$ is an extriangulated category. This construction gives extriangulated categories which are not exact nor triangulated in general. For more details, see [11], Proposition 3.30.

Example 2.8. Let $D^b(\mathcal{T}_n)$ be the bounded derived category of the abelian category $\mathcal{T}_n$. It is triangulated with shift functor [1], the shift of complexes. It has Auslander-Reiten triangles, and the Auslander-Reiten translation is the derived functor of the Auslander-Reiten translation τ of $\mathcal{T}_n$. By abuse of notation, we also denote it by τ. The *cluster tube of rank* n is defined in [2], as the orbit category

$$\mathscr{C}_n := D^b(\mathcal{T}_n)/\tau^{-1}[1].$$

The category $\mathscr{C}_n$ has a triangulated structure such that the canonical projection functor $\pi\colon D^b(\mathcal{T}_n) \to \mathscr{C}_n$ is triangulated, by Keller [9], Theorem 9.9. It is

2-Calabi-Yau, that is to say, there is a bifunctorial isomorphism $\mathsf{D}\mathrm{Hom}_{\mathscr{C}_n}(M, N) \cong \mathrm{Hom}_{\mathscr{C}_n}(N, M[2])$ for objects M and N in $\mathscr{C}_n$, where $\mathsf{D} = \mathrm{Hom}_k(-, k)$ denotes duality with respect to the base field k. It has Auslander-Reiten triangles, and the Auslander-Reiten translation τ is naturally equivalent to the shift functor [1]. The Auslander-Reiten quiver of $\mathscr{C}_n$ is depicted as



where the leftmost and rightmost columns are identified.

Take $\mathscr{X}$ to be the additive subcategory of $\mathscr{C}_n$ consisting of objects whose indecomposable summands are of quasi-length one or two. We know that $\mathscr{X}$ is a functorially finite subcategory of $\mathscr{C}_n$ satisfying $\tau\mathscr{X} = \mathscr{X}$, but $\{0\} \neq \mathscr{X} \subsetneqq \mathscr{C}_n$. By [12], Corollary 4.12, we obtain that $(\mathscr{C}_n, \mathbb{F}, \mathfrak{s}')$ is a Frobenius extriangulated category whose projective-injective objects are precisely $\mathscr{X}$. When $\mathscr{X} \neq \{0\}$, the Frobenius extriangulated category $(\mathscr{C}, \mathbb{F}, \mathfrak{s}')$ is not triangulated, since $\mathscr{X}$ is projective and injective and non-zero. When $\mathscr{X} \neq \mathscr{C}$, it is easy to see that the Frobenius extriangulated category $(\mathscr{C}, \mathbb{F}, \mathfrak{s}')$ is not exact. Otherwise any $\mathbb{F}$-extension splits and then any object in $\mathscr{C}$ is projective and injective. Then $\mathscr{X} = \mathscr{C}$, a contradiction. Hence the Frobenius extriangulated category $(\mathscr{C}, \mathbb{F}, \mathfrak{s}')$ is neither triangulated nor exact.

Remark 2.9. (1) As in Example 2.7(1), an exact category can be regarded as an extriangulated category, whose inflations are monomorphic and whose deflations are epimorphic. Conversely, let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category, in which any inflation is monomorphic, and any deflation is epimorphic. If we let $\mathcal{S}$ be the class of conflations given by the $\mathbb{E}$-triangles, then $(\mathscr{C}, \mathcal{S})$ is an exact category in the sense of [4]. For more details, see [11], Corollary 3.18.

(2) In a triangulated category $\mathscr{C}$, it is easy to see that $\mathcal{P} = 0$ and $\mathcal{I} = 0$, $\mathscr{C}$ has enough projectives and enough injectives. Thus $\mathscr{C}$ is a Frobenius extriangulated category. Conversely, let $\mathscr{C}$ be a Frobenius extriangulated category. If $\mathcal{P} = \mathcal{I} = 0$, then $\mathscr{C}$ is triangulated. For more details, see [11], Corollary 7.6.

Lemma 2.10 ([11], Proposition 3.3)**.** *Let $\mathscr{C}$ be an extriangulated category and let*

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

be an $\mathbb{E}$-triangle. Then we have the following exact sequence:

$$\mathscr{C}(-,A) \xrightarrow{\mathscr{C}(-,x)} \mathscr{C}(-,B) \xrightarrow{\mathscr{C}(-,y)} \mathscr{C}(-,C) \to \mathbb{E}(-,A) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-,B) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(-,C)\ ,$$

$$\mathscr{C}(C,-) \xrightarrow{\mathscr{C}(y,-)} \mathscr{C}(B,-) \xrightarrow{\mathscr{C}(x,-)} \mathscr{C}(A,-) \to \mathbb{E}(C,-) \xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B,-) \xrightarrow{\mathbb{E}(x,-)} \mathbb{E}(A,-)\ ,$$

Lemma 2.11 ([11], Corollary 3.16)**.** *Let $x\colon A \to B$, $y\colon D \to C$ and let $f\colon A \to C$ be any morphisms in an extriangulated category $\mathscr{C}$.*
(1) *If x is an inflation, then $\binom{f}{x}\colon A \to C \oplus B$ is an inflation in $\mathscr{C}$.*
(2) *If y is a deflation, then $(y, f)\colon D \oplus A \to C$ is a deflation in $\mathscr{C}$.*

Lemma 2.12. *Let $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Suppose we are given $\mathbb{E}$-triangles*

$$A \xrightarrow{f} B \xrightarrow{f'} C \dashrightarrow$$

$$A \xrightarrow{g} E \xrightarrow{g'} G \dashrightarrow$$

$$D \xrightarrow{h} B \xrightarrow{h'} E \dashrightarrow$$

satisfying $h'f = g$. Then there exists an $\mathbb{E}$-triangle

$$D \xrightarrow{d} C \xrightarrow{e} G \dashrightarrow$$

which makes

$$\begin{array}{ccccc} & & D & = & D \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{f} & B & \xrightarrow{f'} & C \\ \| & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle e} \\ A & \xrightarrow{g} & E & \xrightarrow{g'} & G \end{array}$$

commutative in $\mathscr{C}$.

P r o o f. See the dual of Proposition 3.17 in [11]. □

3. Proof of the main result

We introduce the notion of homotopy cartesian in an extriangulated category.

Definition 3.1. Let $\mathscr{C}$ be an extriangulated category. Then a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X & \xrightarrow{u} & Y \end{array}$$

is called *homotopy cartesian square* if there exists an $\mathbb{E}$-triangle

$$A \xrightarrow{\binom{f}{-a}} B \oplus X \xrightarrow{(b,\ u)} Y \overset{\delta}{\dashrightarrow}$$

in $\mathscr{C}$.

In order to prove our main result, we need the following lemma.

Lemma 3.2. *Let $\mathscr{C}$ be an extriangulated catgeory and*

$$A \xrightarrow{f} B \xrightarrow{g} C \dashrightarrow$$

be an $\mathbb{E}$-triangle in $\mathscr{C}$. Then for every morphism $a\colon A \to X$, there exists a commutative diagram of the form,

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \dashrightarrow \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \| & \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & C & \dashrightarrow \end{array}$$

where the left hand square is homotopy cartesian.

Proof. Since f is an inflation, by Lemma 2.11, we know that the morphism $\binom{f}{-a}\colon A \to B \oplus X$ is an inflation. Then there exists an $\mathbb{E}$-triangle

$$A \xrightarrow{\binom{f}{-a}} B \oplus X \xrightarrow{(b,\ u)} Y \dashrightarrow$$

in $\mathscr{C}$. By Lemma 2.12, we obtain the commutative diagram

$$\begin{array}{ccccc} & & X & = & X \\ & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle u} \\ A & \xrightarrow{\binom{f}{-a}} & B \oplus X & \xrightarrow{(b,\ u)} & Y \\ \| & & \downarrow{\scriptstyle (1,\ 0)} & & \downarrow{\scriptstyle v} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in which $X \xrightarrow{u} Y \xrightarrow{v} C \dashrightarrow$ is an $\mathbb{E}$-triangle. It follows that $bf = ua$ and $vb = g$. Thus we have the commutative diagram of $\mathbb{E}$-triangles,

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \dashrightarrow \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & C \dashrightarrow \end{array}$$

where the left hand square is homotopy cartesian. □

Let $\mathscr{C}$ be an extriangulated category, and let $\mathscr{X}$ and $\mathscr{Y}$ be two subcategories. Let $\mathscr{X} * \mathscr{Y}$ be the *extension subcategory*, that is, the subcategory consisting of objects B such that there exists an $\mathbb{E}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$ for some $A \in \mathscr{X}$ and $C \in \mathscr{Y}$. We have the following result.

Theorem 3.3. *Let $\mathscr{C}$ be an extriangulated category with enough projectives. If $\mathscr{X}$ and $\mathscr{Y}$ are covariantly finite subcategories of $\mathscr{C}$, then the extension subcategory $\mathscr{X} * \mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$.*

Proof. Let C be an arbitrary object in $\mathscr{C}$. Take a left $\mathscr{Y}$-approximation $f\colon C \to Y_C$. Since $\mathscr{C}$ has enough projective objects, there exists a deflation $n\colon P \to Y_C$, where P is a projective object. By Lemma 2.11, the morphism $(n, f)\colon P \oplus C \to Y_C$ is a deflation. So there exists an $\mathbb{E}$-triangle

$$M \xrightarrow{\binom{m}{g}} P \oplus C \xrightarrow{(n,\ f)} Y_C \dashrightarrow$$

in $\mathscr{C}$. Take a left $\mathscr{X}$-approximation $a\colon M \to X_M$. By Lemma 3.2, we have the commutative diagram of $\mathbb{E}$-triangles,

$$\begin{array}{ccccc} M & \xrightarrow{\binom{m}{g}} & P \oplus C & \xrightarrow{(n,\ f)} & Y_C \dashrightarrow \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle (b,\ c)} & & \| \\ X_M & \xrightarrow{u} & N & \xrightarrow{v} & Y_C \dashrightarrow \end{array}$$

where the left hand square is homotopy cartesian. That is, there exists an $\mathbb{E}$-triangle

$$M \xrightarrow{\begin{pmatrix} m \\ g \\ -a \end{pmatrix}} P \oplus C \oplus X_M \xrightarrow{(b,\ c,\ u)} N \dashrightarrow$$

in $\mathscr{C}$. We claim that $c\colon C\to N$ is a left $(\mathscr{X}*\mathscr{Y})$-approximation of C. Indeed, suppose we are given a morphism $k\colon C\to Z$ where $Z\in\mathscr{X}*\mathscr{Y}$ such that there exists an $\mathbb{E}$-triangle

$$X\xrightarrow{x}Z\xrightarrow{y}Y\dashrightarrow,$$

where $X\in\mathscr{X}$ and $Y\in\mathscr{Y}$. Since f is a left $\mathscr{Y}$-approximation of C and $Y\in\mathscr{Y}$, there exists a morphism $k_1\colon Y_C\to Y$ such that $k_1f=yk$. Since P is a projective object, there exists a morphism $t\colon P\to Z$ such that $yt=k_1n$. It follows that

$$k_1(n,f)=(yt,yk)=y(t,k).$$

By (ET3)$^{\rm op}$, we obtain a commutative diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\binom{m}{g}} & P\oplus C & \xrightarrow{(n,\ f)} & Y_C & \dashrightarrow & \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle (t,\ k)} & & \downarrow{\scriptstyle k_1} & & \\ X & \xrightarrow{x} & Z & \xrightarrow{y} & Y & \dashrightarrow & \end{array}$$

of $\mathbb{E}$-triangles. Since a is a left $\mathscr{X}$-approximation of M and $X\in\mathscr{X}$, there exists a morphism $k_3\colon X_M\to X$ such that $k_3a=k_2$. It follows that

$$xk_3a=xk_2=(t,k)\begin{pmatrix}m\\ g\end{pmatrix}=tm+kg$$

and then $(t,k,xk_3)\begin{pmatrix}m\\ g\\ -a\end{pmatrix}=0$. By Lemma 2.10, there exists a morphism $d\colon N\to Z$ such that $d(b,c,u)=(t,k,xk_3)$. Hence we have $dc=k$. This shows that $c\colon C\to N$ is a left $(\mathscr{X}*\mathscr{Y})$-approximation of C.

Therefore, $\mathscr{X}*\mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$. □

This theorem immediately yields the following important conclusion.

Corollary 3.4 ([6], Theorem 1.1)**.** *Let $\mathscr{C}$ be an abelian category with enough projective objects. If $\mathscr{X}$ and $\mathscr{Y}$ are covariantly finite subcategories of $\mathscr{C}$, then the extension subcategory $\mathscr{X}*\mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$.*

P r o o f. Since an abelian category can be viewed as an extriangulated category. □

Corollary 3.5 ([5], Theorem 1.3)**.** *Let $\mathscr{C}$ be a triangulated category. If $\mathscr{X}$ and $\mathscr{Y}$ are covariantly finite subcategories of $\mathscr{C}$, then the extension subcategory $\mathscr{X}*\mathscr{Y}$ is a covariantly finite subcategory of $\mathscr{C}$.*

P r o o f. In a triangulated category $\mathscr{C}$, it is easy to see that $\mathcal{P}$ consists of zero objects. Moreover, it always has enough projectives. □

Under suitable conditions, we prove the following partial converse of Theorem 3.3.

Proposition 3.6. *Let $\mathscr{C}$ be an extriangulated category, and let $\mathscr{X}$ and $\mathscr{Y}$ be two subcategories of $\mathscr{C}$. Assume that $\mathscr{X} * \mathscr{Y}$ is a covariantly finite subcategory in $\mathscr{C}$. If* $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, \mathscr{Y}) = 0$, *then $\mathscr{Y}$ is a covariantly finite subcategory in $\mathscr{C}$.*

P r o o f. Assume that C is an arbitrary object in $\mathscr{C}$. Take its left $(\mathscr{X} * \mathscr{Y})$-approximation $\varphi_C\colon C \to M_C$ of C. As $M_C \in \mathscr{X} * \mathscr{Y}$, there exists an $\mathbb{E}$-triangle

$$X \xrightarrow{a} M_C \xrightarrow{b} Y \dashrightarrow,$$

where $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. We claim that the morphism $b\varphi_C\colon C \to Y$ is a left $\mathscr{Y}$-approximation of C. Indeed, let $g\colon C \to Y'$ with $Y' \in \mathscr{Y}$ be any morphism in $\mathscr{C}$. Note that $Y' \in \mathscr{Y} \subseteq \mathscr{X} * \mathscr{Y}$. Since φ_C is a left $(\mathscr{X} * \mathscr{Y})$-approximation, there exists a morphism $f\colon M_C \to Y'$ such that $g = f\varphi_C$. Since $\mathrm{Hom}_{\mathscr{C}}(\mathscr{X}, \mathscr{Y}) = 0$, we have $fa = 0$. By Lemma 2.10, there exists $f_1\colon Y \to Y'$ such that $f = f_1 b$. Thus $g = f\varphi_C = f_1(b\varphi_C)$, namely, the morphism g factors through $b\varphi_C$, as required. Therefore $\mathscr{Y}$ is a covariantly finite subcategory in $\mathscr{C}$. □

Example 3.7. Let $\mathscr{C}$ be a triangulated category or an exact category, and let $\mathscr{X}$ and $\mathscr{Y}$ be two subcategories of $\mathscr{C}$. If $(\mathscr{X}, \mathscr{Y})$ is a torsion pair on $\mathscr{C}$, where the notion of a torsion pair is in the sense of Beligiannis and Reiten [3], then $\mathscr{Y}$ is a covariantly finite subcategory in $\mathscr{C}$.

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References

[1] *M. Auslander, I. Reiten*: Applications of contravariantly finite subcategories. Adv. Math. *86* (1991), 111–152. zbl MR doi
[2] *M. Barot, D. Kussin, H. Lenzing*: The Grothendieck group of a cluster category. J. Pure Appl. Algebra *212* (2008), 33–46. zbl MR doi
[3] *A. Beligiannis, I. Reiten*: Homological and homotopical aspects of torsion theories. Mem. Amer. Math. Soc. *188* (2007). zbl MR doi
[4] *T. Bühler*: Exact categories. Expo. Math. *28* (2010), 1–69. zbl MR doi
[5] *X. W. Chen*: Extensions of covariantly finite subcategories. Arch. Math. *93* (2009), 29–35. zbl MR doi
[6] *R. Gentle, G. Todorov*: Extensions, kernels and cokernels of homologically finite subcategories. Representation theory of algebras. Seventh international conference, August 22-26, 1994, Cocoyoc, Mexico (Bautista, Raymundo et al., eds.). vol. 18, American Mathematical Society. CMS Conf. Proc., 1996, pp. 227–235. zbl MR

[7] *L. A. Hügel, F. Marks, J. Vitória*: Silting modules and ring epimorphisms. Adv. Math. *303* (2016), 1044–1076. zbl MR doi
[8] *O. Iyama, B. Lerner*: Tilting bundles on orders on $\mathbb{P}^d$. Isr. J. Math. *211* (2016), 147–169. zbl MR doi
[9] *B. Keller*: On triangulated orbit categories. Doc. Math. *10* (2005), 21–56. zbl MR
[10] *O. Mendoza, V. Santiago*: Homological systems in triangulated categories. Appl. Categ. Struct. *24* (2016), 1–35. zbl MR doi
[11] *H. Nakaoka, Y. Palu*: Mutation via Hovey twin cotorsion pairs and model structures in extriangulated categories. Available at https://arxiv.org/abs/1605.05607v2.
[12] *P. Y. Zhou, B. Zhu*: Triangulated quotient categories revisited. J. Algebra *502* (2018), 196–232. zbl MR doi

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