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SUFFICIENT CONDITIONS FOR A T-PARTIAL ORDER OBTAINED FROM TRIANGULAR NORMS TO BE A LATTICE

Lifeng Li, Jianke Zhang and Chang Zhou

For a t-norm T on a bounded lattice \((L, \leq)\), a partial order \(\leq_T\) was recently defined and studied. In [11], it was pointed out that the binary relation \(\leq_T\) is a partial order on \(L\), but \((L, \leq_T)\) may not be a lattice in general. In this paper, several sufficient conditions under which \((L, \leq_T)\) is a lattice are given, as an answer to an open problem posed by the authors of [11]. Furthermore, some examples of t-norms on \(L\) such that \((L, \leq_T)\) is a lattice are presented.

Keywords: bounded lattice, triangular norm, T-partial order
Classification: 03E72, 03B52

1. INTRODUCTION

In [7], a partial order generated by a commutative semigroup was introduced by Clifford. There have been numerous attempts to extend this ordering to other semigroups (such as [10, 15]). Especially Mitsch [17] succeeded introducing a natural partial order of semigroups. This order was extended to t-norms on a bounded lattice \((L, \leq, 0, 1)\) by Karaçal and Kesicioğlu in [11], and named T-order. Let \((L, \leq, 0, 1)\) be a bounded lattice, the T-order is defined as follows:

\[
x \leq_T y :\iff T(l, y) = x \text{ for some } l \in L
\]

(1)

for any elements \(x, y\) of \(L\) and \(T\) is a t-norm on \(L\). In addition, in [11] it was given the relationship between T-order and partial order “\(\leq\)” of \(L\):

\[
\text{If } x \leq_T y \text{ then } x \leq y
\]

(2)

T-order is a pretty interesting partial order, in resent years, many scholars focus on T-order and other partial orders on \((L, \leq, 0, 1)\). An equivalence relation on the class of t-norms on a bounded lattice was introduced by Kesicioğlu, Karaçal and Mesiar (see [13]) based on T-partial orders, and they also characterized the equivalence classes linked to some special t-norms. Later on, in [3, 9], V and U-partial orders, respectively induced by nullnorms and uninorms were introduced and some basic properties of them

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were investigated. In [16], it was introduced an equivalence relations induced by the U-partial order.

And the authors in [11] pointed out that the binary relation $\leq_T$ is a partial order on $L$, but $(L, \leq_T)$ may not be a lattice in general. One of the open problems posed in the study [11] is: let $(L, \leq, 0, 1)$ be a bounded lattice and $T$ be a t-norm on $L$, give examples that $(L, \leq_T)$ is a lattice, where $T \neq T_W$.

If $L = [0, 1]$, in [13] the following example was given to answer this open problem:

**Example 1.1.** (Kesicioğlu et al. [13]) Consider the function $T^\circ : [0, 1]^2 \to [0, 1]$ defined by

$$T^\circ(x, y) = \begin{cases} 
0, & (x, y) \in (0, k)^2; \\
\min(x, y), & \text{otherwise},
\end{cases} \quad 0 < k < 1. \quad (3)$$

Then $([0, 1], \leq_{T^\circ})$ is a lattice.

In [2], the following theorem was given to answer this open problem:

**Theorem 1.2.** (Aşıcı and Karaçal [2]) Let $T$ be a t-norm on $[0, 1]$ and the family $(T_\lambda)_{\lambda \in (0, 1)}$ be given by

$$T_\lambda(x, y) = \begin{cases} 
0, & T(x, y) \leq \lambda \text{ and } x, y \neq 1; \\
T(x, y), & \text{otherwise}.
\end{cases} \quad (4)$$

Then

(i) $(T_\lambda)_{\lambda \in (0, 1)}$ is a t-norm.

(ii) If $T$ is divisible on $[0, 1]$, then $(L, \leq_{T_\lambda})$ is complete lattice.

Followed by [2] and [13], in the present paper, we continue to answer this open problem on the condition of $L$ is a complete lattice. The rest of this paper is organized as follows. It reviews fundamental notions and properties of t-norm in Sect.2. In Sect.3, some kinds of t-norms such that $(L, \leq_T)$ is a lattice are given. The paper is concluded with a brief summary and an outlook for further research in Sect.4.

### 2. PRELIMINARIES

**Definition 2.1.** (Birkhoff [1], Drygaś [8]) A bounded lattice $(L, \leq, 0, 1)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

**Definition 2.2.** (Birkhoff [1]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of $L$ is defined as

$$[a, b] = \{x \in L, a \leq x \leq b\}$$

Similarly, $[a, b) = \{x \in L, a \leq x < b\}$, $(a, b] = \{x \in L, a < x \leq b\}$ and $(a, b) = \{x \in L, a < x < b\}$. 
Definition 2.3. (Çaylı et al. [5], Karaçal [12]) Let \((L, \leq, 0, 1)\) be a bounded lattice. An operation \(T : L^2 \to L\) is called a triangular norm (t-norm) if it is commutative, associative, increasing with respect to both variables and it satisfies
\[ T(x, 1) = x, \forall x \in L. \]

Definition 2.4. (Casasnovas and Mayor [4]) A t-norm \(T\) on \(L\) is divisible if the following condition holds: \(\forall x, y \in L\) with \(x \leq y\) there is a \(z \in L\) such that \(x = T(y, z)\).

Example 2.5. The following are two basic t-norms \(T_M\) and \(T_W\) which are the strongest and the weakest t-norms, respectively, on a bounded lattice \(L\)
\[ T_M(x, y) = x \land y, \]
\[ T_W(x, y) = \begin{cases} x \land y, & x, y \in \{1\}; \\ 0, & \text{otherwise}. \end{cases} \]

If \(L = [0, 1]\), the following are the four basic t-norms \(T_M, T_P, T_L, T_D\) given by, respectively:
\[ T_M(x, y) = \min\{x, y\}, \]
\[ T_P(x, y) = xy, \]
\[ T_L(x, y) = \max\{x + y - 1, 0\}, \]
\[ T_D(x, y) = \begin{cases} 0, & (x, y) \in [0, 1)^2; \\ \min\{x, y\}, & \text{otherwise}. \end{cases} \]

Theorem 2.6. (Casasnovas and Mayor [4]) Let \((L, \leq, 0, 1)\) be a bounded lattice, \(T\) be a t-norm on \(L\). Then the binary relation \(\leq_T\) is a partial order on \(L\).

3. SOME KINDS OF T-NORMS SUCH THAT \((L, \leq_T)\) IS A LATTICE

Let \((L, \leq, 0, 1)\) be a bounded lattice. Consider a t-norm \(T\) on \(L\). For \(X \subseteq L\), we denote the set of the upper bounds of \(X\) and lower bounds of \(X\) with respect to “\(\leq_T\)” on \(L\) by \(\overline{X}_T\) and \(\underline{X}_T\) respectively. We generalize Theorem 1.1 from the unit interval \([0, 1]\) to an arbitrary complete lattice.

Theorem 3.1. Let \(T\) be a t-norm on a complete lattice \(L\) and the family \((T_a)_{a \in L}\) be given by
\[ T_a(x, y) = \begin{cases} 0, & T(x, y) \leq a \text{ and } 1 \notin \{x, y\}; \\ T(x, y), & \text{otherwise}. \end{cases} \]

Then the following statements hold:
(i) \((T_a)_{a \in L}\) is a t-norm.
(ii) If \(T\) is divisible on \(L\), then \((L, \leq_{T_a})\) is complete lattice.
Proof. (i) (a) Since $T$ is a t-norm on $L$, then $T$ is commutative. It leads to $T_a$ is commutative.

(b) We will show that $T_a$ is associative. For any $x, y, z \in L$, if one of $x, y, z$ is 1, then $T_a(T_a(x, y), z) = T_a(x, T_a(y, z))$. For any $x, y, z \in L \setminus \{1\}$.

(b1) Let $T(x, y) \leq a$, then $T_a(T_a(x, y), z) = T_a(0, z) = 0$ and $T(x, T(y, z)) = T(T(x, y), z) \leq T(a, z) \leq T(a, 1) = a$. Therefore

$$T_a(x, T_a(y, z)) = \begin{cases} T_a(x, 0), & T(y, z) \leq a; \\ T_a(x, T(y, z)), & \text{otherwise.} \end{cases} = 0.$$  

Then $T_a(T_a(x, y), z) = T_a(x, T_a(y, z))$.

(b2) Let $T(x, y) \not\leq a$. If $T(T(x, y), z) \leq a$, then $T_a(T_a(x, y), z) = T_a(T(x, y), z) = 0$ and $T(x, T(y, z)) = T(T(x, y), z) \leq a$. Therefore

$$T_a(x, T_a(y, z)) = \begin{cases} T_a(x, 0), & T(y, z) \leq a; \\ T_a(x, T(y, z)), & \text{otherwise.} \end{cases} = 0.$$  

If $T(T(x, y), z) \not\leq a$, then $T_a(T_a(x, y), z) = T_a(T(x, y), z) = T(T(x, y), z) \leq T(1, T(y, z)) = T(y, z)$. Therefore $T_a(x, T_a(y, z)) = T_a(x, T(y, z))$.

Then $T_a(T_a(x, y), z) = T_a(x, T_a(y, z))$.

(c) We show that $T_a$ satisfies the monotonicity. Let $x_1 \leq x_2$, if $T(x_1, y) \leq a$ and $x_1 \neq 1, y \neq 1$ (The case $x_1 = 1$ or $y = 1$ is trivial), then $0 = T_a(x_1, y) \leq T_a(x_2, y)$. If $T(x_1, y) \not\leq a$, from $T(x_1, y) \leq T(x_2, y)$, we have $T(x_2, y) \not\leq a$. Thus $T_a(x_1, y) = T(x_1, y) \leq T(x_2, y) = T_a(x_2, y)$.

(d) Since $T_a(x, 1) = T(x, 1) = x$ for all $x \in L$, we have that 1 is neutral element. Thus, $T_a$ is a t-norm on $L$.

(ii) Since $0 = T_a(0, x)$ and $x = T_a(x, 1)$, then $0 \leq T_a x \leq T_a 1$ for any $x \in L$. Thus,

$$\bigvee_{T_a} \{x_\tau \mid \tau \in \Phi\} = \bigvee_{T_a} \{x_\tau \mid \tau \in \Phi, x_\tau \neq 0\}$$

$$\bigwedge_{T_a} \{x_\tau \mid \tau \in \Phi\} = \bigwedge_{T_a} \{x_\tau \mid \tau \in \Phi, x_\tau \neq 1\}.$$  

Let $T$ be a divisible t-norm on complete lattice $L$ and $\{x_\tau \mid \tau \in \Phi\} \subseteq L \setminus \{0, 1\}$ be arbitrary.

(a) We will show existence of $\bigvee_{T_a} \{x_\tau \mid \tau \in \Phi\}$.

(a1) Suppose that there exists $x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\}$ such that $x_{\tau_0} \leq a$. Since $T_a(x_{\tau_0}, 1) = T(x_{\tau_0}, 1) = x_{\tau_0}$, then $\{x_{\tau_0}, 1\} \subseteq \{x_{\tau_0}\}_{T_a}$. Suppose $k \in \{x_{\tau_0}\}_{T_a}$, then there exists $z \in L$ such that $x_{\tau_0} = T_a(z, k)$. Because of $0 \neq x_{\tau_0} \leq a$, it leads to $0 \neq T_a(z, k) = T(z, k) \leq a$. From the definition of $T_a$, we have $z = 1$ or $k = 1$. If $z = 1$, $x_{\tau_0} = T_a(z, k) = T(1, k) = k$.  

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Thus $\{x_{\tau_0}\}_{T_a} = \{x_{\tau_0}, 1\}$. Therefore $\lor_{T_a}\{x_{\tau} \mid \tau \in \Phi\}$ exists and $\lor_{T_a}\{x_{\tau} \mid \tau \in \Phi\} = 1$.

(a2) Suppose $x_{\tau} \not\leq a$ for all $\tau \in \Phi$. Let $k = \lor\{x_{\tau} \mid \tau \in \Phi\}$, then $x_{\tau} \leq k$. Since $T$ is a divisible t-norm, then there exist $z_{\tau} \in L$ such that $x_{\tau} = T(z_{\tau}, k)$. Because of $x_{\tau} \not\leq a$ and from the definition of $T_a$, we have $x_{\tau} = T(z_{\tau}, k) = x_{\tau_0}(z_{\tau}, k)$, therefore $x_{\tau} \leq T_a k$, i.e. $k \in \{x_{\tau} \mid \tau \in \Phi\}_{T_a}$. Suppose $s \in \{x_{\tau} \mid \tau \in \Phi\}_{T_a}$ and $s \neq k$, then $x_{\tau} \leq T_a s$, it leads to $x_{\tau} \leq s$. Therefore $k = \lor\{x_{\tau} \mid \tau \in \Phi\} \leq s$. Because of $x_{\tau} \not\leq a$ for all $\tau \in \Phi$ and $T$ is a divisible t-norm on complete lattice $L$, it leads to $k \not\leq a$ and there exists $z \in L$ such that $k = T(z, s)$, therefore $k \leq T_a s$. Thus $\lor_{T_a}\{x_{\tau} \mid \tau \in \Phi\} = \lor\{x_{\tau} \mid \tau \in \Phi\}$.

(b) We will show existence of $\land_{T_a}\{x_{\tau} \mid \tau \in \Phi\}$.

(b1) Suppose that there exists $x_{\tau_0} \in \{x_{\tau} \mid \tau \in \Phi\}$ such that $x_{\tau_0} \leq a$. Since $T_a(x_{\tau_0}, 0) = 0$ and $T_a(x_{\tau_0}, 1) = T(x_{\tau_0}, 1) = x_{\tau_0}$, then $\{0, x_{\tau_0}\} \subseteq \{x_{\tau_0}\}_{T_a}$. Suppose $k \in \{x_{\tau_0}\}_{T_a}$, then there exists $z \in L$ such that $k = T_a(z, x_{\tau_0})$. If $k \neq 0$, from the definition of $T_a$ we have $T_a(z, x_{\tau_0}) = T(z, x_{\tau_0})$. Combing with $T(z, x_{\tau_0}) \leq T(1, x_{\tau_0}) = x_{\tau_0} \leq a$, we have $z = 1$. Therefore $k = T_a(z, x_{\tau_0}) = T(1, x_{\tau_0}) = x_{\tau_0} = 0$. Thus $\lor_{T_a}\{x_{\tau_0}\}_{T_a} = \{0, x_{\tau_0}\}$. Therefore $\land_{T_a}\{x_{\tau} \mid \tau \in \Phi\}$ exists and $\land_{T_a}\{x_{\tau} \mid \tau \in \Phi\} = 0$.

(b2) Suppose $x_{\tau} \not\leq a$ for all $\tau \in \Phi$. Let $k = \land\{x_{\tau} \mid \tau \in \Phi\}$, then $k \not\leq a$, i.e. $k \not\leq a$ or $k = a$.

(b21) In the case of $k \not\leq a$. Since $k = \land\{x_{\tau} \mid \tau \in \Phi\}$, then $k \leq x_{\tau}$, therefore $k = T(l_{\tau}, x_{\tau})$ for some $l_{\tau} \in L$. Because of $k \not\leq a$, then $k \not= 0$. Therefore $k = T(l_{\tau}, x_{\tau}) = T_a(l_{\tau}, x_{\tau})$ for some $l_{\tau} \in L$, i.e. $k \leq T_a x_{\tau}$. Thus $k \in \{x_{\tau} \mid \tau \in \Phi\}_{T_a}$. Suppose $0 \neq s \in \{x_{\tau} \mid \tau \in \Phi\}_{T_a}$, then $s \leq T_a x_{\tau}$, therefore $s \leq x_{\tau}$, i.e. $s \leq \land\{x_{\tau} \mid \tau \in \Phi\} = k$.

Thus, $s = T(l, k) = T_a(l, k)$ for some $l \in L$. Therefore $s \leq T_a k$. That is to say $\land_{T_a}\{x_{\tau} \mid \tau \in \Phi\} = \land\{x_{\tau} \mid \tau \in \Phi\}$.

(b22) In the case of $k = a$. Obviously $0 \in \{x_{\tau} \mid \tau \in \Phi\}_{T_a}$. Suppose $0 \neq s \in \{x_{\tau} \mid \tau \in \Phi\}_{T_a}$, then $s \leq T_a x_{\tau}$. Thus $s \leq x_{\tau}$, i.e. $s \leq \land\{x_{\tau} \mid \tau \in \Phi\} = a$ on the one hand. On the other hand, $0 \neq s \leq x_{\tau}$ implies $s = T(l, x_{\tau}) = T_a(l, x_{\tau})$ for some $l \in L$. From the definition of $T_a$, we have $s = T(l, x_{\tau}) \not\leq a$, which is a contradiction. Thus $0 = \land_{T_a}\{x_{\tau} \mid \tau \in \Phi\}$. 

Example 3.2. Consider the lattice $(L = \{0, a, b, c, d, 1\}, \leq, 0, 1)$ given in Figure 1, and the function $T_b$ on $L$ defined by

$$T_b(x, y) = \begin{cases} 0, & x \land y \leq b \text{ and } 1 \not\in \{x, y\}; \\ x \land y, & \text{otherwise}. \end{cases}$$

then $T_b$ is a t-norm and $T_b$ can also been described in Table 1. The order $\leq_{T_b}$ on $L$ is given in Figure 2.
Suppose $H = (0, k) \subseteq [0, 1)$, Let $*: H^2 \to H$ be an operation on $H$ which is commutative, associative, increasing with respect to both variables and

$$x * y \leq \min\{x, y\}$$

the function $T: [0, 1]^2 \to [0, 1]$ is defined by

$$T(x, y) = \begin{cases} x * y, & (x, y) \in H^2; \\ \min\{x, y\}, & \text{otherwise.} \end{cases}.$$ (6)

Then, $T$ is a t-norm (Proposition 3.60 in [14]).

If $x * y = 0$, then $T(x, y) = T^\circ(x, y)$ (Example 1.1 of this paper or Example 7 in [13]), and authors in [13] proved that $([0, 1], \leq_{T^\circ})$ is a lattice, and

$$x \lor_{T^\circ} y = \begin{cases} k, & (x, y) \in H^2; \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

$$x \land_{T^\circ} y = \begin{cases} 0, & (x, y) \in H^2; \\ \min\{x, y\}, & \text{otherwise.} \end{cases}.$$
Example 3.3. Let $H = (0, k) \subseteq [0, 1)$, consider the t-norm on $[0, 1]$ defined as follows:

$$T(x, y) = \begin{cases} 
\max\{x + y - k, 0\}, & (x, y) \in H^2; \\
\min\{x, y\}, & \text{otherwise};
\end{cases}$$

then $([0, 1], \leq_T)$ is lattice, and for all $x, y \in [0, 1]$,

$$x \vee_T y = \max\{x, y\},$$
$$x \wedge_T y = \min\{x, y\}.$$  

Suppose $x, y \in (0, 1)$ and $x < y$.

(a) We will show existence of $x \vee_T y$.

(a1) If $y \geq k$. Since $x = \min\{x, y\} = T(x, y)$, then $x \leq_T y$, i.e. $x \vee_T y = y$.

(a2) If $y < k$. Let $z = x + k - y < k$, then $x = \max\{y + z - k, 0\} = T(z, y)$. Therefore, $x \leq_T y$, i.e. $x \vee_T y = y$.

(b) We shall show existence of $x \wedge_T y$.

(b1) If $y \geq k$. Since $x = \min\{x, y\} = T(x, y)$, then $x \leq_T y$, i.e. $x \wedge_T y = x$.

(b2) If $y < k$. Let $z = x + k - y < k$, then $x = \max\{y + z - k, 0\} = T(z, y)$. Therefore, $x \leq_T y$, i.e. $x \wedge_T y = x$.

In general, $([0, 1], \leq_T)$ is not a lattice, which illustrated by the following example.

Example 3.4. Let $H = (0, \frac{1}{2})$, consider the t-norm on $[0, 1]$ defined as follows:

$$T(x, y) = \begin{cases} 
xy, & (x, y) \in H^2, \\
\min\{x, y\}, & \text{otherwise}.
\end{cases}$$

For any $x \in (0, 1)$, since $x = \min\{x, 1\} = T(x, 1)$, then $x \in \overline{\{x\}}_T$ and $x \in \overline{\{x\}}_T$.

(a) Let $y \in \overline{\{x\}}_T$, i.e. $x \leq_T y$, then $x \leq y$, thus $\overline{\{x\}}_T \subseteq [x, 1]$.

(a1) Suppose $x \geq \frac{1}{2}$. $x \leq y$ implies $x = \min\{x, y\} = T(x, y)$ by definition of $T$, then $x \leq_T y$, therefore, $[x, 1] \subseteq \overline{\{x\}}_T$, thus $\overline{\{x\}}_T = [x, 1]$.

(a2) Suppose $\frac{1}{4} \leq x < \frac{1}{2}$. If $y \geq \frac{1}{2} > x$, then $x = \min\{x, y\} = T(x, y)$, we have $x \leq_T y$, therefore $[\frac{1}{2}, 1] \subseteq \overline{\{x\}}_T$. If $x < y < \frac{1}{2}$ and $y \in \overline{\{x\}}_T$, then there exists $0 \leq l < \frac{1}{2}$ such that $x = T(l, y) = ly$. $(l, y) \in H^2$ implies $ly < \frac{1}{4}$, it is contradict with $\frac{1}{4} \leq x = T(l, y) = ly < \frac{1}{2}$. Therefore, $\overline{\{x\}}_T = \{x\} \cup [\frac{1}{2}, 1]$.

(a3) Suppose $x < \frac{1}{4}$. If $y \geq \frac{1}{2}$, then $x = \min\{x, y\} = T(x, y)$, we have $x \leq_T y$, therefore, $[\frac{1}{2}, 1] \subseteq \overline{\{x\}}_T$. If $2x < y < \frac{1}{2}$, then there exists $0 \leq l < \frac{1}{2}$ such that $x = ly = T(l, y)$. $0 \leq l < \frac{1}{2}$ implies $x = ly < \frac{1}{2}y$, i.e., $2x < y$, therefore, $(2x, \frac{1}{2}) \subseteq \overline{\{x\}}_T$. If $x < y \leq 2x < \frac{1}{2}$, then there is no exists $l$ such that $x = T(l, y)$. Therefore, $\overline{\{x\}}_T = \{x\} \cup (2x, 1]$. 
Then we have:

\[
\{x\}_T = \begin{cases} 
[x, 1], & \frac{1}{2} \leq x \\
\{x\} \cup [\frac{1}{2}, 1], & \frac{1}{4} \leq x < \frac{1}{2} \\
\{x\} \cup (2x, 1), & x < \frac{1}{4}
\end{cases}
\]

(b) Let \(z \in \{x\}_T\), i.e. \(z \leq_T x\), then \(z \leq x\), thus \(\{x\}_T \subseteq [0, x]\).

(b1) Suppose \(x \geq \frac{1}{2}\). \(z \leq x\) implies \(z = \min\{z, x\} = T(z, x)\) by definition of \(T\), then \(z \leq_T x\), therefore, \([0, x] \subseteq \{x\}_T\), thus \(\{x\}_T = [0, x]\).

(b2) Suppose \(x < \frac{1}{2}\). If \(\frac{1}{2}x \leq z < x\), then there is no exists \(l\) such that \(z = T(l, x)\). If \(z < \frac{1}{2}x\), then there exists \(0 \leq l < \frac{1}{2}\) such that \(z = T(l, x)\), therefore, \([0, \frac{1}{2}x] \subseteq \{x\}_T\). Thus, \(\{x\}_T = \{x\} \cup [0, \frac{1}{2}x]\).

Then we have:

\[
\{x\}_T = \begin{cases} 
[0, x], & \frac{1}{2} \leq x \\
\{x\} \cup [0, \frac{1}{2}x], & x < \frac{1}{2}
\end{cases}
\]

Taking \(x = \frac{1}{8}\) and \(y = \frac{1}{6}\), however, does not exist, since \(\{x\}_T = \{0, \frac{1}{8}\} \cup (\frac{1}{8}, 1]\) and \(\{x\}_T = \{0, \frac{1}{6}\} \cup (\frac{1}{6}, 1]\), however, does not exist, since \(\{x\}_T = \{0, \frac{1}{8}\} \cup (\frac{1}{8}, 1]\) and \(\{x\}_T = \{0, \frac{1}{6}\} \cup (\frac{1}{6}, 1]\).

From Example 3.3, we have that \(([0, 1], \leq_T)\) is neither a join-semilattice nor a meet-semilattice.

**Remark 3.5.** Example 1.1 cannot be generalized from the unit interval \([0, 1]\) to arbitrary complete lattice. For arbitrary bounded lattice \((L, \leq, 0, 1)\), the function \(T\) defined by the formula (3) in Example 1.1 needs not generate a \(t\)-norm on \(L\). For example, consider the lattice \((L = \{0, a, b, c, d, e, 1\}, \leq, 0, 1)\) given in Figure 3. \(H = (0, e)\), the function \(T\) be given by

\[
T(x, y) = \begin{cases} 
0, & (x, y) \in H^2, \\
x \land y, & \text{otherwise}
\end{cases}
\]

**Fig. 3.** The order \(\leq\) on \(L\).
Then \( T(T(a,c),d) = T(a,d) = a \) and \( T(a,T(c,d)) = T(a,b) = 0 \). Hence, \( T \) is not a t-norm on \( L \) depicted in Figure 3, since the associativity is violated.

Let \((L,\leq,0,1)\) be a bounded lattice and \( a \in L \setminus \{0,1\} \). Çaylı̀ in [6] gave a new t-norm \( T_V : L^2 \rightarrow L \) on \( L \), where \( V \) is a t-norm on \([a,1]\), and

\[
T_V(x,y) = \begin{cases} 
V(x,y), & (x,y) \in [a,1)^2; \\
x \land y, & 1 \in \{x,y\}; \\
0, & \text{otherwise.}
\end{cases}
\]  

(7)

**Theorem 3.6.** If \( L \) is a complete lattice, \( a \in L \setminus \{0,1\} \) and \( V \) is a divisible t-norm on \([a,1]\), then \((L,\leq_{T_V})\) is a complete lattice.

**Proof.** Let \( V \) be divisible on \([a,1]\), and \( \{x_\tau \mid \tau \in \Phi\} \) be an arbitrary subset of \( L \setminus \{0,1\} \).

(a) We will show existence of \( \bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\} \).

(a1) Suppose that there exists \( x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\} \) such that \( x_{\tau_0} \notin [a,1) \). Since \( T_V(x_\tau_0,1) = x_\tau_0 \land 1 = x_\tau_0 \), then \( \{x_\tau_0\} \subseteq \{x_{\tau_0}\}_{T_V} \). Assume \( k \in \{x_{\tau_0}\}_{T_V} \), then there exists \( z \in L \) such that \( x_{\tau_0} = T_V(z,k) \). Because of \( x_{\tau_0} \neq 0 \), it leads to \( T_V(z,k) = \begin{cases} V(z,k), & (z,k) \in [a,1)^2; \\
z \land k, & 1 \in \{z,k\}. \end{cases} \). If \( (z,k) \in [a,1)^2 \), since \( V \) is a t-norm on \([a,1]\), then \( x_{\tau_0} = T_V(z,k) = V(z,k) \in [a,1) \), which contradicts with \( x_{\tau_0} \notin [a,1) \), and thus \( z = 1 \) or \( k = 1 \). If \( z = 1 \), \( x_{\tau_0} = T_V(z,k) = 1 \land k = k \). Thus \( \{x_{\tau_0}\}_{T_V} = \{x_{\tau_0},1\} \). Therefore \( \bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\} \) exists and \( \bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\} = 1 \).

(a2) Suppose \( x_\tau \in [a,1) \) for all \( \tau \in \Phi \). Let \( k = \bigvee \{x_\tau \mid \tau \in \Phi\} \), then \( x_\tau \leq k \). Since \( V \) is a divisible t-norm on \([a,1]\), then there exist \( z_\tau \in [a,1] \) such that \( x_\tau = V(z_\tau,k) = T_V(z_\tau,k) \). Therefore \( x_\tau \leq_{T_V} k \), i.e., \( k \in \{x_\tau \mid \tau \in \Phi\}_{T_V} \). Suppose \( s \in \{x_\tau \mid \tau \in \Phi\}_{T_V} \), then \( x_\tau \leq_{T_V} s \), it leads to \( x_\tau \leq s \). Therefore \( k \leq s \). Because of \( x_\tau \in [a,1) \) for all \( \tau \in \Phi \) and \( V \) is a divisible t-norm on complete lattice \( L \), it leads to \( k \in [a,1] \) and there exists \( z \in L \) such that \( k = T_V(z,s) \), therefore \( k \leq_{T_V} s \). Thus \( \bigvee_{T_V} \{x_\tau \mid \tau \in \Phi\} = \bigvee \{x_\tau \mid \tau \in \Phi\} \).

(b) We will show existence of \( \bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\} \).

(b1) Suppose that there exists \( x_{\tau_0} \in \{x_\tau \mid \tau \in \Phi\} \) such that \( x_{\tau_0} \notin [a,1) \). Since \( T_V(x_\tau_0,0) = 0 \) and \( T_V(x_\tau,1) = x_\tau \), we have \( \{0,x_\tau\} \subseteq \{x_\tau\}_{T_V} \). Suppose \( k \in \{x_{\tau_0}\}_{T_V} \), then there exists \( z \in L \) such that \( k = T_V(z,x_{\tau_0}) \). Since \( x_{\tau_0} \notin [a,1) \), then \( k = T_V(z,x_{\tau_0}) = \begin{cases} x_{\tau_0}, & z = 1; \\
0, & z < 1. \end{cases} \). Thus, \( \{x_{\tau_0}\}_{T_V} = \{0,x_{\tau_0}\} \). Therefore \( \bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\} \) exists and \( \bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\} = 0 \).

(b2) Suppose \( x_\tau \in [a,1) \) for all \( \tau \in \Phi \). Let \( k = \bigwedge \{x_\tau \mid \tau \in \Phi\} \), then \( a \leq k \leq x_\tau \). Since \( V \) is a divisible t-norm on \([a,1]\), we obtain that \( \bigwedge_{T_V} \{x_\tau \mid \tau \in \Phi\} = \bigwedge \{x_\tau \mid \tau \in \Phi\} \).

\[\square\]

**Example 3.7.** Consider the lattice \((L = \{0,a,b,c,d,1\},\leq,0,1)\) given in Figure 1. The function \( T_{V_1} \) on \( L \) is defined by

\[
T_{V_1}(x,y) = \begin{cases} 
x \land y, & (x,y) \in [b,1)^2 \text{ or } 1 \in \{x,y\}; \\
0, & \text{otherwise.}
\end{cases}
\]
<table>
<thead>
<tr>
<th>$T_{V_1}$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>d</td>
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<td>0</td>
<td>b</td>
<td>0</td>
<td>d</td>
<td>d</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>1</td>
</tr>
</tbody>
</table>

**Tab. 2.** T-norm $T_{V_1}$.

then $T_{V_1}$ is a t-norm and $T_{V_1}$ can also be described in Table 2. The order $\leq_{T_{V_1}}$ on $L$ has its diagram as given in Figure 4.

![Diagram](image)

**Fig. 4.** The order $\leq_{T_{V_1}}$ and $\leq_{T_{V_2}}$ on $L$.

The following example shows that converse of Theorem 3.2 is not true in general.

**Example 3.8.** Consider the lattice $(L = \{0, a, b, c, d, 1\}, \leq, 0, 1)$ given in Figure 1. Taking t-norm $V$ on $[b, 1]$ as

$$V(x, y) = \begin{cases} b, & (x, y) \in [b, 1)^2; \\ x \land y, & \text{otherwise.} \end{cases}$$

Consider the function $T_{V_2}$ on $L$ defined by

$$T_{V_2}(x, y) = \begin{cases} b, & (x, y) \in [b, 1)^2; \\ x \land y, & 1 \in \{x, y\}; \\ 0, & \text{otherwise.} \end{cases}$$

$T_{V_2}$ described in Table 3 is a t-norm. The order $\leq_{T_{V_2}}$ on $L$ is given in Figure 4. Hence $(L, \leq_{T_{V_2}})$ is a complete lattice, but $V(x, y)$ is not a divisible t-norm on $[b, 1]$.

**Example 3.9.** Consider the t-norm on $[0, 1]$ defined as follows:

$$T_{V_3}(x, y) = \begin{cases} \min\{x, y\}, & (x, y) \in [\frac{1}{2}, 1)^2; \\ \min\{x, y\}, & 1 \in \{x, y\}; \\ 0, & \text{otherwise.} \end{cases}$$
Sufficient conditions for a T-partial order obtained from triangular norms to be a lattice

\[
\begin{array}{cccccc}
T_{V^2} & 0 & a & b & c & d & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 & a & 0 \\
b & 0 & 0 & b & 0 & b & b \\
c & 0 & 0 & 0 & 0 & c & 0 \\
d & 0 & 0 & b & 0 & b & b \\
1 & 0 & a & b & c & d & 1 \\
\end{array}
\]

Tab. 3. T-norm \(T_{V^2}\).

Since \(V(x, y) = \min\{x, y\}\) is a divisible t-norm on \([\frac{1}{2}, 1]\), then \(([0, 1], \leq_{T_{V^3}})\) is lattice. And

\[
x \vee_{T_{V^3}} y = \begin{cases} 
1, & (x, y) \notin \left[\frac{1}{2}, 1\right]^2; \\
\max\{x, y\}, & (x, y) \in \left[\frac{1}{2}, 1\right]^2.
\end{cases}
\]

\[
x \wedge_{T_{V^3}} y = \begin{cases} 
0, & (x, y) \notin \left[\frac{1}{2}, 1\right]^2; \\
\min\{x, y\}, & (x, y) \in \left[\frac{1}{2}, 1\right]^2.
\end{cases}
\]

4. CONCLUSION

The objective of this paper is to give some sufficient conditions for a T-partial order obtained from triangular norms to be a lattice. Sufficient conditions for other partial order (for example U-partial order and V-partial order) to be a lattice will be considered in future work.

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Lifeng Li, 1. School of Science, Xi’an University of Posts and Telecommunications, Xi’an, Shaanxi 710121. P. R. China. 2. Shaanxi Key Laboratory of Network Data Analysis and Intelligent Processing, Xi’an University of Posts and Telecommunications, Xi’an Shaanxi 710121. P. R. China.
e-mail: lilifeng80@163.com

Jianke Zhang, 1. School of Science, Xi’an University of Posts and Telecommunications, Xi’an, Shaanxi 710121. P. R. China. 2. Shaanxi Key Laboratory of Network Data Analysis and Intelligent Processing, Xi’an University of Posts and Telecommunications, Xi’an Shaanxi 710121. P. R. China.
e-mail: jiankezh@163.com

Chang Zhou, School of Science, Xi’an University of Posts and Telecommunications, Xi’an, Shaanxi 710121. P. R. China.
e-mail: maytheday@163.com