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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS IN VECTOR-VALUED FUNCTION SPACE

BAHLOLUL RACHID

ABSTRACT. The aim of this work is to study the existence and uniqueness of solutions of the fractional integro-differential equations
\[ \frac{D_t}{\Gamma(\alpha)} [x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} \left( \int_{-\infty}^{s} a(s-\xi)x(\xi)d\xi \right)ds + f(t), \]
\((\alpha > 0)\) with the periodic condition \(x(0) = x(2\pi)\), where \(a \in L^1(\mathbb{R}_+)\). Our approach is based on the R-boundedness of linear operators \(L^p\)-multipliers and UMD-spaces.

1. Introduction

The aim of this paper is to study the existence and uniqueness of solutions for some retarded fractional integro-differential equations with delay by using methods of maximal regularity in spaces of vector valued functions. Motivated by the fact that neutral functional integro-differential equations with finite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors. We refer the readers to papers ([3], [8], [14], [24]) and the references listed therein for information on this subject. One of the most important tools to prove maximal regularity is the theory of Fourier multipliers. They play an important role in the analysis of parabolic problems. In recent years it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow to treat certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For some recent papers on the subjet, we refer to Weis [17], Poblete [26], Lizama [24], Keyantue [19], Hernan et al [21] et Arendt-Bu [4].

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of the most powerful modern theorems are valid in UMD spaces, i.e., Banach spaces
in which martingale are unconditional differences. The probabilistic definition
of UMD spaces turns out to be equivalent to the \( L^p \)-boundedness of the Hilbert
transform, a transformation which is, in a sense, the typical representative example
of a multiplier operator. On the other hand the notion of R-boundedness has
played an important role in the functional analytic approach to partial differential
equations.

In this work, we study the existence of periodic solutions for the following
integro-differential equations
\[
\frac{d}{dt} [x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + 1 + 1
\]
where \( A: D(A) \subseteq X \to X \) is a linear closed operator on Banach space \((X, \|\cdot\|)\),
\( \Gamma(\cdot) \) is the Euler gamma function \((\alpha > 0)\), and \( f \in L^p(T, X) \) for all \( p \geq 1 \). For
\( r_{2\pi} := 2\pi N \) (some \( N \in \mathbb{N} \)) \( L \) and \( G \) are in \( B(L^p([-r_{2\pi}, 0], X); X) \)
the space of all bounded linear operators and \( x_t \) is an element of \( L^p([-r_{2\pi}, 0], X) \)
which is defined as follows
\[
x_t(\theta) = x(t + \theta) \quad \text{for} \quad \theta \in [-r_{2\pi}, 0].
\]
Initially, Arendt and Bu \cite{3} dealt with the problem \( u'(t) = Au(t) + f(t), u(0) = u(2\pi) \). Maximal regularity for the evolution problem in \( L^p \) was treated earlier by
Weis \cite{28, 29} (see also \cite{29} for a different proof of the operator-valued Mikhlin
multiplier theorem using a transference principle). The study in the \( L^p \) frame-
work (when \( 1 < p < \infty \)) was made possible thanks to the introduction of the concept of randomized boundedness (hereafter \( R \)-boundedness, also known as
Riesz-boundedness or Rademacher-boundedness). With this, necessary conditions
for operator-valued Fourier multipliers were found in this context. In addition, the
space \( X \) must have the UMD property. This was done initially by L. Weis \cite{28, 29}
for the evolutionary problem and then by Arendt-Bu \cite{3} for periodic boundary
conditions. For non-degenerate integro-differential equations both in the periodic
and non periodic cases, operator-valued Fourier multipliers have been used by
various authors to obtain well-posedness in various scales of function spaces: see
\cite{7, 8, 9, 19, 20, 21, 25, 26} and the corresponding references. The well-posedness
or maximal regularity results are important in that they allow for the treatment
of nonlinear problems. Earlier results on the application of operator-valued Fou-
rier multiplier theorems to evolutionary integral equations can be found in \cite{11}. More recent examples of second order integro-differential equations with frictional
damping and memory terms have been studied in the paper \cite{10}.

In \cite{1}, Aparicio et al. studied the existence of periodic solution of degenerate
t integro-differential equations in function spaces described in the following form:
\[
(Mu')'(t) - \Lambda u'(t) - \frac{d}{dt} \int_{-\infty}^{t} c(t-s)u(s)ds = \gamma u(t) + Au(t)
\]
\[ + \int_{-\infty}^{t} b(t-s)Bu(s)ds + f(t), \]

and periodic boundary conditions \( u(0) = u(2\pi), \) \((Mu')(0) = (Mu')(2\pi).\) Here, \(A, B, \Lambda\) and \(M\) are closed linear operators in a Banach space \(X\) satisfying the assumption \(D(A) \cap D(B) \subset D(\Lambda) \cap D(M)\), \(b, c \in L^1(\mathbb{R}_+), f\) is an \(X\)-valued function defined on \([0, 2\pi]\), and \(\gamma\) is a constant.

In \cite{22}, S. Koumla, Kh. Ezzinbi, R. Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay

\[
\frac{d}{dt} x(t) = Ax(t) + \int_{0}^{t} B(t-s)x(s)ds + f(t, x_t) + h(t, x_t)
\]

where \(A : D(A)X \rightarrow X\) is the infinitesimal generator of a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on a Banach space \(X\), for \(t \geq 0\), \(B(t)\) is a closed linear operator with domain \(D(B) \supset D(A)\).

This work is organized as follows: In Section 2 we collect some preliminary results and definitions. In Section 3, we study the existence and uniqueness of strong \(L^p\)-solution of the Eq. \((1.1)\) solely in terms of a property of \(R\)-boundedness for the sequence of operators \(ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))^{-1}\). We obtain that the following assertion are equivalent in UMD space:

1. \((ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))\) is invertible and \(\{ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\tilde{a}(ik))^{-1}, k \in \mathbb{Z}\}\) is \(R\)-bounded.

2. For every \(f \in L^p(T; X)\) there exist a unique function \(u \in H^{1,p}(T; X)\) such that \(u \in D(A)\) and equation \((1.1)\) holds for a.e. \(t \in [0, 2\pi]\).

## 2. Preliminaries

In this section, we collect some results and definitions that will be used in the sequel. Let \(X\) be a complex Banach space. We denote as usual by \(L^1(0, 2\pi; X)\) the space of Bochner integrable functions with values in \(X\). For a function \(f \in L^1(0, 2\pi; X)\), we denote by \(\hat{f}(k), k \in \mathbb{Z}\) the \(k\)-th Fourier coefficient of \(f\):

\[
\hat{f}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ikt}f(t)dt,
\]

where \(e_k(t) = e^{ikt}, t \in \mathbb{R}\).

Let \(u \in L^1(0, 2\pi; X)\). We denote again by \(u\) its periodic extension to \(\mathbb{R}\). Let \(a \in L^1(\mathbb{R}_+)\). We consider the function

\[
F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds, \quad t \in \mathbb{R}.
\]

Since

\[
(2.1) \quad F(t) = \int_{-\infty}^{t} a(t-s)u(s)ds = \int_{0}^{\infty} a(s)u(t-s)ds,
\]
we have $\|F\|_{L^1} \leq \|a\|_1 \|u\|_{L^1} = \|a\|_{L^1(\mathbb{R}, \lambda)} \|u\|_{L^1(0, 2\pi; X)}$ and $F$ is periodic of period $T = 2\pi$ as $u$. Now using Fubini’s theorem and (2.1) we obtain, for $k \in \mathbb{Z}$, that

$$\hat{F}(k) = \hat{a}(ik) \hat{u}(k), \quad k \in \mathbb{Z}$$

where $\hat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t) dt$ denotes the Laplace transform of $a$. This identity plays a crucial role in the paper.

Let $X, Y$ be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from $X$ to $Y$. When $X = Y$, we write simply $\mathcal{L}(X)$.

**Proposition 2.1 (Fejer’s Theorem).** Let $f \in L^p(0, 2\pi; X)$, then one has

$$f = \lim_{n \to \infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=m}^{m} e_k \hat{f}(k)$$

with convergence in $L^p(0, 2\pi; Y)$.

**R-boundedness-UMD space, $L^p$-multiplier and Riemann-Liouville fractional integral.** For results on operator-valued Fourier multipliers and $R$-boundedness (used in the next section), as well as some applications to evolutionary partial differential equations, we refer to Bourgain [3, 6], Clément-de Pagter-Sukochev-Witvliet [12], Weis [28, 29], Girardi-Weis [17, 18], Kunstmann-Weis [23], Clément-Prüss [13], Arendt [2], Arendt-Bu [3], Ataricio-Keyantuo [1] and Suresh [27].

We shall frequently identify the spaces of (vector or operator-valued) functions defined on $[0, 2\pi]$ to their periodic extensions to $\mathbb{R}$.

For $j \in \mathbb{N}$, denote by $r_j$ the $j$-th Rademacher function on $[0, 1]$, i.e. $r_j(t) = \text{sgn}(\sin(2^j \pi t))$. For $x \in X$ we denote by $r_j \otimes x$ the vector valued function $t \to r_j(t)x$.

The important concept of $R$-bounded for a given family of bounded linear operators is defined as follows.

**Definition 2.2.** A family $T \subset \mathcal{L}(X, Y)$ is called $R$-bounded if there exists $c_q \geq 0$ such that

$$\left\| \sum_{j=1}^{n} r_j \otimes T_j x_j \right\|_{L^q(0, 1; X)} \leq c_q \left\| \sum_{j=1}^{n} r_j \otimes x_j \right\|_{L^q(0, 1; X)}$$

for all $T_1, \ldots, T_n \in T, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$, where $1 \leq q < \infty$. We denote by $R_q(T)$ the smallest constant $c_q$ such that (2.3) holds.

**Remark 2.3.** Several useful properties of $R$-bounded families can be found in the monograph of Denk-Hieber-Prüss [16, Section 3], see also [2, 3, 12, 15, 23]. We collect some of them here for later use.

(a) Any finite subset of $\mathcal{L}(X)$ is is $R$-bounded.

(b) If $S \subset T \subset \mathcal{L}(X)$ and $T$ is $R$-bounded, then $S$ is $R$-bounded and $R_p(S) \leq R_p(T)$.

(c) Let $S, T \subset \mathcal{L}(X)$ be $R$-bounded sets. Then $S \cdot T := \{S \cdot T : S \in S, T \in T\}$ is $R$-bounded and

$$R_p(S \cdot T) \leq R_p(S) \cdot R_p(T).$$
(d) Let \( S, T \subset \mathcal{L}(X) \) be \( R \)-bounded sets. Then \( S + T := \{ S + T : S \in S, T \in T \} \) is \( R \)-bounded and
\[
R_p(S + T) \leq R_p(S) + R_p(T).
\]
(e) If \( T \subset \mathcal{L}(X) \) is \( R \)-bounded, then \( T \cup \{ 0 \} \) is \( R \)-bounded and
\[
R_p(T \cup \{ 0 \}) = R_p(T).
\]
(f) If \( S, T \subset \mathcal{L}(X) \) are \( R \)-bounded, then \( T \cup S \) is \( R \)-bounded and
\[
R_p(T \cup S) \leq R_p(S) + R_p(T).
\]
(g) Also, each subset \( M \subset \mathcal{L}(X) \) of the form \( M = \{ \lambda I : \lambda \in \Omega \} \) is \( R \)-bounded whenever \( \Omega \subset \mathbb{C} \) is bounded (\( I \) denotes the identity operator on \( X \)).

The proofs of (a), (e), (f), and (g) rely on Kahane’s contraction principle.

We make the following general observation which will be valid throughout the paper, notably in Section 4. Whenever we wish to establish \( R \)-boundedness of a family of operators \( \{ M_k \}_{k \in \mathbb{Z}} \), if at some point we make an exception such as \( (k \neq 0), (k \notin \{ -1, 0 \}) \) and so on, then later we recover the property for the entire family using items (a), (c) and (f) of the foregoing remark. The corresponding observation for boundedness is clear.

**Definition 2.4.** Let \( \varepsilon \in ]0, 1[ \) and \( 1 < p < \infty \). Define the operator \( H_\varepsilon \) by: for all \( f \in L^p(\mathbb{R}; X) \)
\[
(H_\varepsilon f)(t) := \frac{1}{\pi} \int_{\varepsilon < |s| < \frac{1}{\varepsilon}} \frac{f(t-s)}{s} ds
\]
if \( \lim_{\varepsilon \to 0} H_\varepsilon f := Hf \) exists in \( L^p(\mathbb{R}; X) \). Then \( Hf \) is called the Hilbert transform of \( f \) on \( L^p(\mathbb{R}, X) \).

**Definition 2.5.** A Banach space \( X \) is said to be UMD space if the Hilbert transform is bounded on \( L^p(\mathbb{R}, X) \) for all \( 1 < p < \infty \).

**Definition 2.6.** For \( 1 \leq p < \infty \), a sequence \( \{ M_k \}_{k \in \mathbb{Z}} \subset \mathcal{B}(X,Y) \) is said to be an \( L^p \)-multiplier if for each \( f \in L^p(\mathbb{T}, X) \), there exists \( u \in L^p(\mathbb{T}, Y) \) such that
\[
\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.
\]

**Proposition 2.7.** Let \( X \) be a Banach space and \( \{ M_k \}_{k \in \mathbb{Z}} \) be an \( L^p \)-multiplier, where \( 1 \leq p < \infty \). Then the set \( \{ M_k \}_{k \in \mathbb{Z}} \) is \( R \)-bounded.

**Theorem 2.8** (Marcinkiewicz operator-valued multiplier Theorem). Let \( X, Y \) be UMD spaces and \( \{ M_k \}_{k \in \mathbb{Z}} \subset \mathcal{B}(X,Y) \). If the sets \( \{ M_k \}_{k \in \mathbb{Z}} \) and \( \{ k(M_{k+1} - M_k) \}_{k \in \mathbb{Z}} \) are \( R \)-bounded, then \( \{ M_k \}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier for \( 1 < p < \infty \).

**Definition 2.9.** The Riemann-Liouville fractional integral operator of order \( \alpha > 0 \) is defined by
\[
I_{-\infty}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s) ds
\]
\[
\frac{d}{dt}(k) = ik \hat{x}(k), \quad \forall k \in \mathbb{Z}
\]

and more generally,
\[
\frac{d^n x}{dt^n}(k) = (ik)^n \hat{x}(k), \quad \forall k \in \mathbb{Z}
\]

A similar identity holds for anti-derivatives
\[
I_{-\infty}^{-s} f(k) = (ik)^{-s} \hat{x}(k), \quad \forall k \in \mathbb{Z}
\]

**Remark 2.10.** If we set \( u(x) = e^{ikx} \) for \( k \in \mathbb{Z} \) we have
1) \( I_{-\infty}^{a} u(t) = (ik)^{-\alpha} e^{ikx} \)
2) \( I_{-\infty}^{a}(a \ast u)(t) = (ik)^{-\alpha} e^{ikx} \hat{a}(ik) \).

### 3. Periodic solutions in UMD space

For \( a \in L^1(\mathbb{R}_+) \), we denote by \( a \ast x \) the function
\[
(a \ast x)(t) := \int_{-\infty}^{t} a(t-s)x(s)ds
\]
and \( D\varphi = \varphi(0) - L(\varphi) \), with this notation we may rewrite Eq. (1.1) in the following way:
\[
(3.1) \quad \frac{d}{dt}(Dx_t) = A(Dx_t) + I_{-\infty}^{a}(a \ast x)(t) + G(x_t) + f(t) \quad \text{for } t \in \mathbb{R}
\]

we have \( a \ast x(k) = \hat{a}(ik) \hat{x}(k) \) and \( I_{-\infty}^{a}(a \ast x)(k) = (ik)^{-\alpha} \hat{a}(ik) \hat{x}(k) \).

Denote by \( L_k(x) := L(e_k x) ; G_k(x) := G(e_k x) \) and \( e_k(\theta) := e^{ik\theta} \). \( D_k = I - L_k \) for all \( k \in \mathbb{Z} \). We define
\[
\Delta_k = (ikD_k - AD_k - G_k - (ik)^{-\alpha} \hat{a}(ik)) \text{ and } \sigma_{\mathbb{Z}}(\Delta) = \{ k \in \mathbb{Z} : \Delta_k \text{ is not bijective} \}
\]

the periodic vector-valued space is defined by
\[
H^{1,p}(\mathbb{T}; X) = \{ u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), v(k) = ik \hat{u}(k) \text{ for all } k \in \mathbb{Z} \}
\]

**Lemma 3.1.** Let \( f \in L^1(\mathbb{T}; X) \). If \( g(t) = \int_{0}^{t} f(s)ds \) and \( k \in \mathbb{Z} \), \( k \neq 0 \). Then
\[
\hat{g}(k) = \frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k).
\]

**Definition 3.2.** For \( 1 \leq p < \infty \), we say that a sequence \( \{M_k\}_{k\in \mathbb{Z}} \subset \mathcal{B}(X,Y) \) is an \( (L^p, H^{1,p}) \)-multiplier, if for each \( f \in L^p(\mathbb{T}, X) \) there exists \( u \in H^{1,p}(\mathbb{T}, Y) \) such that
\[
\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}.
\]
Lemma 3.3. Let $1 \leq p < \infty$ and $(M_k)_{k \in \mathbb{Z}} \subset B(X)$ $(B(X)$ is the set of all bounded linear operators from $X$ to $X$). Then the following assertions are equivalent:

(i) $(M_k)_{k \in \mathbb{Z}}$ is an $(L^p, H^{1,p})$-multiplier.

(ii) $(ikM_k)_{k \in \mathbb{Z}}$ is an $(L^p, L^p)$-multiplier.

We begin by establishing our concept of strong solution for Eq. (3.1).

Definition 3.4. Let $f \in L^p(\mathbb{T}; X)$. A function $x \in H^{1,p}(\mathbb{T}; X)$ is said to be a $2\pi$-periodic strong $L^p$-solution of Eq. (3.1) if $D^p_t x \in D(A)$ for all $t \geq 0$ and Eq. (3.1) holds almost everywhere.

Lemma 3.5 ([24]). Let $G: L^p(\mathbb{T}, X) \rightarrow X$ be a bounded linear operator. Then

$$\widehat{G(u)}(k) = G(e_k \hat{u}(k)) := G_k \hat{u}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$ 

Proposition 3.6. Let $A$ be a closed linear operator defined on an UMD space $X$. Suppose that $\sigma_\mathcal{D}(\Delta) = \phi$. Then the following assertions are equivalent:

(i) $(ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\hat{a}(ik))^{-1})_{k \in \mathbb{Z}}$ is an $L^p$-multiplier for $1 < p < \infty$

(ii) $(ik(ikD_k - AD_k - G_k - (ik)^{-\alpha}\hat{a}(ik))^{-1})_{k \in \mathbb{Z}}$ is $R$-bounded.

Proof. (i) $\Rightarrow$ (ii) Define $M_k = ik(C_k - AD_k)^{-1}$, where $C_k := ikD_k - b_k - G_k$ such that $b_k = (ik)^{-\alpha}\hat{a}(ik)$. By Theorem (2.8) it is sufficient to prove that the set \{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}} is $R$-bounded. Since

$$k[M_{k+1} - M_k] = k[i(k + 1)(C_{k+1} - AD_{k+1})^{-1} - ik(C_k - AD_k)^{-1}]$$

$$= k(C_{k+1} - AD_{k+1})^{-1}[i(k + 1)(C_k - AD_k) - ik(C_{k+1} - AD_{k+1})] (C_k - AD_k)^{-1}$$

$$= k(C_{k+1} - AD_{k+1})^{-1}[ik(C_k - C_{k+1}) + i(C_k - AD_k) + ik(AD_{k+1} - AD_k)]$$

$$\times (C_k - AD_k)^{-1}$$

$$= k(C_{k+1} - AD_{k+1})^{-1}(C_k - C_{k+1})ik(C_k - AD_k)^{-1} + ik(C_{k+1} - AD_{k+1})^{-1}$$

$$+ k(C_{k+1} - AD_{k+1})^{-1}(AD_{k+1} - AD_k)ik(C_k - AD_k)^{-1}$$

we have

$$C_k - C_{k+1} = ik(D_k - D_{k+1}) - iD_{k+1} + (G_{k+1} - G_k) - (b_{k+1} - b_k)$$

$$= ik(L_{k+1} - L_k) + (G_{k+1} - G_k) + iL_k - (b_{k+1} - b_k).$$

Therefore

$$k(C_{k+1} - AD_{k+1})^{-1}(AD_{k+1} - AD_k)ik(C_k - AD_k)^{-1}$$

$$= k(C_{k+1} - AD_{k+1})^{-1}AD_{k+1}ik(C_k - AD_k)^{-1}$$

$$- k(C_{k+1} - AD_{k+1})^{-1}ikAD_k(C_k - AD_k)^{-1}$$

$$= k[C_{k+1}(C_k - AD_k)^{-1} - I]ik(C_k - AD_k)^{-1}$$

$$+ k(C_{k+1} - AD_{k+1})^{-1}ik[I - C_k(C_k - AD_k)^{-1}]$$
Since products and sums of $R$-bounded sequences is $R$-bounded [24, Remark 2.2]. Then the proof is complete.

Lemma 3.7. Let $1 \leq p < \infty$. Suppose that $\sigma_\mathcal{Z}(\Delta) = \phi$ and that for every $f \in L^p(\mathbb{T}; X)$ there exists a $2\pi$-periodic strong $L^p$-solution $x$ of Eq. (3.1). Then $x$ is the unique $2\pi$-periodic strong $L^p$-solution.

Proof. Suppose that $x_1$ and $x_2$ two strong $L^p$-solution of Eq. (3.1) then $x = x_1 - x_2$ is a strong $L^p$-solution of Eq. (3.1) corresponding to $f = 0$. Taking Fourier transform in (3.1), we obtain that

$$ikD_k\hat{x}(k) = AD_k\hat{x}(k) + (ik)^{-\alpha}\tilde{a}(ik)\hat{x}(k) + G_k\hat{x}(k), \quad k \in \mathbb{Z}.$$ 

Then

$$(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)\hat{x}(k) = 0.$$ 

It follows that $\hat{x}(k) = 0$ for every $k \in \mathbb{Z}$ and therefore $x = 0$. Then $x_1 = x_2$.

Theorem 3.8. Let $X$ be a Banach space. Suppose that for every $f \in L^p(\mathbb{T}; X)$ there exists a unique strong solution of Eq. (3.1) for $1 \leq p < \infty$. Then

1. for every $k \in \mathbb{Z}$ the operator $\Delta_k = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$ has bounded inverse
2. $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is $R$-bounded.

Before to give the proof of Theorem 3.8 we need the following lemma.

Lemma 3.9. If $ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)(x) = 0$ for all $k \in \mathbb{Z}$, then $u_t(\cdot) = e^{ikt}e_k(\cdot)x$ is a $2\pi$-periodic strong $L^p$-solution of the following equation

$$\frac{d}{dt}(Du_t) = A(Du_t) + I^\alpha_{-\infty}(a * u)(t) + Gu_t.$$

Proof. $(ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)(x) = 0 \Rightarrow ikD_kx = AD_kx + (ik)^{-\alpha}\tilde{a}(ik)x + G_kx$. Then

$$ikx = ikL_kx + AD_kx + (ik)^{-\alpha}\tilde{a}(ik)x + G_kx.$$ 

We have $u_t = e^{ikt}e_kx$ and by Remark 2.10 (2),

$$u_t' = ike^{ikt}e_kx = e^{ikt}e_k(ikx)$$

$$= e^{ikt}e_k[ikL_kx + AD_kx + (ik)^{-\alpha}\tilde{a}(ik)x + G_kx]$$

$$= ike^{ikt}e_kL_kx + e^{ikt}e_kAD_kx + e_k e^{ikt}(ik)^{-\alpha}\tilde{a}(ik)x + e^{ikt}e_kG_kx$$

$$= ike^{ikt}e_kL_kx + e^{ikt}e_kAD_kx + e^{ikt}e_kG_kx + I^\alpha_{-\infty}(a * u_t) + G(e^{ikt}e_kx)$$

$$= ikL(e^{ikt}e_kx) + AD(e^{ikt}e_kx) + I^\alpha_{-\infty}(a * u_t) + G(e^{ikt}e_kx)$$

$$= ikL(e^{ikt}e_kx) + AD(e^{ikt}e_kx) + I^\alpha_{-\infty}(a * u_t) + G(e^{ikt}e_kx)$$

$$= ikL(u_t) + A(Du_t) + I^\alpha_{-\infty}(a * u_t) + G(u_t)$$

$$(u_t - Lu_t)' = A(Du_t) + I^\alpha_{-\infty}(a * u_t) + G(u_t),$$

where $\Delta_k = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$. Then

$$(u_t - Lu_t)' = A(Du_t) + I^\alpha_{-\infty}(a * u_t) + G(u_t),$$

where $\Delta_k = (ikD_k - AD_k - (ik)^{-\alpha}\tilde{a}(ik) - G_k)$. Then
Theorem 4.1. Let $a \ast x \in L^p$ be a closed linear operator. Then the following assertions are equivalent for all $x$:

\[
(Du_t)' = A(Du_t) + I_{-\infty}^\alpha(a \ast u_t) + G(u_t)
\]

Proof of Theorem 3.8. 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $x \in H^1_p(T; X)$ such that:

\[
\frac{d}{dt} Dx_t = A(Dx_t) + I_{-\infty}^\alpha(a \ast x)(t) + G(x_t) + f(t).
\]

Taking Fourier transform, $G$ and $D$ are bounded. We have $(\hat{Dx}')(k) = \hat{x}'(k) - (Lx)'(k)$ and $I_{-\infty}^\alpha(a \ast x)(k) = (ik)^{-\alpha} \hat{a}(ik) \hat{x}(k)$ by Lemma 3.5, we deduce that:

\[
\hat{x}'(k) - (Lx)'(k) = i k \hat{x}(k) - ikL_k \hat{x}(k) = ik(I - L_k) \hat{x}(k) = ikD_k \hat{x}(k).
\]

Consequently, we have

\[
(ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k) \hat{x}(k) = \hat{f}(k) = y \Rightarrow (ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k)\text{ is surjective if } (ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k)(u) = 0,
\]

then by Lemma 3.9, $x_t = e^{ikt}e_k u$ is a $2\pi$-periodic strong $L^p$-solution of Eq. (3.1) corresponding to the function $f(t) = 0$. Hence $x_t = 0$ and $u = 0$ then $(ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k)$ is injective.

2) Let $f \in L^p(T; X)$. By hypothesis, there exists a unique $x \in H^1_p(T, X)$ such that the Eq. (3.1) is valid. Taking Fourier transforms, we deduce that

\[
\hat{x}(k) = (ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k)^{-1} \hat{f}(k)
\]

Hence

\[
(ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k) \hat{x}(k) = (ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k)^{-1} \hat{f}(k)
\]

Since $x \in H^1_p(T; X)$, then there exists $v \in L^p(T; X)$ such that

\[
\hat{v}(k) = ik \hat{x}(k) = ik(ikD_k - AD_k - (ik)^{-\alpha} \hat{a}(ik) - G_k)^{-1} \hat{f}(k)
\]

Then \(\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}\) is an $L^p$-multiplier and \(\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}\) is $R$-bounded.

4. Main result

Our main result in this work is to establish that the converse of Theorem 3.8 are true, provided $X$ is a UMD space.

Theorem 4.1. Let $X$ be an UMD space and $A: D(A) \subset X \rightarrow X$ be an closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.

1) for every $f \in L^p(T, X)$ there exists a unique $2\pi$-periodic strong $L^p$-solution of Eq. (3.1);

2) $\sigma(\Delta) = \phi$ and \(\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}\) is $R$-bounded.

Lemma 4.2 (3). Let $f, g \in L^p(T; X)$. If $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$. Then

\[
f(t) \in D(A) \text{ and } Af(t) = g(t) \text{ for all } t \in [0, 2\pi].
\]
Proof. 1) $\Rightarrow$ 2) see Theorem 3.8

1) $\Leftarrow$ 2) Let $f \in L^p(\mathbb{T}; X).$ Define $\Delta_k = (ikD_k - AD_k - (ik)^{-\alpha} \tilde{a}(ik) - G_k).

By Lemma 3.3, the family $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an $L^p$-multiplier it is equivalent to the family $\{\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an $L^p$-multiplier that maps $L^p(\mathbb{T}; X)$ into $H^{1,p}(\mathbb{T}; X),$ namely there exists $x \in H^{1,p}(\mathbb{T}, X)$ such that

\begin{equation}
\hat{x}(k) = \Delta_k^{-1} \hat{f}(k) = (ikD_k - AD_k - (ik)^{-\alpha} \tilde{a}(ik) - G_k)^{-1} \hat{f}(k).
\end{equation}

In particular, $x \in L^p(\mathbb{T}; X)$ and there exists $v \in L^p(\mathbb{T}; X)$ such that

\begin{equation}
\hat{\dot{v}}(k) = ik \hat{x}(k)
\end{equation}

By Theorem 2.1, we have

\[ x_t(\theta) = x(t + \theta) = \lim_{n \to +\infty} \frac{1}{n + 1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e^{ik\theta} \hat{x}(k). \]

Hence in $L^p(\mathbb{T}; X),$ we obtain that

\[ x_t = \lim_{n \to +\infty} \frac{1}{n + 1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e^{ik\theta} \hat{x}(k). \]

Since $G$ is bounded, then

\[ Gx_t = \lim_{n \to +\infty} \frac{1}{n + 1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} G(e_k \hat{x}(k)) \]

\[ = \lim_{n \to +\infty} \frac{1}{n + 1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} G_k \hat{x}(k) \]

Using now (4.1) and (4.2) we have:

\[ \widehat{(Dx_t)}'(k) = ikD_k \hat{x}(k) = AD_k \hat{x}(k) + I_{-\infty}^{\alpha}(a \ast x)(k) + G_k \hat{x}(k) + \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}. \]

Since $A$ is closed, then $Dx_t \in D(A)$ [Lemma 4.2] and from the uniqueness theorem of Fourier coefficients, that Eq. (3.1) is valid. □

References


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