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NOTE ON $\alpha$-FILTERS IN DISTRIBUTIVE NEARLATTICES

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Abstract. In this short paper we introduce the notion of $\alpha$-filter in the class of distributive nearlattices and we prove that the $\alpha$-filters of a normal distributive nearlattice are strongly connected with the filters of the distributive nearlattice of the annihilators.

Keywords: distributive nearlattice; annihilator; $\alpha$-filter

MSC 2010: 06A12, 03G10, 06D50

1. Introduction and preliminaries

A nearlattice is a join-semilattice with greatest element in which every principal filter is a bounded lattice. These structures are a natural generalization of the implication algebras studied by Abbott in [1] and the bounded distributive lattices. The nearlattices form a variety and has been studied by Cornish and Hickman in [14] and [16], and by Chajda, Halaš, Kühr and Kolařík in [8], [9], [10] and [11]. A particular class of nearlattices are the distributive nearlattices. In [6] and [7], a full duality is developed for distributive nearlattices and some applications are given, and recently in [15], the author proposes a sentential logic associated with the class of distributive nearlattices.

On the other hand, Cornish in [13] introduced the notion of $\alpha$-ideal in the class of distributive lattices and characterizes Stone lattices in terms of $\alpha$-ideals. These results were extended to the Hilbert algebras in [4] and [5]. We can study a dual notion of $\alpha$-ideal in the class of distributive nearlattices, i.e. the concept of $\alpha$-filter. The main objective of this paper is to introduce the notion of $\alpha$-filter in the variety of distributive nearlattices. We see that the $\alpha$-filters of a normal distributive near-
lattice $A$ are strongly connected with the filters of the distributive nearlattice $R(A)$ of the annihilators. This result extends those obtained by Cornish.

Let $A = \langle A, \lor, 1 \rangle$ be a join-semilattice with greatest element. A filter is a subset $F$ of $A$ such that $1 \in F$, if $a \leq b$ and $a \in F$, then $b \in F$ and if $a, b \in F$, then $a \land b \in F$ whenever $a \land b$ exists. If $X$ is a nonempty subset of $A$, the smallest filter containing $X$ is called the filter generated by $X$ and will be denoted by $F(X)$. A filter $G$ is said to be finitely generated if $G = F(X)$ for some finite nonempty subset $X$ of $A$. If $X = \{a\}$, then $F(\{a\}) = [a] = \{x \in A : a \leq x\}$, called the principal filter of $a$. We denote by $F_0(A)$ the set of all filters of $A$. A subset $I$ of $A$ is called an ideal if for every $a, b \in A$, if $a \leq b$ and $b \in I$, then $a \in I$ and for all $a, b \in I$, $a \lor b \in I$. We say that a nonempty proper ideal $P$ is prime if for every $a, b \in A$, $a \land b \in I$ implies $a \in I$ or $b \in I$ whenever $a \land b$ exists. We denote by $Id(A)$ and $X(A)$ the set of all ideals and prime ideals of $A$, respectively. Finally, we say that a nonempty ideal $I$ of $A$ is maximal if it is proper and for every $J \in Id(A)$, if $I \subseteq J$, then $J = I$ or $J = A$. We denote by $Idm(A)$ the set of all maximal ideals of $A$. Note that every maximal ideal is prime.

**Definition 1.** Let $A$ be a join-semilattice with greatest element. Then $A$ is a nearlattice if each principal filter is a bounded lattice with respect to the induced order.

Note that the operation meet is defined only in a corresponding principal filter. We indicate this fact by indices, i.e. $\land_a$ denotes the meet in $[a]$. Then the operation meet is not defined everywhere. However, the nearlattices can be regarded as total algebras through a ternary operation. This fact was first proved by Hickman in [16] and independently by Chajda and Kolařík in [11]. Araújo and Kinyon in [2] found a smaller equational base.

**Theorem 2** ([2]). Let $A$ be a nearlattice. Let $m: A^3 \to A$ be a ternary operation given by $m(x, y, z) = (x \lor z) \land (y \lor z)$. The following identities are satisfied:

1. $m(x, y, x) = x$,
2. $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$,
3. $m(x, x, 1) = 1$.

Conversely, let $A = \langle A, m, 1 \rangle$ be an algebra of type $(3, 0)$ satisfying the identities (1)–(3). If we define $x \lor y = m(x, y, x)$, then $A$ is a join-semilattice with greatest element. Moreover, for each $a \in A$, $[a]$ is a bounded lattice, where for every $x, y \in [a]$ their infimum is $x \land_a y = m(x, y, a)$. Hence, $A$ is a nearlattice.

**Definition 3.** Let $A$ be a nearlattice. Then $A$ is distributive if each principal filter is a bounded distributive lattice with respect to the induced order.
Example 4 ([1]). An implication algebra can be defined as a join-semilattice with greatest element such that each principal filter is a Boolean lattice. If $A = \langle A, \to, 1 \rangle$ is an implication algebra, then the join of two elements $x$ and $y$ is given by $x \lor y = (x \to y) \to y$ and for each $a \in A$, $[a] = \{ x \in A : a \leq x \}$ is a Boolean lattice, where for $x, y \in [a]$ the meet is given by $x \land_a y = (x \to (y \to a)) \to a$ and $x \to a$ is the complement of $x$ in $[a]$. Thus, every implication algebra is a distributive nearlattice.

From the results given in [14], we have the following characterization of the filter generated by a nonempty subset $X$ in a distributive nearlattice $A$:

$$F(X) = \{ a \in A : \exists x_1, \ldots, x_n \in X, \exists x_1 \land \ldots \land x_n, a = x_1 \land \ldots \land x_n \}.$$

In [3] it was shown that if $A$ is a distributive nearlattice, then the set of all filters $\text{Fi}(A) = \langle \text{Fi}(A), \lor, \land, \to, \{1\}, A \rangle$ is a Heyting algebra, where the least element is $\{1\}$, the greatest element is $A$, $G \lor H = F(G \cup H)$, $G \land H = G \cap H$ and $(\star) \quad G \to H = \{ a \in A : [a] \cap G \subseteq H \}$

for all $G, H \in \text{Fi}(A)$. So, the pseudocomplement of $F \in \text{Fi}(A)$ is $F^* = F \to \{1\}$.

Theorem 5 ([9]). Let $A$ be a distributive nearlattice. Let $I \in \text{Id}(A)$ and let $F \in \text{Fi}(A)$ such that $I \cap F = \emptyset$. Then there exists $P \in X(A)$ such that $I \subseteq P$ and $P \cap F = \emptyset$.

The following definition given in [3] is an alternative definition of relative annihilator in distributive nearlattices different from that given in [10].

Definition 6. Let $A$ be a join-semilattice with greatest element and $a, b \in A$. The annihilators of $a$ relative to $b$ is the set

$$a \circ b = \{ x \in A : b \leq x \lor a \}.$$

In particular, the relative annihilator $a^\top = a \circ 1 = \{ x \in A : x \lor a = 1 \}$ is called the annihilator of $a$.

It follows that a nearlattice $A$ is distributive if and only if $a \circ b \in \text{Fi}(A)$ for all $a, b \in A$. Also note that by $(\star)$, we have that $[a]^* = \{ x \in A : x \lor a = 1 \}$, i.e. $[a]^* = a^\top$, which is the dual notion of annulet given by Cornish in [13]. The following result will be useful.

Lemma 7 ([3]). Let $A$ be a distributive nearlattice. Let $a, b \in A$ and $I \in \text{Id}(A)$.

1. $I \cap a^\top = \emptyset$ if only if there exists $U \in \text{Idm}(A)$ such that $I \subseteq U$ and $a \in U$.
2. $U \in \text{Idm}(A)$ if only if for every $a \in A$, $a \notin U$ if only if $U \cap a^\top \neq \emptyset$.

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We are interested in a particular class of distributive near lattices which generalize the normal lattices given in [12].

**Definition 8.** Let $A$ be a distributive nearlattice. Then $A$ is *normal* if each prime ideal is contained in a unique maximal ideal.

**Theorem 9 ([3]).** Let $A$ be a distributive nearlattice. The following conditions are equivalent:

1. $A$ is normal,
2. $(a \lor b)^\top = a^\top \lor b^\top$ for all $a, b \in A$.

2. $\alpha$-filters

In this section we study the notion of $\alpha$-filter in the class of distributive nearlattices. First, we see some characteristics of annihilators. Let $A$ be a distributive nearlattice, $a \in A$ and we consider the set

$$a^{\top\top} = \{y \in A : \forall x \in a^\top, y \lor x = 1\}.$$

**Lemma 10.** Let $A$ be a distributive nearlattice. The following properties are satisfied for every $a, b \in A$:

1. $[a] \subseteq a^{\top\top}$.
2. $a^{\top\top\top} = a^\top$.
3. $a \leq b$ implies $a^\top \subseteq b^\top$.
4. $a^\top \subseteq b^\top$ if only if $b^{\top\top} \subseteq a^{\top\top}$.
5. $(a \land b)^\top = a^\top \land b^\top$ whenever $a \land b$ exists.
6. $(a \lor b)^{\top\top} = a^{\top\top} \land b^{\top\top}$.

**Proof.** We prove only the assertions (2), (4) and (6).

(2) Let $y \in a^{\top\top\top}$. Thus, for every $x \in a^{\top\top}$ we have $y \lor x = 1$. In particular, $a \in a^{\top\top}$ and $y \lor a = 1$. Therefore $y \in a^\top$. The reciprocal is similar.

(4) Suppose that $a^\top \subseteq b^\top$. Let $y \in b^{\top\top}$. If $x \in a^\top$, then $x \in b^\top$ and $y \lor x = 1$. So, $y \in a^{\top\top}$ and $b^{\top\top} \subseteq a^{\top\top}$. Conversely, suppose that $b^{\top\top} \subseteq a^{\top\top}$ and let $x \in a^\top$. Since $b \in b^{\top\top}$, $b \in a^{\top\top}$ and $b \lor x = 1$. Therefore $x \in b^\top$ and $a^\top \subseteq b^\top$.

(6) Since $a, b \leq a \lor b$, we have $(a \lor b)^{\top\top} \subseteq a^{\top\top} \lor b^{\top\top}$ and $(a \lor b)^{\top\top} \subseteq a^{\top\top} \land b^{\top\top}$. Let $y \in a^{\top\top} \land b^{\top\top}$ and suppose that $y \not\in (a \lor b)^{\top\top}$. Then there is $x \in (a \lor b)^\top$ such that $y \lor x < 1$ and by Theorem 5, there exists $P \in X(A)$ such that $y \lor x \in P$. So, $x, y \in P$. Since $y \in a^{\top\top} \land b^{\top\top}$, we have that for every $z \in a^\top$, $y \lor z = 1$ and for every $w \in b^\top$, $y \lor w = 1$. On the other hand, as $x \in (a \lor b)^\top$, it follows that
a ∨ b ∨ x = 1 and a ∨ x ∈ b^T. Consequently, y ∨ a ∨ x = 1. We have two cases: if
P ∩ a^T ≠ ∅, then there is t ∈ a^T such that t ∈ P. Thus, y ∨ t = 1 ∈ P, which is a
contradiction. If P ∩ a^T = ∅, then by Lemma 7 there exists U ∈ Idm(A) such that
P ⊆ U and a ∈ U. So, x, y, a ∈ U and y ∨ a ∨ x = 1 ∈ U, which is a contradiction.
Therefore, we conclude that (a ∨ b)^T = a^T ∩ b^T. □

If A is a distributive nearlattice, then an element a ∈ A is dense if a^T = {1}. We
denote by D(A) the set of all dense elements of A. By Lemma 10, it is easy to prove
that D(A) ∈ Id(A) and a^T ⊆ P for all a ∈ A. The following result gives an
equivalence of the implication algebras in terms of annihilators.

**Theorem 11.** Let A be a distributive nearlattice. The following conditions are
equivalent:

(1) A is an implication algebra,

(2) [a] ∨ a^T = A for all a ∈ A.

**Proof.** (1) ⇒ (2): Suppose that A is an implication algebra. By the results
developed in [1], we know that X(A) = Idm(A). Let a ∈ A. Obviously [a] ∨ a^T ⊆ A.
We prove the other inclusion. Let c ∈ A and suppose that c ∉ [a] ∨ a^T. So, by
Theorem 5 there exists P ∈ X(A) such that c ∈ P and P ∩ ([a] ∨ a^T) = ∅. Then
a ∉ P and P ∩ a^T = ∅. Thus, P is maximal and by Lemma 7 it follows that
P ∩ a^T ≠ ∅, which is a contradiction. Therefore [a] ∨ a^T = A.

(1) ⇒ (2): Let a ∈ A and b ∈ [a] such that b ≠ a and b ≠ 1. Let us prove that b
has a complement in [a]. We know that a ∈ [b] ∨ b^T = F([b] ∪ b^T). If only there is
x ∈ [b] such that a = x, then b ≤ x = a and b = a, which is a contradiction. On the
other hand, if only there is x ∈ b^T such that a = x, then x ∨ b = a ∨ b = 1. Since
a ≤ b, it follows that a ∨ b = b and b = 1, which is a contradiction. Thus, there exists
x ∈ [b] and there exists y ∈ b^T such that x ∨ y exists and a = x ∨ y. Then

\[ a = a ∨ b = (x ∧ y) ∧ b = (x ∧ b) ∧ y = b ∧ y, \]

i.e. a = b ∧ y. Moreover, y ∈ b^T and b ∨ y = 1. As y ∈ [a], then y is the complement
of b in [a] and A is an implication algebra. □

Let A be a normal distributive nearlattice and we consider the family

\[ R(A) = \{ a^T : a ∈ A \}. \]

Let \( \overline{m} : R(A)^3 \to R(A) \) be a map given by \( \overline{m}(a^T, b^T, c^T) = (a^T \land c^T) \land (b^T \land c^T) \).

By Theorems 9 and 2 and Lemma 10, it follows that the structure

\[ R(A) = (R(A), \overline{m}, A) \]

is a distributive nearlattice.
Corollary 12. Let $A$ be a normal distributive nearlattice. Then the relation $\theta^T$ on $A$ defined by

\[(a, b) \in \theta^T \text{ if only if } a^T = b^T\]

is a congruence on $A$.

Corollary 13. Let $A$ be a normal distributive nearlattice and $\theta^T$ be the congruence given by $(*)$. Then $R(A)$ is isomorphic to $A/\theta^T$.

Proof. Let $\varrho: A \rightarrow R(A)$ be the map defined by $\varrho(a) = a^T$. By Theorem 9 and Lemma 10 we have that $\varrho(m(a, b, c)) = \mathfrak{m}(\varrho(a), \varrho(b), \varrho(c))$, where the ternary operation $m(a, b, c)$ is given by Theorem 2. So, $\varrho$ is an homomorphism onto such that $\theta^T = \text{Ker}(\varrho)$. It follows by Isomorphism Theorem. □

Example 14. Let $A$ be the normal distributive nearlattice from Figure 1. Then $R(A) = \{1^T, a^T, b^T, c^T\}$. On the other hand, the congruence $\theta^T$ is given by the partition $\{1\}, \{b\}, \{a, d\}$ and $\{c, e\}$. Hence, $R(A)$ and $A/\theta^T$ are isomorphic.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

Definition 15. Let $A$ be a distributive nearlattice and $F \in \text{Fi}(A)$. We say that $F$ is an $\alpha$-filter if $a^{TT} \subseteq F$ for all $a \in F$.

We denote by $\text{Fi}_\alpha(A)$ the set of all $\alpha$-filters of $A$.

Example 16. If $A$ is a normal distributive nearlattice, then $\text{Ker}(\theta^T)$ is an $\alpha$-filter.

Example 17. If $A$ is a distributive nearlattice, then $a^T$ is an $\alpha$-filter for all $a \in A$. Let $x \in a^T$. We prove that $x^{TT} \subseteq a^T$. If $y \in x^{TT}$, then $x^T \subseteq y^T$ and since $a^T$ is a filter, we have $x \lor y \in a^T$ and $x \lor y \lor a = 1$, i.e. $y \lor a \in x^T$. So, $y \lor a \in y^T$ and $y \lor a = 1$. It follows that $y \in a^T$ and $a^T$ is an $\alpha$-filter.
Remark 18. Not every filter is an $\alpha$-filter. In Example 14, we consider the filter $F = \{1, a, b\}$. Thus, $a^{\top+} = \{1, a, d\}$ and $a^{\top+} \not\subseteq F$.

**Theorem 19.** Let $A$ be a distributive nearlattice and $F \in \text{Fi}(A)$. The following conditions are equivalent:

(1) $F$ is an $\alpha$-filter.
(2) If $a^{\top} = b^{\top}$ and $a \in F$, then $b \in F$ for all $a, b \in A$.
(3) $F = \bigcup \{a^{\top+}: a \in F\}$.

**Proof.** (1) $\Rightarrow$ (2): Let $a, b \in A$ such that $a^{\top} = b^{\top}$ and $a \in F$. Then $a^{\top+} = b^{\top+}$ and since $F$ is an $\alpha$-filter, $a^{\top+} \subseteq F$. Then by Lemma 10, $b^{\top} \subseteq b^{\top+} \subseteq F$, i.e. $b \in F$.

(2) $\Rightarrow$ (3): Since $a \in a^{\top+}$ for all $a \in A$, we have $F \subseteq \bigcup \{a^{\top+}: a \in F\}$. We see the other inclusion. If $x \in \bigcup \{a^{\top+}: a \in F\}$, then there is $b \in F$ such that $x \in b^{\top+}$. So, $b^{\top} \subseteq x^{\top}$ and $x^{\top} \subseteq b^{\top+}$. Then by Lemma 10, $x^{\top+} = x^{\top+} \cap b^{\top+} = (x \vee b)^{\top+}$ and $x^{\top} = (x \vee b)^{\top}$. As $x \vee b \in F$, by hypothesis we have $x \in F$.

(3) $\Rightarrow$ (1): Let $b \in F$. If $x \in b^{\top+}$, then $x \in \bigcup \{a^{\top+}: a \in F\}$ and $x \in F$. Therefore $b^{\top+} \subseteq F$ and $F$ is an $\alpha$-filter. $\square$

**Theorem 20.** Let $A$ be a normal distributive nearlattice and $F \in \text{Fi}(A)$. Then

$$\alpha(F) = \{x \in A: \exists a \in F, a^{\top} \subseteq x^{\top}\}$$

is the smallest $\alpha$-filter containing $F$.

**Proof.** It is clear that $F \subseteq \alpha(F)$. Let $x, y \in A$ such that $x \leq y$ and $x \in \alpha(F)$. Then by Lemma 10, $x^{\top} \subseteq y^{\top}$ and there exists $a \in F$ such that $a^{\top} \subseteq x^{\top}$. So, $a^{\top} \subseteq y^{\top}$ and $y \in \alpha(F)$. Let $x, y \in \alpha(F)$ and suppose that $x \wedge y$ exists. Then there exist $a, b \in F$ such that $a^{\top} \subseteq x^{\top}$ and $b^{\top} \subseteq y^{\top}$. Since $F$ is a filter, $m(a, b, x \wedge y) \in F$, where the ternary operation $m(a, b, x \wedge y)$ is given by Theorem 2. On the other hand, $a^{\top} \vee (x \wedge y)^{\top} \subseteq x^{\top}$ and $b^{\top} \vee (x \wedge y)^{\top} \subseteq y^{\top}$. As $A$ is normal,

$$m(a, b, x \wedge y)^{\top} = m(a^{\top}, b^{\top}, (x \wedge y)^{\top}) \subseteq x^{\top} \wedge y^{\top} = (x \wedge y)^{\top}.$$ 

Thus, $m(a, b, x \wedge y)^{\top} \subseteq (x \wedge y)^{\top}$ and $x \wedge y \in \alpha(F)$. Then $\alpha(F)$ is a filter. Let $x \in \alpha(F)$. We see that $x^{\top+} \subseteq \alpha(F)$. If $y \in x^{\top+}$, then $x^{\top} \subseteq y^{\top}$. Since $x \in \alpha(F)$, there exists $a \in F$ such that $a^{\top} \subseteq x^{\top}$. So, $a^{\top} \subseteq y^{\top}$ and $y \in \alpha(F)$. Then $x^{\top+} \subseteq \alpha(F)$ and $\alpha(F)$ is an $\alpha$-filter. Let $H \in \text{Fi}_a(A)$ such that $F \subseteq H$. If $x \in \alpha(F)$, then there exists $a \in F$ such that $a^{\top} \subseteq x^{\top}$, i.e. $x^{\top+} \subseteq a^{\top+}$. As $a \in H$ and $H$ is an $\alpha$-filter, we have $a^{\top+} \subseteq H$. Consequently, $x \in H$ and $\alpha(F) \subseteq H$. $\square$
Remark 21. Let $A$ be a normal distributive nearlattice.

(1) Note that the map $\alpha : \text{Fi}(A) \to \text{Fi}(A)$ of Theorem 20 is a closure operator and the $\alpha$-filters are closed elements with respect to $\alpha$.

(2) A proper $\alpha$-filter contains non-dense elements. Indeed, if $F$ is a proper $\alpha$-filter and $x \in F \cap D(A)$, then $F = \alpha(F)$ and $x^\top = \{1\}$. Thus, there exists $a \in F$ such that $a^\top \subseteq x^\top$. So, $a^\top = \{1\}$ and $a^{\top\top} = A$. On the other hand, since $F$ is an $\alpha$-filter, $a^{\top\top} \subseteq F$, i.e. $A = F$ which is a contradiction.

Now, we define the operations of infimum $\sqcap$, supremum $\sqcup$, and implication $\Rightarrow$ in $\text{Fi}_\alpha(A)$ as:

$$F \sqcap G = F \cap G, \quad F \sqcup G = \alpha(F \lor G), \quad F \Rightarrow G = \alpha(F \to G)$$

for each pair $F, G \in \text{Fi}_\alpha(A)$. By Theorem 20, we have that $F \sqcap G, F \sqcup G, F \Rightarrow G \in \text{Fi}_\alpha(A)$ for all $F, G \in \text{Fi}_\alpha(A)$. Consider the structure

$$\text{Fi}_\alpha(A) = \langle \text{Fi}_\alpha(A), \sqcup, \sqcap, \Rightarrow, \{1\}, A \rangle.$$ 

Theorem 22. Let $A$ be a normal distributive nearlattice. Then $\text{Fi}_\alpha(A)$ is a Heyting algebra.

Proof. It is easy to verify that $\langle \text{Fi}_\alpha(A), \sqcup, \sqcap, \{1\}, A \rangle$ is a bounded lattice. Let $F, H, K \in \text{Fi}_\alpha(A)$. Suppose that $F \sqcap H \subseteq K$. If $x \in F$, then $[x] \cap H \subseteq F \sqcap H \subseteq K$. Thus, $[x] \cap H \subseteq K$ and $x \in H \Rightarrow K$. Hence, $x \in H \Rightarrow K$ and $F \subseteq H \Rightarrow K$.

Reciprocally, we assume that $F \subseteq H \Rightarrow K$. Let $x \in F \sqcap H$. So, $x \in F \subseteq H \Rightarrow K$ and there exists $a \in H \Rightarrow K$ such that $a^\top \subseteq x^\top$. It follows that $x \lor a \in [a] \cap H \subseteq K$ and $x^\top = x^\top \lor a^\top = (x \lor a)^\top$, i.e. $x^\top = (x \lor a)^\top$ and $x \lor a \in K$. By Theorem 19, we have $x \in K$. Therefore, $F \sqcap H \subseteq K$ and $\text{Fi}_\alpha(A)$ is a Heyting algebra.

Let $A$ be a nearlattice. Following the results developed in [15], we introduce the next notation. For each natural number $n$ we define inductively for every $a_1, \ldots, a_n, b \in A$, the element $m^{n-1}(a_1, \ldots, a_n, b)$ as follows:

1. $m^0(a_1, b) = m(a_1, a_1, b)$,
2. for $n > 1$, $m^{n-1}(a_1, \ldots, a_n, b) = m(m^{n-2}(a_1, \ldots, a_{n-1}, b), a_n, b)$.

Then $m^{n-1}(a_1, \ldots, a_n, b) = (a_1 \lor b) \land_b \ldots \land_b (a_n \lor b)$ and in particular, $m^0(a_1, b) = a_1 \lor b$ and $m^1(a_1, a_2, b) = m(a_1, a_2, b)$, where the operation $m(a_1, a_2, b)$ is given by Theorem 2. We are able to formulate our main result.

Theorem 23. Let $A$ be a normal distributive nearlattice. Then $\text{Fi}_\alpha(A)$ is isomorphic to the Heyting algebra $\text{Fi}(\text{R}(A))$. 248
Proof. We consider the map \( \psi : \text{Fi}_\alpha(A) \to \text{Fi}(R(A)) \) defined by \[
\psi(F) = \{a^\top : a \in F \}.
\]

We prove that \( \psi \) is well-defined. Let \( F \in \text{Fi}_\alpha(A) \). It is clear that \( 1^\top \in \psi(F) \). Let \( a^\top, b^\top \in R(A) \) such that \( a^\top \subseteq b^\top \) and \( a^\top \in \psi(F) \). Then \( b^\top \cap a^\top \subseteq a^\top \) and \( a^\top \in \psi(F) \). Thus, \( b^\top \cap a^\top \subseteq a^\top \) and if \( F \) is an \( \alpha \)-filter, \( a^\top \subseteq F \). So, \( b^\top \cap a^\top \subseteq F \). Let \( a^\top, b^\top \in \psi(F) \) and suppose that \( a^\top \cap b^\top \) exists in \( R(A) \), i.e. there is \( c \in A \) such that \( a^\top \cap b^\top = c^\top \). Then \( a, b \in F \) and as \( F \) is a filter, \( m(a, b, c) \in F \). It follows that

\[
m(a, b, c)^\top = \overline{m(a^\top, b^\top, c^\top)} = (a^\top \cap b^\top) \cap c^\top = c^\top
\]

and \( c^\top \in \psi(F) \). Thus, \( a^\top \cap b^\top \subseteq \psi(F) \) and \( \psi(F) \in \text{Fi}(R(A)) \).

Let \( F, H \in \text{Fi}_\alpha(A) \). It is immediate that \( \psi(F \cap H) = \psi(F) \cap \psi(H) \). We see that \( \psi(F \cup H) = \psi(F) \cup \psi(H) \). Let \( x^\top \in \psi(F \cup H) \). Then \( x \in \alpha(F \cup H) \) and there exists \( a \in F \cup H \) such that \( a^\top \subseteq x^\top \). So, there exist \( x_1, \ldots, x_n \in F \cup H \) such that \( x_1 \land \ldots \land x_n \exists \) and \( a = x_1 \land \ldots \land x_n \). Then \( x_1^\top, \ldots, x_n^\top \in \psi(F) \cap \psi(H) \). On the other hand, \( a^\top = (x_1 \land \ldots \land x_n)^\top = x_1^\top \cap \ldots \cap x_n^\top \) and \( a^\top \in \psi(F) \land \psi(H) \). Since \( \psi(F) \land \psi(H) \) is a filter, we have \( x^\top \in \psi(F) \land \psi(H) \) and \( \psi(F \cup H) \subseteq \psi(F) \land \psi(H) \). Conversely, if \( x^\top \in \psi(F) \land \psi(H) \), then there exist \( x_1^\top, \ldots, x_n^\top \in \psi(F) \cup \psi(H) \) such that \( x_1^\top \cap \ldots \cap x_n^\top \) exists and \( x^\top = x_1^\top \cap \ldots \cap x_n^\top \). It follows that \( x_1, \ldots, x_n \in F \cup H \) and \( m^{-1}(x_1, \ldots, x_n, x) \in F \cup H \). So,

\[
m^{-1}(x_1, \ldots, x_n, x)^\top = \overline{m^{-1}(x_1^\top, \ldots, x_n^\top, x)} = (x_1^\top \cap \ldots \cap x_n^\top) \cap x^\top = x^\top
\]

and \( m^{-1}(x_1, \ldots, x_n, x)^\top \subseteq x^\top \). Thus, \( x \in \alpha(F \cup H) = F \cup H \), i.e. \( x^\top \in \psi(F \cup H) \) and \( \psi(F) \lor \psi(H) \subseteq \psi(F \cup H) \). Therefore, \( \psi(F \cup H) = \psi(F) \lor \psi(H) \).

Now, we prove that \( \psi(F \Rightarrow H) = \psi(F) \Rightarrow \psi(H) \). Let \( x^\top \in \psi(F \Rightarrow H) \). Then \( x \in \alpha(F \Rightarrow H) = F \Rightarrow H \), i.e. \( x^\top \subseteq \psi(F \Rightarrow H) \) and \( \psi(F) \lor \psi(H) \subseteq \psi(F \Rightarrow H) \). Therefore, \( \psi(F \Rightarrow H) = \psi(F) \lor \psi(H) \).

We have \( x \subseteq \psi(F) \Rightarrow \psi(H) \). Let \( y \in F \Rightarrow H \). Then \( y^\top \in \psi(F) \Rightarrow \psi(H) \) and \( y \in F \Rightarrow H \). So, \( a \lor y \in \alpha(F \Rightarrow H) = F \Rightarrow H \), i.e. \( a \lor y \cap \psi(F) \subseteq \psi(H) \). If \( y^\top \in [x^\top] \cap \psi(F) \), then \( x^\top \subseteq y^\top \) and \( y \in F \Rightarrow H \). Thus, \( a \lor y \in \alpha(F \Rightarrow H) = F \Rightarrow H \), i.e. \( a \lor y \cap \psi(F) \subseteq \psi(H) \). Therefore, \( \psi(F \Rightarrow H) = \psi(F) \Rightarrow \psi(H) \).

Let \( \pi : \text{Fi}(R(A)) \to \text{Fi}_\alpha(A) \) be the map given by \( \pi(G) = \{a : a^\top \subseteq G\} \). By Lemma 10, it follows that \( \pi(G) \in \text{Fi}_\alpha(A) \). So, \( \psi \) and \( \pi \) are the inverses of each other and \( \psi \) is 1-1 and onto. Therefore \( \psi \) is an isomorphism. \( \square \)
References


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