Archivum Mathematicum

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*Archivum Mathematicum*, Vol. 55 (2019), No. 4, 225–228

Persistent URL: `http://dml.cz/dmlcz/147875`

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ARCHIVUM MATHEMATICUM (BRNO)
Tomus 55 (2019), 225–228

# RECOGNIZABILITY OF FINITE GROUPS BY SUZUKI GROUP

Alireza Khalili Asboei and Seyed Sadegh Salehi Amiri

Abstract. Let $G$ be a finite group. The main supergraph $\mathcal{S}(G)$ is a graph with vertex set $G$ in which two vertices $x$ and $y$ are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. In this paper, we will show that $G \cong Sz(q)$ if and only if $\mathcal{S}(G) \cong \mathcal{S}(Sz(q))$, where $q = 2^{2m+1} \geq 8$.

## 1. Introduction

Let $G$ be a finite group and $x \in G$. The order of $x$ is denoted by $o(x)$. The set of all element orders of $G$ is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. It is clear that $\pi_e(G)$ is determined by the subset $\mu(G)$ of maximal element orders with respect to divisibility. We set $m_i = m_i(G) = |\{g \in G | o(g) = i\}|$.

The *main supergraph* $\mathcal{S}(G)$ is the graph whose vertices are the group elements and two elements $x$ and $y$ are connected if either $o(x) \mid o(y)$ or $o(y) \mid o(x)$. We also denote the subgraph of $\mathcal{S}(G)$ with the identity removed by $\mathcal{S}^*(G)$ [4]. We write $x \sim y$ when two vertices $x$ and $y$ are adjacent.

For each finite group $G$ and each integer $d \geq 1$, let $G(d) = \{x \in G | x^d = 1\}$. We say that the groups $G_1$ and $G_2$ are of *the same order type* if $|G_1(d)| = |G_2(d)|$, for all $d \in \mathbb{N}$. By the definition of the main supergraph, it is clear that if $G_1$ and $G_2$ are groups with the same order type, then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. The example $G_1 = Z_4 \times Z_4$ and $G_2 = Z_4 \times Z_2 \times Z_2$ shows that the converse statement is not true in general.

In 1987, J.G. Thompson [9, Problem 12.37] posed the following Problem:
**Thompson's Problem.** Suppose that $G_1$ and $G_2$ are two groups of the same order type. If $G_1$ is solvable, is it true that $G_2$ is also necessarily solvable?

In [5], the set of $m_i$ (also known as nse) was used to prove no solvable group has the same order type as $Sz(2^{2m+1})$, where $2^{2m+1} - 1$ is a prime power. In this paper we use instead the supergraph to remove the requirement that $2^{2m+1} - 1$ is a prime power. As two groups having the same order type implies their supergraphs coincide, if a solvable group is uniquely determined by $\mathcal{S}(G)$, then Thompson's conjecture holds for $G$. In [7, 8, 10], the authors proved that alternating groups of degree $p$, $p+1$ and $p+2$, the symmetric groups of degree $p$, the small Ree groups ${}^2G_2(3^{2n+1})$ and $\mathrm{PSL}_2(q)$, where $q$ is a prime power are uniquely determined by

2010 *Mathematics Subject Classification*: primary 20D08; secondary 05C25.
*Key words and phrases*: main supergraph, Suzuki group.
Received November 10, 2018, revised February 2019. Editor C. Greither.
DOI: 10.5817/AM2019-4-225

their main supergraph. Furthermore, in [6], it was also proven for sporadic simple groups and for $\mathrm{PSL}_2(q)$ and $\mathrm{PGL}_2(p)$, where $p$ is a prime. Our main theorem is as follows:

**Main Theorem.** Let $\mathcal{S}(G) \cong \mathcal{S}(Sz(q))$, where $q = 2^{2m+1} \geq 8$. Then $G \cong Sz(q)$.

We construct the *prime graph* of $G$, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices $p$ and $q$ are joined by an edge if and only if $G$ has an element of order $pq$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$. Throughout this paper, we denote by $\phi$ the Euler's totient function.

## 2. Preliminary results

In this section, we present some preliminary results which will turn out to be useful in what follows. First, we quote some known results about Frobenius group and 2-Frobenius group, which are useful in the sequel.

**Lemma 2.1** ([2]). *Let $G$ be a 2-Frobenius group of even order, i.e., $G$ is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. Then:*
*(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;*
*(b) $G/K$ and $K/H$ are cyclic, $|G/K| \mid (|K/H| - 1)$, $(|G/K|, |K/H|) = 1$ and $G/K \lesssim \mathrm{Aut}(K/H)$.*

**Lemma 2.2** ([2]). *Suppose that $G$ is a Frobenius group of even order and $H$, $K$ are the Frobenius kernel and the Frobenius complement of $G$, respectively. Then $t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$.*

**Lemma 2.3** ([12]). *If $G$ is a finite group such that $t(G) \geq 2$, then $G$ has one of the following structures:*
(a) *$G$ is a Frobenius group or a 2-Frobenius group;*
(b) *$G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and $K/H$ is a non-abelian simple group. In particular, $H$ is nilpotent, $G/K \lesssim \mathrm{Out}(K/H)$ and the odd order components of $G$ are the odd order components of $K/H$.*

**Lemma 2.4** ([3]). *The Suzuki groups are only non-abelian simple groups of order prime to* 3.

**Lemma 2.5** ([1, 11]). *Let $S = Sz(q)$ with $q = 2^{2m+1} \geq 8$, $m \geq 1$. Then $m_2(S) = (q-1)(q^2+1)$, $m_4(S) = (q^2 - q)(q^2 + 1)$ and $m_{2^r} = 0$ for $r \geq 3$.*

## 3. Proof of the Main Theorem

Now we are ready to prove the main theorem of this paper.

**Proof of the main theorem.** It is well known that $Sz(q)$ has no elements of order $2r$ with $r$ an odd number (see [11]). Therefore, by Lemma 2.5, $\mathcal{S}^*(Sz(q))$ has a complete component consisting of the $(q^2+1)(q^2-1)$ elements whose order is a power of 2. Let $K_1$ denote this component in $\mathcal{S}^*(G)$. Since $K_1$ is complete, it follows that $K_1$ contains only elements of prime power order for a fixed prime $p$. Moreover, as 2 divides the number of elements who order is a power of $p$ (by pairing $g$ and $g^{-1}$) but does not divide $(q^2+1)(q^2-1)$, it follows that $p=2$. Therefore, 2 is an isolated vertex in the prime graph $\Gamma(G)$.

If $G$ is a Frobenius group with complement $H$, then by Lemma 2.2 $|H|=q^2$ or $(q^2+1)(q-1)$. However $|H| \mid |G|/|H|-1$ which gives a contradiction. While, if $G$ is a 2-Frobenius group with series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ as in Lemma 2.1, then $(q^2+1)(q-1) = |K/H| \mid |H|-1 = 2^t-1$ for some $t$ which is a contradiction. Thus by Lemma 2.3 $G$ has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, with $K/H \cong Sz(q^{'})$ for some $q^{'} < q$ as $3 \nmid |G|$ by Lemma 2.4. Furthermore, Lemma 2.3 implies $H$ and $G/K$ are 2-groups and therefore $(q^2+1)(q-1) \mid (q^{'2}+1)(q^{'}-1)$ showing that $q^{'}=q$, $K=G$ and $H=1$. In particular, $G \cong Sz(q)$. $\square$

**Acknowledgement.** The authors would like to thank the referee for his/her valuable comments which helped to improve the manuscript.

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