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ON n-EXACT CATEGORIES

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Abstract. An n-exact category is a pair consisting of an additive category and a class of sequences with $n+2$ terms satisfying certain axioms. We introduce n-weakly idempotent complete categories. Then we prove that an additive n-weakly idempotent complete category together with the class $\mathcal{C}_n$ of all contractible sequences with $n+2$ terms is an n-exact category. Some properties of the class $\mathcal{C}_n$ are also discussed.

Keywords: n-exact category; contractible sequence; idempotent complete category

MSC 2010: 18E99, 18E10

1. Introduction

The notion of n-cluster-tilting subcategories was introduced by Iyama et al. in [8], and was developed further in the sense of higher dimensional Auslander-Reiten theory by Iyama in [5], [6], [7]. Geiß et al. in [4] introduced $(n+2)$-angulated categories and showed that an n-cluster-tilting subcategory (in the sense of Iyama) of a triangulated category which satisfies a certain condition is an $(n+2)$-angulated category. This allows them to build a broad class of $(n+2)$-angulated categories. Recently, Jasso in [9] introduced n-exact categories and provided several results and examples which illustrate the importance of such a class of categories. In particular, he showed that the n-cluster-tilting subcategories of exact categories are n-exact, and that the stable category of a Frobenius n-exact category is an $(n+2)$-angulated category. An n-exact category is a pair consisting of an additive category and a class (called n-exact structure) of sequences with $n+2$ terms satisfying certain axioms. The n-exact categories are higher analogues of exact categories, see [2], [12]. In this paper, we introduce n-weakly idempotent complete categories; these are both generalizations (see Corollary 3.5) and higher analogues of the weakly idempotent complete categories introduced by Thomason in [13] and further investigated by

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Freyd and Neeman in [3] and [11]. We prove that an additive n-weakly idempotent complete category together with the class $\mathcal{C}_n$ of all contractible sequences with $n+2$ terms is an n-exact category (see Theorem 3.7). As a consequence, the class $\mathcal{C}_n$ is closed under direct sum and direct summand (see Corollary 3.8). In analogy with the case of n-abelian categories (see [9], Corollary 3.10) we show that the admissible monomorphisms with respect to two exact structures, on an additive category, with different orders are split monomorphisms (see Proposition 3.9).

2. Preliminaries

2.1. Notation. Mainly, we follow the notation and definitions of [9]. Throughout this paper, $\mathcal{A}$ denotes an additive category and n denotes a positive integer. By $\mathcal{A}(A,B)$, we denote the class of morphisms $A \to B$ in $\mathcal{A}$ while 1_A denotes the identity morphism of the object $A \in \mathcal{A}$. We will use the symbol $\mathrm{Ch}(\mathcal{A})$ to denote the category of (cochain) complexes in $\mathcal{A}$. We write $\mathrm{Ch}^n(\mathcal{A})$ for the full subcategory of $\mathrm{Ch}(\mathcal{A})$ consisting of all complexes

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \dots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}$$

that are concentrated in degrees $0, 1, \dots, n+1$.

Definition 2.1.

(D1) A morphism of complexes $f \in \mathrm{Ch}(\mathcal{A})(X,Y)$ is said to be *homotopic* to a morphism $g \in \mathrm{Ch}(\mathcal{A})(X,Y)$ if there is a morphism $h = (h^k\colon X^k \to Y^{k-1})_{k\in\mathbb{Z}}$, called a homotopy, satisfying

$$f^k - g^k = d_Y^{k-1}h^k + h^{k+1}d_X^k$$

for all $k \in \mathbb{Z}$. In such a case, we write $h\colon f \to g$. The "homotopic" relation gives easily rise to an equivalence relation on $\mathrm{Ch}(\mathcal{A})(X,Y)$.

(D2) A category whose objects are those of $\mathrm{Ch}(\mathcal{A})$ and whose morphisms are, up to homotopy, those of $\mathrm{Ch}(\mathcal{A})$, is called the *homotopy category* of $\mathcal{A}$ and is denoted by $\mathrm{H}(\mathcal{A})$.

(D3) A complex $X \in \mathrm{Ch}(\mathcal{A})$ is said to be *contractible* if the identity morphism 1_X is homotopic to the zero morphism 0_X of X, i.e., $1_X = 0_X$ in $\mathrm{H}(\mathcal{A})$.

(D4) A *weak cokernel* of a morphism $f \in \mathcal{A}(A,B)$ is a morphism $g \in \mathcal{A}(B,C)$ such that $gf = 0$ and for every morphism $q \in \mathcal{A}(B,C')$ satisfying $qf = 0$, there is a morphism $p \in \mathcal{A}(C,C')$ such that $q = pg$. The notion of the *weak kernel* is defined dually.

(D5) A sequence $(d_X^1, \dots, d_X^n)$ of morphisms in the complex

$$X\colon\ X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \dots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}$$

is said to be an *n-cokernel* of a morphism $d_X^0 \in \mathcal{A}(X^0, X^1)$ if d_X^n is a cokernel of d_X^{n-1} and d_X^k is a weak cokernel of d_X^{k-1} for $k = 1, \dots, n-1$. In that case, the complex X is called a *right n-exact sequence*. The definitions of the *n-kernel* of a morphism and that of the *left n-exact sequence* are given dually. The terminologies "right n-exact sequence" and "left n-exact sequence" are borrowed from [10].

(D6) We say that a complex $X \in \mathrm{Ch}^n(\mathcal{A})$ is an *n-exact sequence* if $(d_X^1, \dots, d_X^n)$ is an n-cokernel of d^0 and $(d_X^0, \dots, d_X^{n-1})$ is an n-kernel of d_X^n or, equivalently, X is both a right n-exact sequence and a left n-exact sequence.

(D7) A morphism $f \in \mathrm{Ch}^n(\mathcal{A})(X, Y)$ is called a *weak isomorphism* if there is an integer $0 \leqslant k \leqslant n+1$ (with $n+2 := 0$) such that f^k and f^{k+1} are isomorphisms.

(D8) We say that a morphism $m \in \mathcal{A}(A, B)$ is a *split monomorphism* if there is a morphism $e \in \mathcal{A}(B, A)$ such that $em = 1_A$. A *split epimorphism* is defined dually.

(D9) The category $\mathcal{A}$ is said to be *n-weakly idempotent complete* if every split monomorphism has an n-cokernel and every split epimorphism has an n-kernel. When $n = 1$, $\mathcal{A}$ is called a weakly idempotent category.

(D10) A morphism of complexes $f \in \mathrm{Ch}^{n-1}(\mathcal{A})(X, Y)$

$$\begin{array}{ccccccccc}
X & & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f^0} & & \big\downarrow{\scriptstyle f^1} & & & & \big\downarrow{\scriptstyle f^n} \\
Y & & Y^0 & \xrightarrow{d_Y^0} & Y_1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n
\end{array}$$

is said to be an *n-pushout diagram* (*n-pushout for short*) *of X along f^0* if the *mapping cone $C = C(f)$* of f:

$$X^0 \xrightarrow{d_C^{-1}} X^1 \oplus Y^0 \xrightarrow{d_C^0} \dots \xrightarrow{d_C^{n-2}} X^n \oplus Y^{n-1} \xrightarrow{d_C^{n-1}} Y^n$$

is a right n-exact sequence, where

$$d_C^{-1} = \begin{bmatrix} -d_X^0 \\ f^0 \end{bmatrix} \quad \text{and} \quad d_C^{n-1} = [\, f^n \quad d_Y^{n-1} \,],$$

and for $k = 0, 1, \dots, n-2$,

$$d_C^k := \begin{bmatrix} -d_X^{k+1} & 0 \\ f^{k+1} & d_Y^k \end{bmatrix} :\ X^{k+1} \oplus Y^k \to X^{k+2} \oplus Y^{k+1}.$$

The definition of *n-pullback diagram* is given dually.

Remark 2.2.

(1) It is known and easily verified that a complex $X \in \mathrm{Ch}(\mathcal{A})$ is contractible if and only if X is isomorphic to the zero complex in $\mathrm{H}(\mathcal{A})$.

(2) The class of n-abelian categories is contained in that of n-weakly idempotent complete categories, see [9], Definition 3.1.

2.2. n-exact categories. Let $\mathcal{X}$ be a class of n-exact sequences in $\mathcal{A}$. The elements of $\mathcal{X}$ are called *$\mathcal{X}$-admissible n-exact sequences* ($\mathcal{X}$-admissibles for short). If the sequence

$$X\colon\ X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \dots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}$$

is $\mathcal{X}$-admissible, then the morphisms d_X^0 and d_X^n are called, respectively, an *$\mathcal{X}$-admissible monomorphism* and an *$\mathcal{X}$-admissible epimorphism.* Let $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{E}_{\mathcal{X}}$ denote, respectively the classes of all $\mathcal{X}$-admissible monomorphisms and all $\mathcal{X}$-admissible epimorphisms. The class $\mathcal{X}$ is said to be an *n-exact structure on $\mathcal{A}$* provided the following axioms hold:

(E0) $\mathcal{X}$ is closed under weak isomorphisms of n-exact sequences.

(E1) The zero sequence in $\mathrm{Ch}^n(\mathcal{A})$ is $\mathcal{X}$-admissible.

(E2) The class $\mathcal{M}_{\mathcal{X}}$ is closed under composition.

(E2)$^{\mathrm{op}}$ The class $\mathcal{E}_{\mathcal{X}}$ is closed under composition.

(E3) For every $\mathcal{X}$-admissible X and every morphism $f^0 \in \mathcal{A}(X^0, Y^0)$, there is an n-pushout f of $(d_X^0, \dots, d_X^{n-1})$ along f^0 such that $d_Y^0 \in \mathcal{M}_{\mathcal{X}}$:

$$\begin{array}{ccccccccc}
X & & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n \\
\big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f^0} & & \vdots & & & & \vdots \\
Y & & Y^0 & \xrightarrow{d_Y^0} & Y_1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n
\end{array}$$

(E3)$^{\mathrm{op}}$ For every $\mathcal{X}$-admissible X and every morphism $g^{n+1} \in \mathcal{A}(Y^{n+1}, X^{n+1})$, there is an n-pullback g of $(d_X^1, \dots, d_X^n)$ along g^{n+1} such that $d_Y^n \in \mathcal{E}_{\mathcal{X}}$:

$$\begin{array}{ccccccccc}
X & & Y^1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} \\
\big\downarrow{\scriptstyle g} & & \vdots & & & & \vdots & & \big\downarrow{\scriptstyle g^{n+1}} \\
Y & & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1}
\end{array}$$

A pair $(\mathcal{A}, \mathcal{X})$ is said to be an *n-exact category* if $\mathcal{X}$ is an n-exact structure on $\mathcal{A}$.

2.3. Preparatory results. We need the following important results, due to Jasso in [9], which play the key role in our proofs in the next section:

Proposition 2.3 ([9], Proposition 2.7). *Let $f \in \mathrm{Ch}^n(\mathcal{A})(X,Y)$ be a morphism of n-exact sequences such that f^k and f^{k+1} are isomorphisms for some $1 \leqslant k \leqslant n$. Then f induces an isomorphism in $\mathrm{H}(\mathcal{A})$.*

Combining this proposition with Remark 2.2, we obtain:

Corollary 2.4. *Under the assumptions of Proposition 2.3, X is contractible if and only if Y is.*

Proposition 2.5 ([9], Proposition 2.5). *Let $X, Y \in \mathrm{Ch}^n(\mathcal{A})$ be two isomorphic n-sequences in $\mathrm{H}(\mathcal{A})$. Then the following statements hold.*

(1) *The complex X is an n-exact sequence if and only if Y is an n-exact sequence.*
(2) *Every contractible complex in $\mathrm{Ch}^n(\mathcal{A})$ is an n-exact sequence.*

Proposition 2.6 ([9], Proposition 2.6). *Let $X \in \mathrm{Ch}^n(\mathcal{A})$ be a right n-exact sequence. Then d_X^0 is a split monomorphism if and only if X is a contractible n-exact sequence.*

3. Main results

From now on, $\mathcal{C}_n$ will denote the class of all contractible complexes in $\mathrm{Ch}^n(\mathcal{A})$. By Proposition 2.5, every complex in $\mathcal{C}_n$ is an n-exact sequence. We start this section with the following lemma which we will use extensively in the proofs of our results. This is an adapted form of [9], Comparison-Lemma 2.1.

Lemma 3.1. *Let $X \in \mathrm{Ch}^n(\mathcal{A})$ be a right n-exact sequence. If $f, g \in \mathrm{Ch}^n(\mathcal{A})(X,Y)$ are two morphisms of complexes such that $f^0 = g^0$, then there exists a homotopy $h\colon f \to g$ such that $h^1 = 0$ and $d_Y^n h^{n+1} = f^{n+1} - g^{n+1}$.*

Proof. Given that d_X^k is a weak cokernel of d_X^{k-1} for $k = 1, \ldots, n$, and $d_X^{n+1} = 0$ is a weak cokernel of the cokernel d_X^n of d_X^{n-1}, a construction similar to that in the proof of [9], Comparison-Lemma 2.1 gives morphisms $h^k \in \mathcal{A}(X^k, Y^{k-1})$ for all $k \leqslant n+2$, such that $f^k - g^k = d_Y^{k-1} h^k + h^{k+1} d_X^k$ for $k = 1, \ldots, n+1$, and $h^k = 0$ for all $k \leqslant 1$. Note that $h^{n+2} = 0$ simply because $X^{n+2} = 0$. This yields

$$\begin{aligned}
[(f^{n+1} - g^{n+1}) - d_Y^n h^{n+1}] d_X^n &= (f^{n+1} - g^{n+1}) d_X^n - d_Y^n (h^{n+1} d_X^n) \\
&= d_Y^n (f^n - g^n) - d_Y^n (h^{n+1} d_X^n) \\
&= d_Y^n (f^n - g^n) - d_Y^n [(f^n - g^n) - d_Y^{n-1} h^n] \\
&= d_Y^n (f^n - g^n) - d_Y^n (f^n - g^n) + d_Y^n d_Y^{n-1} h^n = 0.
\end{aligned}$$

Therefore,

$$f^{n+1} - g^{n+1} = d_Y^n h^{n+1},$$

since the morphism d_X^n, being a cokernel of d_X^{n-1}, is an epimorphism. Writing $h^k = 0$ for all $k \geqslant n+3$, we obtain a homotopy $h\colon f \to g$ as required. □

Proposition 3.2. *Let $X, Y \in \mathrm{Ch}^n(\mathcal{A})$ be two exact sequences and assume that $f \in \mathrm{Ch}^n(\mathcal{A})(X,Y)$ is a morphism. Then f induces an isomorphism in $\mathrm{H}(\mathcal{A})$ provided one of the following two conditions holds*

(1) *X is contractible and f^{n+1} is an isomorphism.*

(2) *Y is contractible and f^0 is an isomorphism.*

Proof. (1) Assume X is contractible and f^{n+1} is an isomorphism. Let g^{n+1} be the inverse of f^{n+1}. According to the dual of Proposition 2.6, d_X^n is a split epimorphism. Let $d_n \in \mathcal{A}(X^{n+1}, X^n)$ be a morphism such that $d_X^n d_n = 1_{X^{n+1}}$. Writing $g^n = d_n g^{n+1} d_Y^n$, we get $d_X^n g^n = g^{n+1} d_Y^n$. It then follows, by the factorization property of weak kernels, that there exists a complex morphism $g \in \mathrm{Ch}^n(\mathcal{A})(Y,X)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
Y & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} \\
\vdots\downarrow g & \vdots\downarrow g^0 & & \vdots\downarrow g^1 & & & \downarrow g^n & & \downarrow g^{n+1} \\
X & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1}
\end{array}$$

The dual of Lemma 3.1 applied to both $(gf, 1_X)$ and $(fg, 1_Y)$ implies that $gf = 1_X$ and $fg = 1_Y$ in $\mathrm{H}(\mathcal{A})$.

(2) Assume Y is contractible and f^0 is an isomorphism. Let g^0 be the inverse of f^0. By Proposition 2.6, d_Y^0 is a split monomorphism. Let $d_0 \in \mathcal{A}(Y^1, Y^0)$ be a morphism such that $d_0 d_Y^0 = 1_{Y^0}$. Writing $g^1 = d_X^0 g^0 d_0$, we get $d_X^0 g^0 = g^1 d_Y^0$. It then follows, by the factorization property of weak cokernels, that there exists a complex morphism $g \in \mathrm{Ch}^n(\mathcal{A})(Y,X)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
Y & Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots \longrightarrow & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} \\
\vdots\downarrow g & \downarrow g^0 & & \downarrow g^1 & & & \vdots\downarrow g^n & & \vdots\downarrow g^{n+1} \\
X & X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots \longrightarrow & X^n & \xrightarrow{d_X^n} & X^{n+1}
\end{array}$$

The rest of the proof runs as in (1) using Lemma 3.1. □

Corollary 3.3. *Let $X, Y \in \mathrm{Ch}^n(\mathcal{A})$ be two n-exact sequences and let $f \in \mathrm{Ch}^n(\mathcal{A})(X,Y)$ be a morphism. Assume that f^0 and f^{n+1} are isomorphisms. Then X is contractible if and only if Y is.*

P r o o f. Follows easily from Proposition 3.2 and Remark 2.2. □

The next result states in particular that the contractible n-exact sequences can be extended to contractible m-exact sequences for all $m > n$.

Lemma 3.4. *Let $X \in \mathrm{Ch}^n(\mathcal{A})$ be a right n-exact sequence and m an integer greater than n. Then,*

(1) *the following sequence $\overline{X}$ is a right m-exact sequence*

$$\overline{X}\colon\ X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \ldots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} 0 \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \xrightarrow{d_X^m} 0$$

(2) *the sequence $\overline{X}$ is contractible if X is.*

P r o o f. (1) Note that the morphism d_X^n, being a cokernel of d_X^{n-1}, is an epimorphism. In addition, the morphism d_X^{n+1} is a split epimorphism. By the dual of [1], Proposition 1.1.7 the morphisms d_X^{n+1} and $1_0 = d_X^{n+2}$ are, respectively, the cokernels of d_X^n and d_X^{n+1}, and the isomorphism 1_0 is the cokernel of itself. The rest of the proof follows from the fact that X is a right n-exact sequence.

(2) Since X is contractible, it follows that d_X^0 is a split monomorphism by Proposition 2.6. By the same proposition, $\overline{X}$ is contractible, since $\overline{X}$ is a right m-exact sequence by (1). □

Corollary 3.5.

(1) *If $\mathcal{A}$ is weakly idempotent complete, then $\mathcal{A}$ is n-weakly idempotent complete for every $n \geqslant 2$.*

(2) *The n-sequence $\overline{X}$: $X^0 \overset{d^0=1_{X^0}}{=\!=\!=} X^0 \xrightarrow{d^1} 0 \xrightarrow{d^2=1_0} \cdots \xrightarrow{d^{n-2}=1_0} 0 \xrightarrow{d^{n-1}=1_0} 0$ is contractible.*

P r o o f. (1) Follows from (1) of Lemma 3.4 and its dual.

(2) Observe that in the sequence X: $X^0 \overset{d^0=1_{X^0}}{=\!=\!=} X^0 \xrightarrow{d^1} 0$, the morphism d^1 is a cokernel of the "split monomorphism" 1_{X^0}. By Proposition 2.6, the sequence X is contractible. Hence, by (2) of Lemma 3.4, the sequence $\overline{X}$ is contractible. □

Proposition 3.6. *The class $\mathcal{C}_n$ satisfies axioms* (E0), (E1), (E3) *and* (E3)$^{\mathrm{op}}$.

P r o o f. (E0) Follows from Corollaries 2.4 and 3.3.

(E1) Follows trivially from Remark 2.2.

(E3) Let X be $\mathcal{C}_n$-admissible and let $f^0 \in \mathcal{A}(X^0, Y^0)$ be a morphism. We need to prove that there exists an n-pushout diagram of $(d_X^0, \ldots, d_X^{n-1})$ along f^0:

$$\begin{array}{ccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n \\
\downarrow{\scriptstyle f^0} & & \vdots & & & & \vdots \\
Y^0 & \overset{d_Y^0}{\dashrightarrow} & Y^1 & \overset{d_Y^1}{\dashrightarrow} & \cdots & \overset{d_Y^{n-1}}{\dashrightarrow} & Y^n
\end{array}$$

such that $d_Y^0 \in \mathcal{M}_{\mathcal{C}_n}$. Since X is $\mathcal{C}_n$-admissible, the morphism d_X^0 is a split monomorphism by Proposition 2.6. Let $d \in \mathcal{A}(X^1, X^0)$ be a morphism such that $dd_X^0 = 1_{X^0}$. We claim that the following morphism f is an n-pushout diagram of X along f^0:

$$\begin{array}{c}
Y \\ \downarrow{\scriptstyle f} \\ X
\end{array}
\qquad
\begin{array}{ccccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & X^3 & \longrightarrow & \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n \\
\downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^0 d} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle d_X^n} \\
Y^0 & \overset{1_{Y^0}}{=\!=} & Y^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X^{n+1}
\end{array}$$

Equivalently, the mapping cone $C = C(f)$ is a right n-exact sequence:

$$X^0 \xrightarrow[\begin{bmatrix} -d_X^0 \\ f^0 \end{bmatrix}]{d_C^{-1}=} X^1 \oplus Y^0 \xrightarrow[\begin{bmatrix} -d_X^1 & 0 \\ f^0 d & 1_{Y^0} \end{bmatrix}]{d_C^0=} X^2 \oplus Y^0 \xrightarrow[\begin{bmatrix} -d_X^2 & 0 \\ 0 & 0 \end{bmatrix}]{d_C^1=} X^2 \oplus 0 \longrightarrow \cdots$$

$$\cdots \xrightarrow[\begin{bmatrix} -d_X^{n-1} & 0 \\ 0 & 0 \end{bmatrix}]{d_C^{n-2}=} X^n \oplus 0 \xrightarrow[\begin{bmatrix} -d_X^n & 0 \end{bmatrix}]{d_C^{n-1}=} X^{n+1}$$

By (2) of Corollary 3.5, the morphism 1_{Y^0} belongs to $\mathcal{M}_{\mathcal{C}_n}$. We shall prove that d_C^k is a weak cokernel of d_C^{k-1} for $k = 1, \ldots, n-2$, and d_C^{n-1} is a kernel of d_C^{n-2}.

Let first $[U \ V] \in \mathcal{A}(X^1 \oplus Y^0, Z)$ be a morphism such that

$$0 = [U \quad V]\, d_C^{-1} = [U \quad V] \begin{bmatrix} -d_X^0 \\ f^0 \end{bmatrix}.$$

This is equivalent to $-Ud_X^0 + Vf^0 = 0$. Because $dd_X^0 = 1_{X^0}$, we have

$$(-U + Vf^0 d)d_X^0 = -Ud_X^0 + Vf^0 = 0.$$

But d_X^1 is a weak cokernel of d_X^0. Hence there exists a morphism $W \in \mathcal{A}(X^2, Z)$ such that

$$-U + Vf^0 d = Wd_X^1,$$

which is equivalent to $U = -Wd_X^1 + Vf^0d$. Thus,

$$[\,U \quad V\,] = [\,W \quad V\,]\begin{bmatrix} -d_X^1 & 0 \\ f^0d & 1_{Y^0} \end{bmatrix} = [\,W \quad V\,]\, d_C^0.$$

This shows that d_C^0 is a cokernel of d_C^{-1}.

Let now $[U_0\ V_0] \in \mathcal{A}(X^2 \oplus Y^1, Z)$ be a morphism such that

$$0 = [\,U_0 \quad V_0\,]\, d_C^0 = [\,U_0 \quad V_0\,]\begin{bmatrix} -d_X^1 & 0 \\ f^0d & 1_{Y^0} \end{bmatrix}.$$

This is equivalent to

$$\begin{cases} -U_0d_X^1 + V_0f^0d = 0, \\ V_0 = 1_Y^0V_0 = 0. \end{cases}$$

Hence $-U_0d_X^1 = 0$, because $V_0 = 0$. Since $-d_X^2$ is a weak cokernel of $-d_X^1$, there exists a morphism $W_0 \in \mathcal{A}(X^2, Z)$ such that $U_0 = -W_0d_X^2$ so that

$$[\,U_0 \quad V_0\,] = [\,U_0 \quad 0\,] = [\,W_0 \quad 0\,]\begin{bmatrix} -d_X^2 & 0 \\ 0 & 0 \end{bmatrix} = [\,W_0 \quad 0\,]\, d_C^1.$$

This shows that d_C^1 is a weak cokernel of d_C^0. The proof of the rest of the claim, i.e., that d_C^{k+1} is a weak cokernel of d_C^k for $k = 1, \ldots, n-3$ and d_C^{n-1} is a cokernel of d_C^{n-2}, follows easily from the fact that X is a right n-exact sequence.

Axiom (E3)$^{\rm op}$ follows from (E3) by duality. This concludes the proof. □

Theorem 3.7. *If $\mathcal{A}$ is n-weakly idempotent complete, then $(\mathcal{A}, \mathcal{C}_n)$ is an n-exact category.*

Proof. By Proposition 3.6, we only need to prove axiom (E2); axiom (E2)$^{\rm op}$ can be proved dually using the dual of Proposition 2.6 and the fact that the composite of two split epimorphisms is also a split epimorphism. Let f and g be two composable (i.e., the composite fg exists) $\mathcal{C}_n$-admissible monomorphisms in $\mathcal{A}$. By definition, there exist two $\mathcal{C}_n$-admissible (contractible) n-exact sequences X and Y in $\mathrm{Ch}^n(\mathcal{A})$ such that $d_X^0 = f$ and $d_Y^0 = g$. It follows from Proposition 2.6 that the morphisms f and g are split monomorphisms. Hence, the composite fg is also a split monomorphism. Given that $\mathcal{A}$ is n-weakly idempotent complete, the morphism fg has an n-cokernel $(d^2, d^3, \ldots, d^n)$. Therefore the sequence $\circ \xrightarrow{fg} \circ \xrightarrow{d^1} \circ \ldots \circ \xrightarrow{d^{n-1}} \circ \xrightarrow{d^n} \circ$ is contractible by Proposition 2.6. Consequently, fg is an $\mathcal{C}_n$-admissible monomorphism. Hence $\mathcal{C}_n$ satisfies axiom (E2). □

The following corollary is a consequence of this theorem and [9], Proposition 4.6, Proposition 4.12. We point out here that the term "$\mathcal{X}$-admissible" is missing in the statement "If $X \oplus Y$ is an n-exact sequence" of [9], Proposition 4.12.

Corollary 3.8. *Assume $\mathcal{A}$ is n-weakly idempotemt complete. If X_1 and X_2 are two complexes in $\mathrm{Ch}^n(\mathcal{A})$, then $X_1 \oplus X_2$ is contractible if and only if both X_1 and X_2 are contractible.*

The last result in this paper shows that the morphims which are admissible monomorphisms with respect to two exact structures with different orders are split monomorphisms.

Proposition 3.9. *Let $m < n$ be two distinct positive integers and let $\mathcal{X}$ and $\mathcal{Y}$ be, respectively, an m-exact structure and an n-exact structure on $\mathcal{A}$. Assume there exists a morphism $d^0 \in \mathcal{A}(Z^0, Z^1)$ which is both an $\mathcal{X}$-admissible monomorphism and an $\mathcal{Y}$-admissible monomorphism. Then d^0 is a split monomorphism.*

P r o o f. This is an adaptation of the proof of [9], Corollary 3.10. Since d^0 is both an $\mathcal{X}$-admissible monomorphism and an $\mathcal{Y}$-admissible monomorphism, there exist an $\mathcal{X}$-admissible m-exact sequence X and a $\mathcal{Y}$-admissible n-exact sequence Y so that $d^0 = d^0_X = d^0_Y$. Hence $(d^1_X, \ldots, d^m_X)$ and $(d^1_Y, \ldots, d^n_Y)$ are, respectively, an m-cokernel and an n-cokernel of d^0. It follows, by the factorization property of weak cokernels, that there exist two complex morphisms $f \in \mathrm{Ch}^n(\mathcal{A})(X, Y)$ and $g \in \mathrm{Ch}^n(\mathcal{A})(Y, X)$ such that the following diagram is commutative:

$$
\begin{array}{ccccccccccccccccc}
X & & X^0 & \xrightarrow{d^0} & X^1 & \xrightarrow{d^1_X} & X^2 & \xrightarrow{d^2_X} & \cdots & \xrightarrow{d^m_X} & X^{m+1} & \xrightarrow{d^{m+1}_X} & X^{m+2} & \xrightarrow{d^{m+2}_X} & \cdots & \xrightarrow{d^n_X} & X^{n+1} \\
\big\downarrow f & & \| & & \| & & \vdots & & & & \vdots & & \vdots & & & & \vdots\, f^{n+1} \\
Y & & X^0 & \xrightarrow{d^0} & X^1 & \xrightarrow{d^1_Y} & Y^2 & \xrightarrow{d^2_Y} & \cdots & \xrightarrow{d^m_Y} & Y^{m+1} & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
\big\downarrow g & & \| & & \| & & \vdots & & & & \vdots & & \vdots & & & & \vdots\, g^{n+1} \\
X & & X^0 & \overset{d^0}{=\!=} & X^1 & \xrightarrow{d^1_X} & X^2 & \xrightarrow{d^2_X} & \cdots & \xrightarrow{d^m_X} & X^{m+1} & \xrightarrow{d^{m+1}_X} & X^{m+2} & \xrightarrow{d^{m+2}_X} & \cdots & \xrightarrow{d^n_X} & X^{n+1}
\end{array}
$$

Since $g^0 f^0 = 1_{X^0}$, it follows from Lemma 3.1 that there exists a homotopy $h\colon 1_X \to fg$. Hence $1_{X^{n+1}} = 1_{X^{n+1}} - g^{n+1} f^{n+1} = d^n_X h^{n+1}$, which means that d^n_X is a split epimorphism. Since $(d^0 = d^0_X, \ldots, d^{m-1}_X)$ is an n-kernel d^n_X, the dual of Proposition 2.6 implies that X is contractible. Therefore d^0 is a split monomorphism. This concludes the proof. □

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