Czechoslovak Mathematical Journal

LeRoy B. Beasley

$(0,1)$-matrices, discrepancy and preservers

Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 4, 1123–1131

Persistent URL: `http://dml.cz/dmlcz/147920`

Czechoslovak Mathematical Journal, 69 (144) (2019), 1123–1131

$(0,1)$-MATRICES, DISCREPANCY AND PRESERVERS

LeRoy B. Beasley, Logan

Received February 20, 2018. Published online August 30, 2019.

Abstract. Let m and n be positive integers, and let $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ be nonnegative integral vectors. Let $A(R, S)$ be the set of all $m \times n$ $(0,1)$-matrices with row sum vector R and column vector S. Let R and S be nonincreasing, and let $F(R)$ be the $m \times n$ $(0,1)$-matrix, where for each i, the ith row of $F(R, S)$ consists of r_i 1's followed by $(n - r_i)$ 0's. Let $A \in A(R, S)$. The discrepancy of A, $\operatorname{disc}(A)$, is the number of positions in which $F(R)$ has a 1 and A has a 0. In this paper we investigate linear operators mapping $m \times n$ matrices over the binary Boolean semiring to itself that preserve sets related to the discrepancy. In particular, we show that bijective linear preservers of Ferrers matrices are either the identity mapping or, when $m = n$, the transpose mapping.

Keywords: Ferrers matrix; row-dense matrix; discrepancy; linear preserver; strong linear preserver

MSC 2010: 15A04, 15A21, 15A86, 05B20, 05C50

1. Introduction

Graph theory, or equivalently $(0,1)$-matrix theory, plays an important role in the analysis of biological networks. Some obvious ones are the prey-predator models, the climate-growth models, the pollinator-plant models, etc. In the study of plant species versus biological pollinators, a bipartite graph is an obvious tool for analysis. To study the bipartite graph we often use the reduced adjacency matrix (a $(0,1)$-matrix).

A nested bipartite network has a reduced adjacency matrix that is equivalent to a Ferrers matrix, see [4], [6]. A measure of the "closeness" of a bipartite network to a nested one is the discrepancy, defined as the number of ones in the reduced adjacency matrix that must be interchanged with a zero in the same row to yield a Ferrers matrix. However, finding the discrepancy of a $(0,1)$-matrix is an NP-Complete problem, see [3]. Finding the isomorphic discrepancy is not so difficult, see [2]. In order to study such systems, one method is to identify a set of matrices with that property

DOI: 10.21136/CMJ.2019.0092-18

and apply transformations that preserve that property to expand the known set. In this article we will characterize the linear operators that preserve some sets related to the set of Ferrers matrices. For relevant definitions see the following section.

Let $\mathbb{B}$ denote the binary Boolean semiring. That is, $\mathbb{B} = \{0, 1\}$ with addition and multiplication the same as for the reals except that $1 + 1 = 1$. The semiring $\mathbb{B}$ is equivalent to the Boolean algebra of subsets of the set of one element where $\emptyset = 0$, $\{a\} = 1$, $+$ is union and $\times$ is intersection. We let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries from $\mathbb{B}$. Then $\mathcal{M}_{m,n}(\mathbb{B})$ is a semimodule with multiplication and addition defined as usual.

Let $E_{i,j}$ be the matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ which has exactly one nonzero entry, a one in the (i, j) position. The matrices $E_{i,j}$ are called *cells*. Let $J_{m,n} \in \mathcal{M}_{m,n}(\mathbb{B})$ denote the matrix of all ones, $O_{m,n} \in \mathcal{M}_{m,n}(\mathbb{B})$ denote the zero matrix and I_n denote the $n \times n$ identity matrix. If no confusion arises, we suppress the subscripts and write J, O and I.

A linear operator on $\mathcal{M}_{m,n}(\mathbb{B})$ is a mapping T which is additive, that is $T(A+B) = T(A) + T(B)$, and homogeneous, $T(\alpha A) = \alpha T(A)$. It is easily seen that a linear operator over $\mathbb{B}$ is also any additive map such that $T(O) = O$.

Due to the fact that $\mathcal{M}_{m,n}(\mathbb{B})$ is finite, the following proposition is easily established (see [1]).

Proposition 1.1. *Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. Then the following are equivalent:*
(1) *T is injective;*
(2) *T is surjective;*
(3) *T is bijective.*

Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. We say that T *preserves* a set $\mathcal{X} \subseteq \mathcal{M}_{m,n}(\mathbb{B})$ if $A \in \mathcal{X}$ implies that $T(A) \in \mathcal{X}$. The operator *strongly preserves* the set $\mathcal{X}$ if

$$A \in \mathcal{X} \text{ if and only if } T(A) \in \mathcal{X}. \tag{1.1}$$

Thus, "T strongly preserves the set $\mathcal{X}$" is equivalent to saying "T preserves the set $\mathcal{X}$ and T preserves the complement $\mathcal{M}_{m,n}(\mathbb{B}) \setminus \mathcal{X}$".

When mapping monoids whose addition is union or Boolean sum, such as in $\mathcal{M}_{m,n}(\mathbb{B})$, if T maps the whole monoid to a single element, then T preserves any set that contains that element, thus, in this case, to seriously investigate any set of preservers, additional conditions must be placed on the operator such as being bijective or strongly preserving the set.

Example 1.2. Let $T\colon \mathcal{M}_{n,n}(\mathbb{B}) \to \mathcal{M}_{n,n}(\mathbb{B})$ be defined by $T(X) = I_n$ for all $X \neq O$, and $T(O) = O$. Then T is linear and preserves the set of permutation matrices. But T maps every matrix to a permutaion matrix (I_n) except O. Clearly T does not strongly preserve any set except $\mathcal{M}_{n,n}(\mathbb{B})$ and the set $\{X \in \mathcal{M}_{m,n}(\mathbb{B})\colon\ X \neq O\}$.

Let f be a function on $\mathcal{M}_{m,n}(\mathbb{B})$. We say that T *preserves* f if T preserves the set $\{X \in \mathcal{M}_{m,n}(\mathbb{B});\ f(X) = r\}$ for each r in the image of f. That is,

(1.2) T preserves f if and only if for each r in the image of f, T (strongly) preserves $f^{-1}(r)$.

Given a $(0,1)$-matrix, let r_i be the number of nonzero entries in the ith row. Similarly c_j is the number of nonzero entries in the jth column. These are called the *ith row sum* and the *jth column sum* whether the matrix has real or Boolean entries.

The minimum and maximum value of the discrepancy of $(0,1)$-matrices in $A(R,S)$ was investigated by Brualdi and Shen in [5]. In the next section we shall define several subsets of $\mathcal{M}_{m,n}(\mathbb{B})$ that are related to Ferrers matrices and discrepancy. In Section 3 we will discuss the discrepancy of $(0,1)$-matrices. In Section 4 we will characterize linear preservers of Ferrers matrices and the set of matrices defined in Section 2. In the final section we will summarize the results of the previous sections and ask some relevant questions and state some conjectures.

2. Sets of $(0,1)$-matrices

There are several equivalent definitions for Ferrers matrices. The fact that they are equivalent is easily established.

Definition 2.1. Ferrers matrices:

Ferrers # 1. An $m \times n$ matrix of zeros and ones is called a *Ferrers matrix* if it has nonincreasing row sums and for each $i = 1, \ldots, m$ the ith row consists of r_i ones, followed by $n - r_i$ zeros.

Ferrers # 2. An $m \times n$ matrix A of zeros and ones is called a *Ferrers matrix* if $a_{i,j} = 1$ implies that for all $k \leqslant i$ and $l \leqslant j$, $a_{k,l} = 1$,

Ferrers # 3. An $m \times n$ matrix of zeros and ones is called a *Ferrers matrix* if it is the reduced adjacency matrix of a nested bipartite network. (From ecology).

Note that every 2×2 matrix of zeros and ones which has nonincreasing row and column sums is a Ferrers matrix, and the transpose of a Ferrers matrix is a Ferrers matrix. So, henceforth we assume that $2 \leqslant m \leqslant n$ and $3 \leqslant n$.

Let $\mathcal{P}_k$ denote the set of all $k \times k$ permutation matrices.

Definition 2.2. Let $\mathcal{F}M$ denote the set of all Ferrers matrices in $\mathcal{M}_{m,n}(\mathbb{B})$.

Let $\mathcal{I}_rFM = \{PA\colon P \in \mathcal{P}_m \text{ and } A \in \mathcal{F}M\}$. That is, $\mathcal{I}_rFM$ is the set of all matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ which are row permutations of a matrix in $\mathcal{F}M$.

Let $\mathcal{I}_cFM = \{AQ\colon Q \in \mathcal{P}_n \text{ and } A \in \mathcal{F}M\}$. That is, $\mathcal{I}_cFM$ is the set of all matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ which are column permutations of a matrix in $\mathcal{F}M$.

Let $\mathcal{I}FM = \{PAQ\colon P \in \mathcal{P}_m, Q \in \mathcal{P}_n \text{ and } A \in \mathcal{F}M\}$. That is, $\mathcal{I}FM$ is the set of all matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ which are equivalent (via row and column permutations) to a matrix in $\mathcal{F}M$. That is, members of $\mathcal{I}FM$ are the reduced adjacency matrices of graphs isomorphic to nested bipartite networks.

Let $\mathbb{Z}_+$ denote the set of all nonnegative integers so that $\mathbb{Z}_+^k$ is the set of all k tuples of nonnegative integers. Let $Q_{(m,n)} = \{(R,S);\ R \in \mathbb{Z}_+^m,\ S \in \mathbb{Z}_+^n,\ n \geqslant r_1 \geqslant r_2 \geqslant \ldots \geqslant r_m,\ m \geqslant s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n\}$. That is, $Q_{(m,n)}$ is the set of ordered pairs of nonincreasing sequences of length m and n from $\{0,1,2,\ldots,n\}$ and $\{0,1,2,\ldots,m\}$, respectively.

Let $(R,S) \in Q_{(m,n)}$ and define $A(R,S)$ to be the subset of $\mathcal{M}_{m,n}(\mathbb{B})$ consisting of matrices with r_i nonzero entries in row i and s_j nonzero entries in column j, where r_i is the ith component of R and s_j is the jth component of S. Note that in order that $A(R,S) \neq \emptyset$, we must have that $r_1 + r_2 + \ldots + r_m = s_1 + s_2 + \ldots + s_n$.

3. Discrepancy

Given an $m \times n$ matrix A of zeros and ones which has nonincreasing row and column sums, the discrepancy of A, $\operatorname{disc}(A)$ or $\operatorname{BR}(A)$, is a measure of how near that matrix is to a Ferrers matrix.

Definition 3.1. Let $B \in A(R,S)$ for some $(R,S) \in Q_{(m,n)}$. The *discrepancy of* B, $\operatorname{disc}(B)$, is the minimum number of ones exchanged with zeros in the same row of B that yields a Ferrers matrix. That is, if $F(R,S)$ is the Ferrers matrix whose row sums are the same as the row sums of B, then the discrepancy of B is the number of entries of B that are zero and the corresponding entry of $F(R,S)$ is one.

As seen in the following example, the discrepancy of a $(0,1)$-matrix is not independent of permutation of columns which maintain the nonincreasing nature of the columns.

Example 3.2. Consider the two matrices:

$$A = \begin{bmatrix} 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & \mathbf{0} & \mathbf{1} \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \mathbf{0} & \mathbf{1} & 0 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & \mathbf{0} & 1 & \mathbf{1} \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & \mathbf{0} & 0 & \mathbf{1} \end{bmatrix}.$$

Both are in $A((3,3,3,2,2,2),(6,3,3,3))$ and both can be reduced to the Ferrers matrix

$$F = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

by exchanging the bold ones and zeros in each row. The discrepancy of A is 4 while the discrepancy of A' is 3. Note that A' is achieved from A by permuting the last two columns.

The discrepancy is only defined on the sets $A(R,S)$ for $(R,S) \in Q_{(m,n)}$. We now define a general discrepancy as a function of any member of $\mathcal{M}_{m,n}(\mathbb{B})$. For matrices not in $A(R,S)$ for some $(R,S) \in Q_{(m,n)}$, let the discrepancy be ∞.

Definition 3.3. Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$ and let the *general discrepancy of* A be $\mathrm{Gdisc}(A) = \min\{\mathrm{disc}(PAQ);\ P \in \mathcal{P}_m,\ Q \in \mathcal{P}_n, \text{ and } PAQ \in A(R,S) \text{ for some } (R,S) \in Q_{(m,n)}\}$, that is, $\mathrm{Gdisc}(A)$ is the minimum of the discrepancies of PAQ, where PAQ has nonincreasing row sums and column sums and $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$.

Clearly, $\mathrm{disc}(A) \geqslant \mathrm{Gdisc}(A)$ for any $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

The general discrepancy is equivalent to the isomorphic discrepancy in Berger and Schreck [3] for matrices with nonincreasing row sums and nonincreasing column sums.

4. Preservers

Note that for transformations $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{K}$ for $\mathcal{K}$ a monoid, saying T is nonsingular means that $T(X) = O$ only if $X = O$. Unlike transformations on vector spaces (over a field), nonsingularity does not imply invertibility. In the following we let $R_i = \sum_{j=1}^{n} E_{i,j}$, the ith row of the J matrix, and $C_j = \sum_{i=1}^{m} E_{i,j}$, the jth column of the J matrix. Let $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$. We say that A *dominates* B if $a_{ij} = 0$ implies that $b_{ij} = 0$. This is denoted by $A \sqsupseteq B$ or $B \sqsubseteq A$. The following proposition gives some of the properties of linear operators on Boolean matrices that we shall find useful.

Proposition 4.1. *Let* $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ *be bijective.*

(1) *If T preserves a set $\mathcal{X}$, then T strongly preserves $\mathcal{X}$.*

(2) *The image of a cell is a cell.*

(3) *The image of a matrix A has the same number of nonzero entries as does A.*

Proof. (1) Since T is injective, $\mathcal{X}$ is finite, and $T(\mathcal{X}) \subseteq \mathcal{X}$, we have $T(\mathcal{X}) = \mathcal{X}$.

(2) If E is a cell and $T(E)$ is not, then $T(E)$ dominates at least two cells since T is nonsingular. Thus, there exist at most $mn-2$ cells, $E_1, E_2, \ldots, E_j$ $(j \leqslant mn-2)$ such that $T(J) = T(E+(E_1+\ldots+E_j))$, and $E+(E_1+\ldots+E_j) \neq J$, contradicting the injectivity of T.

(3) This follows from case (2) and from the facts that T is bijective and $\mathcal{M}_{m,n}(\mathbb{B})$ is finite. □

Theorem 4.2. *Let* $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ *be a bijective linear operator that maps $\mathcal{F}M$ to itself. Then either:*

(1) *T is the identity; or*

(2) *$m = n$ and T is the transpose operator.*

Proof. Since T is bijective, by Proposition 4.1, $T(E_{1,1}) = E_{r,s}$ for some r and s. But the only member of $\mathcal{F}M$ that has only one nonzero entry is $E_{1,1}$. Thus, $T(E_{1,1}) = E_{1,1}$.

Consider $T(E_{1,2})$. Since the only two members of $\mathcal{F}M$ with exactly two nonzeros are $E_{1,1} + E_{1,2}$ and $E_{1,1} + E_{2,1}$, suppose that $m < n$ and $T(E_{1,2}) = E_{2,1}$. Then $T(E_{1,3})$ must be $E_{3,1}$ because T is bijective and by Proposition 4.1 T strongly preserves $\mathcal{F}M$, for the only possible other choice would be $E_{1,2}$ which is impossible because then $T(E_{1,1}+E_{1,3})$ would be a member of $\mathcal{F}M$ contracting Proposition 4.1, case (1). Following this pattern, we arrive at $T(R_1) \sqsubseteq C_1$, an impossibility, since T is bijective on the set of cells and $m < n$. It now follows that $T(E_{1,2}) = E_{1,2}$ and that $T(R_1) = R_1$. Parallel to this we get that $T(C_1) = C_1$, and since $T(E_{j,1}) = E_{j,1}$, we get that $T(R_i) = R_i$ and $T(C_j) = C_j$. That is, T is the identity.

If $m = n$, it is possible to have that $T(E_{1,2}) = E_{2,1}$ and substituting rows for columns, following the above proof, we get that T is the transpose operator. □

Given an arbitrary matrix $Q \in \mathcal{M}_{m,n}(\mathbb{B})$ we shall use the notation $J \setminus Q$ to denote the matrix whose entries are zero wherever Q has entry one and one wherever Q

has entry zero. So $J \setminus E_{m,n}$ is the matrix all of whose entries are one except the (m,n)-entry which is zero.

Theorem 4.3. *Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that maps $\mathcal{F}M$ into itself. Then T strongly preserves $\mathcal{F}M$ if and only if T is bijective.*

Proof. If T is bijective, then T strongly preserves $\mathcal{F}M$ by Proposition 4.1, case (1). Thus, suppose that T strongly preserves $\mathcal{F}M$.

Suppose $T(E_{m,n}) = O$. Since $E_{1,1} \in \mathcal{F}M$ and $E_{1,1} + E_{m,n} \notin \mathcal{F}M$, $T(E_{1,1}) \in \mathcal{F}M$ and $T(E_{1,1} + E_{m,n}) \notin \mathcal{F}M$. However, $T(E_{m,n}) = O$, so $T(E_{1,1}) = T(E_{1,1} + E_{m,n})$, a contradiction. Thus $T(E_{m,n}) \neq O$.

Now suppose that $T(J) = T(Z)$ for some $Z \neq J$. Starting with $E_{m,n}$, add cells $F_1, \ldots, F_j$ such that $T(J) = T(E_{m,n} + F_1 + \ldots + F_j)$. Since $T(J) = T(Z)$ and $Z \neq J$, we have that $j \leqslant mn - 2$. Thus, there is some cell Q such that $T(J) = T(J \setminus Q)$ and $Q \neq E_{m,n}$, a contradiction since $J \in \mathcal{F}M$ and $J \setminus Q \notin \mathcal{F}M$.

If the image of a cell is O or if the image of a cell has more than one nonzero entry, then for some cell Q, $T(J) = T(J \setminus Q)$ and $Q \neq E_{m,n}$. Thus, T maps cells to cells and $T(J) = J$, so T is bijective. □

Corollary 4.4. *Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves the discrepancy of every matrix in $\mathcal{M}_{m,n}(\mathbb{B})$. Then either:*

(1) *T is the identity; or*

(2) *$m = n$ and T is the transpose operator.*

Proof. If T preserves the discrepancy of every matrix in $\mathcal{M}_{m,n}(\mathbb{B})$, then T strongly preserves the set of matrices of discrepancy zero, the Ferrers matrices $\mathcal{F}M$. The corollary follows by applying Theorem 4.3 to Theorem 4.2. □

Theorem 4.5. *Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a bijective linear operator. Then*

(1) *T preserves $\mathcal{I}_r FM$ if and only if $T(X) = PX$ for some $P \in \mathcal{P}_m$;*

(2) *T preserves $\mathcal{I}_c FM$ if and only if $T(X) = XQ$ for some $Q \in \mathcal{P}_n$; and*

(3) *T preserves $\mathcal{I}FM$ if and only if*

  (a) *$T(X) = PXQ$ for some $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$ or*

  (b) *$m = n$ and $T(X) = PX^tQ$ for some $P, Q \in \mathcal{P}_n$.*

Proof. (1) If $T(X) = PX$ for some $P \in \mathcal{P}_m$, then clearly T preserves $\mathcal{I}_r FM$. So assume that T preserves $\mathcal{I}_r FM$.

Consider $T(E_{i,1})$. Since T is bijective, $T(E_{i,1})$ is a cell and is in $\mathcal{I}_r FM$, so that the nonzero entry 1 is in the first column. Let $T(E_{i,1}) = E_{\sigma(i),1}$. Then σ is a permutation since T is bijective. Now consider $T(E_{i,2})$. Since $T(E_{i,1} + E_{i,2})$ must be in $\mathcal{I}_r FM$, we must have $T(E_{i,2}) = E_{\sigma(i),2}$. Continuing in this way we get that $T(E_{i,j}) = E_{\sigma(i),j}$ for all i and j. That is, $T(X) = PX$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$, where P is the permutation matrix corresponding to the permutation σ.

(2) The proof is parallel to the proof of case (1), arguing on the columns instead of rows.

(3) The proof is parallel to the proof of case (1) since if $T(E_{i,j}) = E_{r,s}$, then the image of any cell in row i is mapped to a cell in row r and the image of any cell in column j is mapped to a cell in column s, unless $m = n$ in which case we may have that the image of any cell in row i is mapped to a cell in column s and the image of any cell in column j is mapped to a cell in row r. □

Theorem 4.6. *Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. Then*
(1) *T strongly preserves $\mathcal{I}_rFM$ if and only if $T(X) = PX$ for some $P \in \mathcal{P}_m$;*
(2) *T strongly preserves $\mathcal{I}_cFM$ if and only if $T(X) = XQ$ for some $Q \in \mathcal{P}_n$; and*
(3) *T strongly preserves $\mathcal{I}FM$ if and only if*
  (a) *$T(X) = PXQ$ for some $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$ or*
  (b) *$m = n$ and $T(X) = PX^tQ$ for some $P, Q \in \mathcal{P}_n$.*

Proof. The proof of each part is parallel to the proof of Theorem 4.3. □

Corollary 4.7. *Let $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator that preserves the function* Gdisc. *Then either:*
(1) *$T(X) = PXQ$ for some $P \in \mathcal{P}_m$ and $Q \in \mathcal{P}_n$ or*
(2) *$m = n$ and $T(X) = PX^tQ$ for some $P, Q \in \mathcal{P}_n$.*

Proof. If T preserves Gdisc, then T strongly preserves the set of matrices of generalized discrepancy zero, the set of matrices $\mathcal{I}FM$. □

5. Summary, Questions and Conjectures

We know that in general $\operatorname{disc}(A)$ and $\operatorname{Gdisc}(A)$ may be different as seen in Example 3.2. In fact, for $A = \begin{bmatrix} 1&1&0&1 \\ 1&0&1&0 \end{bmatrix}$, which has discrepancy 2, if one interchanges columns 3 and 4, we get $B = \begin{bmatrix} 1&1&1&0 \\ 1&0&0&1 \end{bmatrix}$, which has discrepancy 1. We also know that if $\operatorname{disc}(A) = 0$, then $\operatorname{Gdisc}(A) = 0$. This leads to some questions:

Question 5.1. Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B})$ whose general discrepancy is k, what is the largest value of $\operatorname{disc}(A)$?

Question 5.2. Given $A \in A(R,S)$ for some $(R,S) \in Q_{(m,n)}$, if $\operatorname{disc}(A) = k$ and $\operatorname{Gdisc}(A) = l$, is there a matrix $B \in A(R,S)$ such that $\operatorname{disc}(B) = q$ for any $k \leqslant q \leqslant l$?

And concerning linear preservers, we make the following question and conjecture:

Question 5.3. The discrepancy of a real matrix can be defined, relating a weighted bipartite graph to a reduced real adjacency matrix. What are the preservers of the corresponding matrix sets?

Conjecture 5.4. Let $k \geqslant 0$ be "small". If $T\colon \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is bijective and preserves the set of matrices of discrepancy k, then either

(1) T is the identity; or

(2) $m = n$ and T is the transpose operator.

Acknowledgements. The author wishes to thank the referee whose many suggestions improved the presentation.

References

[1] *L. B. Beasley, N. J. Pullman*: Linear operators preserving properties of graphs. Proc. 20th Southeast Conf. on Combinatorics, Graph Theory, and Computing. Congressus Numerantium 70, Utilitas Mathematica Publishing, Winnipeg, 1990, pp. 105–112. zbl MR

[2] *A. Berger*: The isomorphic version of Brualdies nestedness is in P, 2017, 7 pages. Available at https://arxiv.org/abs/1602.02536v2.

[3] *A. Berger, B. Schreck*: The isomorphic version of Brualdi's and Sanderson's nestedness. Algorithms (Basel) *10* (2017), Paper No. 74, 12 pages. zbl MR doi

[4] *R. A. Brualdi, G. J. Sanderson*: Nested species subsets, gaps, and discrepancy. Oecologia *119* (1999), 256–264. doi

[5] *R. A. Brualdi, J. Shen*: Discrepancy of matrices of zeros and ones. Electron. J. Comb. *6* (1999), Research Paper 15, 12 pages. zbl MR

[6] *S. M. Motlaghian, A. Armandnejad, F. J. Hall*: Linear preservers of row-dense matrices. Czech. Math. J. *66* (2016), 847–858. zbl MR doi

Author's address: LeRoy B. Beasley, Department of Mathematics and Statistics, Utah State University, Logan, Utah 84322-3900, USA, e-mail: leroy.b.beasley@aggiemail.usu.edu.