Tomáš Rusin
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A NOTE ON THE COHOMOLOGY RING OF THE ORIENTED GRASSMANN MANIFOLDS $\tilde{G}_{n,4}$

Tomáš Rusin

ABSTRACT. We use known results on the characteristic rank of the canonical 4–plane bundle over the oriented Grassmann manifold $\tilde{G}_{n,4}$ to compute the generators of the $\mathbb{Z}_2$–cohomology groups $H^j(\tilde{G}_{n,4})$ for $n = 8, 9, 10, 11$. Drawing from the similarities of these examples with the general description of the cohomology rings of $\tilde{G}_{n,3}$ we conjecture some predictions.

1. Introduction

Let us denote $G_{n,k}$ the Grassmann manifold of $k$–dimensional vector subspaces in $\mathbb{R}^n$, i.e. the space $O(n)/(O(k) \times O(n - k))$. Next, denote $\tilde{G}_{n,k}$ the oriented Grassmann manifold of oriented $k$-dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k) \times SO(n - k))$. We may suppose that $k \leq n - k$ for both of them.

The manifolds $G_{n,k}$ and $\tilde{G}_{n,k}$ come equipped with their canonical $k$-plane bundles, which we denote $\gamma_{n,k}$ and $\tilde{\gamma}_{n,k}$ respectively.

For the Grassmann manifold $G_{n,k}$ there is a concise description of its $\mathbb{Z}_2$-cohomology ring as a quotient ring of a polynomial ring (see [2])

$$H^*(G_{n,k}; \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{n,k},$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by $k$ homogeneous polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \ldots, \bar{w}_n$, where each $\bar{w}_i$ denotes the $i$-dimensional component of the formal power series

$$1 + (w_1 + w_2 + \cdots + w_k) + (w_1 + w_2 + \cdots + w_k)^2 + (w_1 + w_2 + \cdots + w_k)^3 + \cdots.$$ 

Each indeterminate $w_i$ is a representative of the $i$th Stiefel-Whitney class $w_i(\gamma_{n,k})$ of the canonical $k$-plane bundle $\gamma_{n,k}$ over $G_{n,k}$.

However, the cohomology ring of the oriented Grassmann manifold $\tilde{G}_{n,k}$ is not fully generated by the characteristic classes $w_i(\tilde{\gamma}_{n,k})$ and is not known in general. There are descriptions of $H^*(\tilde{G}_{n,k}; \mathbb{Z}_2)$ for spheres $\tilde{G}_{n,1} \cong S^{n-1}$, complex quadrics $\tilde{G}_{n,2}$, and in [1] for $\tilde{G}_{n,3}$ as well.

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In this paper we begin the study of the $\mathbb{Z}_2$-cohomology ring of $\tilde{G}_{n,4}$ by considering the cases $n = 8, 9, 10, 11$. We will abbreviate $H^j(X;\mathbb{Z}_2)$ to $H^j(X)$, denote $w_i = w_i(\gamma_{n,k})$ and $\tilde{w}_i = w_i(\tilde{\gamma}_{n,k})$ as usual.

The paper is organized as follows. In the second section we review the general strategy on how to approach the study of $H^*(\tilde{G}_{n,k})$. It contains the tools which will be used later to perform the computations. The third section contains the main result of the paper, which is the complete description of all cohomology groups of $\tilde{G}_{n,4}$ for $n = 8, 9, 10, 11$, along with partial information about the ring structure of $H^*(\tilde{G}_{n,4})$. Some conjectures are also discussed based on these results.

2. Preliminaries

To obtain information about $H^j(\tilde{G}_{n,4})$, we first need to recall some general facts about $H^j(\tilde{G}_{n,k})$. We proceed similarly as in [4].

There is a covering projection $p: \tilde{G}_{n,k} \to G_{n,k}$, which is universal for $(n, k) \neq (2, 1)$. To this 2-fold covering, there is an associated line bundle $\xi$ over $G_{n,k}$, such that $w_1(\xi) = w_1(\gamma_{n,k})$, to which we have Gysin exact sequence (Corollary 12.3)

\[ \begin{array}{cccc}
\psi & H^j-1(G_{n,k}) & w_1 & H^j(G_{n,k}) & p^* \to H^j(\tilde{G}_{n,k}) & \psi & H^j(G_{n,k}) & w_1 \\
\end{array} \]  

where $H^j-1(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel-Whitney class $w_1 = w_1(\gamma_{n,k})$.

Since the pullback $p^*\gamma_{n,k}$ is isomorphic to $\tilde{\gamma}_{n,k}$, the covering projection $p: \tilde{G}_{n,k} \to G_{n,k}$ induces the ring homomorphism $p^*: H^*(G_{n,k}) \to H^*(\tilde{G}_{n,k})$, which maps each Stiefel-Whitney class $w_i$ to $\tilde{w}_i$.

Consequently, the image $\text{Im}(p^*: H^j(G_{n,k}) \to H^j(\tilde{G}_{n,k}))$ is a subspace of the $\mathbb{Z}_2$-vector space $H^j(\tilde{G}_{n,k})$ consisting only of cohomology classes, which can be expressed as polynomials in the Stiefel-Whitney characteristic classes of $\tilde{\gamma}_{n,k}$. We will call it the characteristic subspace and denote it $C(j; n, k)$. Moreover (see [9]), the image $\text{Im}(p^*)$ of the ring homomorphism $p^*: H^*(G_{n,k}) \to H^*(\tilde{G}_{n,k})$ is a self-annihilating subspace of $H^*(\tilde{G}_{n,k})$. That is, we have the following.

**Lemma 2.1.** For any $\tilde{x} \in C(j; n, k)$ and $\tilde{y} \in C(j'; n, k)$ we have $\tilde{x}\tilde{y} = 0$ if $j + j' = k(n - k) = \dim(\tilde{G}_{n,k})$.

From the exactness of the sequence (2.1), we have $\tilde{w}_1 = p^*(w_1) = 0$ and it is clear that a monomial $\tilde{w}_2^{a_2}\tilde{w}_3^{a_3} \ldots \tilde{w}_k^{a_k} = p^*(w_2^{a_2}w_3^{a_3} \ldots w_k^{a_k})$ is zero in $H^j(\tilde{G}_{n,k})$ if and only if $w_2^{a_2}w_3^{a_3} \ldots w_k^{a_k}$ is a $w_1$-multiple of some polynomial in $H^*(G_{n,k})$. Let us therefore denote $g_i \in \mathbb{Z}_2[w_2, \ldots, w_k]$ the reduction of the polynomial $\tilde{w}_i$ (see (1.1)) modulo $w_1$ and by $J_{n,k}$ the ideal in $\mathbb{Z}_2[w_2, \ldots, w_k]$ generated by $g_{n-k+1}, \ldots, g_n$. The following lemma is a formal restatement of the previous observation.

**Lemma 2.2.** Monomial $\tilde{w}_2^{a_2}\tilde{w}_3^{a_3} \ldots \tilde{w}_k^{a_k} \in C(j; n, k)$ is equal to zero iff $w_2^{a_2}w_3^{a_3} \ldots w_k^{a_k} \in J_{n,k}$.

The question whether $C(j; n, k)$ is equal to $H^j(\tilde{G}_{n,k})$ is related to the notion of the characteristic rank of a vector bundle, which was defined in [3, 7].
**Definition 2.3.** Let $X$ be a connected, finite CW-complex and $\xi$ a real vector bundle over $X$. The **characteristic rank** of the vector bundle $\xi$, $\text{charrank}(\xi)$, is the greatest integer $q$, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(\xi)$ of $\xi$.

This implies that the characteristic rank of $\tilde{\gamma}_{n,k}$ is equal to the greatest integer $q$, such that the homomorphism $p^*: H^j(G_{n,k}) \rightarrow H^j(\tilde{G}_{n,k})$ is surjective (that is $C(j; n, k) = H^j(\tilde{G}_{n,k})$) for all $j$, $0 \leq j \leq q$, or equivalently, by (2.1), that the homomorphism $w_1: H^j(G_{n,k}) \rightarrow H^{j+1}(G_{n,k})$ is injective for all $j$, $0 \leq j \leq q$.

Hence, in order to compute the characteristic rank of $\tilde{\gamma}_{n,k}$, it is necessary to study the kernel of $w_1: H^j(G_{n,k}) \rightarrow H^{j+1}(G_{n,k})$. Let us denote $b_j(X)$ the $j$th $\mathbb{Z}_2$–Betti number of a manifold $X$ and then define $\alpha_j(\tilde{G}_{n,k}) = b_j(\tilde{G}_{n,k}) - \dim(C(j; n, k))$, the codimension of the subspace $C(j; n, k) \subseteq H^j(\tilde{G}_{n,k})$.

There is a useful upper bound for this number described in the next proposition.

**Proposition 2.4 (5 Proposition 2.4. (3))).** For a non-negative integer $x$, we associate with $H^{n-k+x+1}(G_{n,k})$ ($2 \leq k \leq n-k$) the set

$$N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} \cdots w_k^{b_k}g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}.$$ 

If $x \leq n - k - 1$ and there are $t$ linearly independent elements in the set $N_x(G_{n,k})$, then

$$\alpha_{n-k+x}(\tilde{G}_{n,k}) \leq |N_x(G_{n,k})| - t,$$

where $|N_x(G_{n,k})|$ is the cardinality of the set $N_x(G_{n,k})$.

When $j \leq \text{charrank}(\tilde{\gamma}_{n,k})$ we have (see [4 (3)])

$$(2.2) \quad b_j(\tilde{G}_{n,k}) = b_j(G_{n,k}) - b_{j-1}(G_{n,k})$$

and the Betti numbers for $G_{n,k}$ are readily calculable from the Poincaré polynomial [2]

$$(2.3) \quad P_t(G_{n,k}) = \frac{(1 - t^{n-k+1}) \cdots (1 - t^n)}{(1-t) \cdots (1-t^k)}.$$ 

3. Computations

Recently, the number $\text{charrank}(\tilde{\gamma}_{n,4})$ was completely determined [8] and adjusted to our notation we have the following.

**Theorem 3.1 (8 Theorem 6.6)).** Let $n \geq 8$ be an integer. If $t \geq 3$ is the unique integer such that $2^{t-1} < n \leq 2^t$, then

$$\text{charrank}(\tilde{\gamma}_{n,4}) = \min \{4n - 3 \cdot 2^{t-1} - 5, 2^t - 5\}.$$ 

For better clarity of the forthcoming proofs we first list generators of the ideals $J_{n,4}$ and derive some additional relations in cohomology implied by Lemma [2,2].
Lemma 3.2. We have

\[ J_{8,4} = J_{9,4} = (g_6, g_7, g_8) = (w_2^3 + w_3^2, w_2 w_3, w_2^2 w_3 + w_2 w_3^2 + w_2^3 w_4 + w_2^5 w_4), \]
\[ J_{10,4} = (w_2^3 w_3, w_2^4 + w_2 w_3^2 + w_2^2 w_4 + w_2^4, w_3^4, w_3^5 + w_3^2 w_4 + w_2 w_2^2), \]
\[ J_{11,4} = (w_2^3 + w_2 w_3^2 + w_2^3 w_4 + w_2^4, w_3^4, w_3^5 + w_3^2 w_4 + w_2 w_2^2, w_2 w_3 + w_3 w_2^2). \]

Additionally, in \( H^*(\tilde{G}_{8,4}) \) and \( H^*(\tilde{G}_{9,4}) \) we have

\[ \tilde{w}_4^2 = \tilde{w}_2^2 \tilde{w}_4, \quad \tilde{w}_3 \tilde{w}_4^2 = 0, \quad \tilde{w}_3^3 = 0, \quad \tilde{w}_5 = 0, \]
\[ \tilde{w}_3^2 \tilde{w}_4 = \tilde{w}_2^4 \tilde{w}_4. \]

In \( H^*(\tilde{G}_{10,4}) \) we have

\[ \tilde{w}_3^3 \tilde{w}_4^2 = \tilde{w}_3^4 \tilde{w}_4, \quad \tilde{w}_3 \tilde{w}_4^3 = 0, \quad \tilde{w}_2^2 + \tilde{w}_2^4 \tilde{w}_4 = \tilde{w}_2^2 \tilde{w}_4^5, \]
\[ \tilde{w}_4^4 = \tilde{w}_2^3 \tilde{w}_4^2. \]

In \( H^*(\tilde{G}_{11,4}) \) we have

\[ \tilde{w}_2^5 \tilde{w}_3^2 + \tilde{w}_3^3 \tilde{w}_4^3 = \tilde{w}_2^5 \tilde{w}_4^2, \quad \tilde{w}_2 \tilde{w}_3^2 \tilde{w}_4 = 0, \quad \tilde{w}_3^3 \tilde{w}_4^2 = \tilde{w}_2^3 \tilde{w}_4^2, \]
\[ \tilde{w}_2^6 \tilde{w}_3^2 = \tilde{w}_2^3 \tilde{w}_4^2, \quad \tilde{w}_2^7 = \tilde{w}_2^4 \tilde{w}_4^2, \quad \tilde{w}_2 \tilde{w}_4^2 = 0, \quad \tilde{w}_2^5 \tilde{w}_4^2 = \tilde{w}_2^4 \tilde{w}_4^2, \]
\[ \tilde{w}_4^2 \tilde{w}_4 = \tilde{w}_2^4 \tilde{w}_4^2. \]

Proof. Direct computation of the polynomials \( g_i \in \mathbb{Z}_2[w_2, w_3, w_4] \) shows that \( g_5 = 0 \) and \( g_6, \ldots, g_9 \) are as claimed. Since \( g_5 = 0 \) we have \((g_5, g_6, g_7, g_8) = (g_6, g_7, g_8)\). However \( g_9 = w_3^3 = w_2 g_7 + w_3 g_6 \), thus also \((g_6, g_7, g_8) = (g_6, g_7, g_8)\).

By definition, both \( J_{8,4} \) and \( J_{9,4} \) are equal to \((g_6, g_7, g_8)\).

Now, since \( J_{8,4} = J_{9,4} \), by Lemma 2.2 the relations in \( H^*(\tilde{G}_{8,4}) \) and \( H^*(\tilde{G}_{9,4}) \) for the elements of the characteristic subspace will be the same. We will check them in \( H^*(\tilde{G}_{8,4}) \). The proof for \( H^*(\tilde{G}_{9,4}) \) is identical. Since \( w_2 g_6 + g_8 \in J_{8,4} \), we have \( \tilde{w}_2^3 \tilde{w}_4 + \tilde{w}_4^2 = 0 \), which is equivalent to \( \tilde{w}_4^3 = \tilde{w}_2^3 \tilde{w}_4 \). We have already shown that \( w_3^3 \in J_{8,4} \), thus \( \tilde{w}_3^3 = 0 \). We have \( w_5^2 = w_2^3 g_6 + w_3 g_7 \in J_{8,4} \), therefore we obtain \( \tilde{w}_2^7 = 0 \). Next \( \tilde{w}_2^3 \tilde{w}_4 = \tilde{w}_2^4 \tilde{w}_3 \tilde{w}_4 \) by the first relation and the latter is zero because \( w_4 g_7 \in J_{8,4} \). Finally, \( \tilde{w}_2^3 \tilde{w}_4^2 = \tilde{w}_2^4 \tilde{w}_4^2 \) by the same reason.

In \( H^*(\tilde{G}_{10,4}) \) we have \( \tilde{w}_3^3 \tilde{w}_4^2 + \tilde{w}_3^2 \tilde{w}_4^3 = 0 \) since \( w_3^3 \tilde{w}_4^2 + w_3^2 \tilde{w}_4^3 = w_3 \tilde{w}_4 + w_3^2 \tilde{w}_4 = w_3 \tilde{w}_4 + w_3 \tilde{w}_4 \in J_{10,4} \). It is easy to check that \( w_2^3 \tilde{w}_4^2 = (w_2^3 + w_4) \tilde{w}_4 + w_3 \tilde{w}_4 \in J_{10,4} \). Next \( \tilde{w}_3^3 \tilde{w}_4^2 + \tilde{w}_3^2 \tilde{w}_4^3 = \tilde{w}_3^3 \tilde{w}_4^2 + \tilde{w}_3^3 \tilde{w}_4^2 = \tilde{w}_3^3 \tilde{w}_4^2 \) by the first relation and \( w_3^3 \tilde{w}_4^2 \in J_{10,4} \). Finally \( \tilde{w}_3^3 = \tilde{w}_3^3 \tilde{w}_4 + \tilde{w}_3^3 \tilde{w}_4^2 + \tilde{w}_3^3 \tilde{w}_4^3 \) since \( w_3^3 \tilde{w}_4^2 \in J_{10,4} \) and the first two summands are equal as they are \( w_2 \)-multiples of equal classes.

In \( H^*(\tilde{G}_{11,4}) \) we have \( \tilde{w}_2^3 \tilde{w}_3^2 + \tilde{w}_3^2 \tilde{w}_4 = \tilde{w}_2^3 \tilde{w}_4^2 \), since \( w_2^3 \tilde{w}_3^2 + w_3^2 \tilde{w}_4^2 = w_2^3 \tilde{w}_3^2 + w_3^2 \tilde{w}_4^2 = w_3^2 w_4 \in J_{11,4} \). Then we have \( w_2^3 \tilde{w}_3^2 w_4 = w_3 \tilde{w}_4, g_9 + g_1 \in J_{11,4} \). Next, we have \( w_3^3 \tilde{w}_3^2 w_4 = (w_3^3 + w_2^3 \tilde{w}_4) g_9 + w_3^2 g_9 + w_3 g_1 \in J_{11,4} \). Since \( w_3^2 g_9 \in J_{11,4} \), we have \( \tilde{w}_2^7 = \tilde{w}_2^4 \tilde{w}_3 \tilde{w}_4 + \tilde{w}_2^3 \tilde{w}_4^2 \), but the first summand is zero and the latter is equal to \( \tilde{w}_2^3 \tilde{w}_4^2 \) by the third relation, which is equal to \( \tilde{w}_2^4 \tilde{w}_4^2 \), because \( w_3 g_9 \in J_{11,4} \). Since \( w_3^2 g_9 + w_3 g_1 \in J_{11,4} \), we have \( \tilde{w}_2^7 + \tilde{w}_2^3 \tilde{w}_4^2 + \tilde{w}_2^4 \tilde{w}_4^2 + \tilde{w}_2^3 \tilde{w}_4^2 = 0 \), but the last two summands are equal by the third relation. Since \( w_2^3 \tilde{w}_4 g_8 + w_2^2 \tilde{w}_4 \in J_{11,4} \), we have \( \tilde{w}_2^3 + \tilde{w}_2^3 \tilde{w}_4 + \tilde{w}_2^3 \tilde{w}_4^2 = 0 \), but the first two summands are equal by the previous relation.
Since $w_2w_3g_{11} \in J_{11,4}$, we have $\tilde{w}_0^{2} \tilde{w}_3^{2} = \tilde{w}_2 \tilde{w}_3 \tilde{w}_4^{2}$ and the RHS is equal to $\tilde{w}_2^{4} \tilde{w}_4^{2}$ by the third relation. Since $w_2w_4g_{10} \in J_{11,4}$, we have $\tilde{w}_0^{2} \tilde{w}_4 + \tilde{w}_2 \tilde{w}_3 \tilde{w}_4^{2} + \tilde{w}_2^{2} \tilde{w}_4^{2} = 0$, but the last summand is $\tilde{w}_2$–multiple of zero and the middle one is equal to $\tilde{w}_2^{4} \tilde{w}_4^{2}$ and we obtain the desired result. \hfill \Box

**Theorem 3.3.** We have the following generators of $H^3(\tilde{G}_{8,4})$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>gen.</th>
<th>$j$</th>
<th>gen.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\tilde{w}_0$</td>
<td>9</td>
<td>$a_4 \tilde{w}_2 \tilde{w}_3, \tilde{w}_2 \tilde{w}_3 \tilde{w}_4$</td>
</tr>
<tr>
<td>1</td>
<td>$\tilde{w}_2$</td>
<td>10</td>
<td>$a_4 \tilde{w}_2^3, a_4 \tilde{w}_2 \tilde{w}_4, \tilde{w}_2^3 \tilde{w}_4$</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{w}_3$</td>
<td>11</td>
<td>$a_4 \tilde{w}_3 \tilde{w}_4$</td>
</tr>
<tr>
<td>3</td>
<td>$a_4, \tilde{w}_2^2, \tilde{w}_4$</td>
<td>12</td>
<td>$a_4 \tilde{w}_2^4, a_4 \tilde{w}_2 \tilde{w}_4, \tilde{w}_2^4 \tilde{w}_4$</td>
</tr>
<tr>
<td>4</td>
<td>$\tilde{w}_2 \tilde{w}_3$</td>
<td>13</td>
<td>$a_4 \tilde{w}_2 \tilde{w}_3 \tilde{w}_4$</td>
</tr>
<tr>
<td>5</td>
<td>$a_4 \tilde{w}_2, \tilde{w}_3 \tilde{w}_4$</td>
<td>14</td>
<td>$a_4 \tilde{w}_2^3 \tilde{w}_4$</td>
</tr>
<tr>
<td>6</td>
<td>$a_4 \tilde{w}_2 \tilde{w}_3, \tilde{w}_2 \tilde{w}_4$</td>
<td>15</td>
<td>$-$</td>
</tr>
<tr>
<td>7</td>
<td>$a_4 \tilde{w}_3 \tilde{w}_4$</td>
<td>16</td>
<td>$a_4 \tilde{w}_2^4 \tilde{w}_4$</td>
</tr>
<tr>
<td>8</td>
<td>$a_4 \tilde{w}_2^2, a_4 \tilde{w}_4, \tilde{w}_2^4 \tilde{w}_4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where $a_4$ is an element in $H^4(\tilde{G}_{8,4}) \setminus C(4; 8, 4)$.

**Proof.** We have $\text{charrank}(\tilde{G}_{8,4}) = 3$, so for $j \leq 3$ we have $C(j; 8, 4) = H^j(\tilde{G}_{8,4})$, but $C(4; 8, 4) \subset H^4(\tilde{G}_{8,4})$ is a proper subspace and thus for the codimension we have $\alpha_4(\tilde{G}_{8,4}) = b_j(\tilde{G}_{n,k}) - \dim(C(j; n, k)) \geq 1$. On the other hand, from Proposition 2.4 we have $\alpha_4(\tilde{G}_{8,4}) \leq 1$ since $x = 0$ and $N_0(G_{8,4}) = \{g_5\}$ is a one element set. Let us denote $a_4 \in H^4(\tilde{G}_{8,4})$ an element outside $C(4; 8, 4)$.

Now, let us first list all generators of $C(j; 8, 4)$ with the help of Lemma 3.2 before continuing further with $H^j(\tilde{G}_{8,4})$. Note that $\tilde{w}_2^3 = \tilde{w}_3^3$ and $\tilde{w}_2 \tilde{w}_3 = 0$, since $g_6, g_7 \in J_{8,4}$. Also note that if $\tilde{x} \in C(j; 8, 4)$ is a nonzero element, it must be a $\tilde{w}_i$–multiple of some nonzero element in $C(j - i; 8, 4)$ for some $i \in \{2, 3, 4\}$.

In $C(5; 8, 4)$ there is only one nonzero element, $\tilde{w}_2 \tilde{w}_3$.

In $C(6; 8, 4)$ there are two, $\tilde{w}_2^3$ and $\tilde{w}_2 \tilde{w}_4$, because $\tilde{w}_2^3 = \tilde{w}_3^3$.

In $C(7; 8, 4)$ we have $\tilde{w}_2^3 \tilde{w}_3 = 0$ and thus $\tilde{w}_3 \tilde{w}_4$ is the only generator.

In $C(8; 8, 4)$ we have $\tilde{w}_2^3 = \tilde{w}_2 \tilde{w}_3^2$ and $\tilde{w}_2^2 \tilde{w}_4 = \tilde{w}_4^3$ as the two generators.

In $C(9; 8, 4)$ we have $\tilde{w}_2^2 \tilde{w}_3 = \tilde{w}_3^2 = 0$, so $\tilde{w}_2 \tilde{w}_3 \tilde{w}_4$ is the only nonzero element.

In $C(10; 8, 4)$ we have $\tilde{w}_5 = \tilde{w}_2^2 \tilde{w}_3^2 = 0$ and $\tilde{w}_2 \tilde{w}_4 = \tilde{w}_2^2 \tilde{w}_4 = \tilde{w}_2 \tilde{w}_4$ as the generator.

In $C(11; 8, 4)$ we have $\tilde{w}_2^2 \tilde{w}_3 \tilde{w}_4 = 0, \tilde{w}_2^3 \tilde{w}_3 = 0, \tilde{w}_3 \tilde{w}_4 = 0$.

In $C(12; 8, 4)$ there is one generator $\tilde{w}_2 \tilde{w}_4$ equal to both $\tilde{w}_2 \tilde{w}_4$ and $\tilde{w}_2 \tilde{w}_3 \tilde{w}_4$.

By Poincaré duality, to each nonzero element $\tilde{x} \in H^j(\tilde{G}_{8,4})$ there exists a nonzero element $\tilde{y} \in H^{16-j}(\tilde{G}_{8,4})$ such that $\tilde{x}\tilde{y} \neq 0$. Thus Lemma 2.1 implies $C(j; 8, 4) = 0$ for all $j > 12$. Additionally, the dual to element $\tilde{w}_2 \tilde{w}_4$ must be $a_4$. Hence $a_4 \tilde{w}_2 \tilde{w}_4 = a_4 \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 \neq 0$. 


It remains to determine $\alpha_j(\tilde{G}_{8,4})$ for $j = 5, 6, 7, 8$. For $j \leq 7$ Proposition 2.4 with $x = j - 4 \leq 3$ implies that $\alpha_j(\tilde{G}_{8,4}) \leq |N_{j-4}(G_{8,4})| - t_{j-4}$, where $t_{j-4}$ is the maximal number of linearly independent elements of $N_{j-4}(G_{8,4})$. For $N_1(G_{8,4}) = \{g_6\}$ we have $t_1 = 1$. For $N_2(G_{8,4}) = \{w_2 g_5, g_7\}$ we have $t_2 = 1$, since $g_5 = 0$. For $N_3(G_{8,4}) = \{w_3 g_5, w_2 g_6, g_8\}$ we have $t_3 = 2$ as all $w_2 g_6, g_8, w_2 g_6 + g_8$ are nonzero. Thus $\alpha_5(\tilde{G}_{8,4}) \leq 0$, $\alpha_6(\tilde{G}_{8,4}) \leq 1$ and $\alpha_7(\tilde{G}_{8,4}) \leq 1$. On the other hand, we have shown $a_4 \tilde{w}_2 \in H^6(\tilde{G}_{8,4})$ and $a_4 \tilde{w}_3 \in H^7(\tilde{G}_{8,4})$ to be nonzero and dual to $\tilde{w}_3^2 \tilde{w}_4$ and $\tilde{w}_2 \tilde{w}_3 \tilde{w}_4$ respectively. Therefore $\alpha_6(\tilde{G}_{8,4}), \alpha_7(\tilde{G}_{8,4}) \neq 0$ and both must be equal to 1.

We determine $\alpha_8(\tilde{G}_{8,4})$ with the help of the Euler characteristic. For the Grassmann manifold $G_{8,4}$ we can compute its Euler characteristic from the Poincaré polynomial [2,3] to obtain $\chi(G_{8,4}) = 6$. As $\tilde{G}_{8,4}$ is a 2-fold cover, we have $\chi(\tilde{G}_{8,4}) = 2 \cdot \chi(G_{8,4}) = 12$. By this point we know the Betti numbers $b_0(\tilde{G}_{8,4}), \ldots, b_7(\tilde{G}_{8,4})$. Poincaré duality and a simple calculation yields $b_8(\tilde{G}_{8,4}) = 4$. Consequently, $\alpha_8(\tilde{G}_{8,4}) = 2$ and since we already know $a_4 \tilde{w}_2^2$ and $a_8 \tilde{w}_3$ are nonzero, we only need to show that they are distinct. That is done by considering their products with $\tilde{w}_2^4$ and realizing one is zero while the other is not.

By obvious adjustment of the last argument we also prove that $a_4 \tilde{w}_2^3 \neq a_4 \tilde{w}_2 \tilde{w}_4$ and $a_4 \tilde{w}_2^4 \neq a_4 \tilde{w}_2^2 \tilde{w}_4$. All the remaining numbers $\alpha_j(\tilde{G}_{8,4})$ are now determined by Poincaré duality combined with the knowledge of all $C(j; 8, 4)$ and the obvious generators suffice to produce the required values.

\textbf{Theorem 3.4.} We have the following generators of $H^j(\tilde{G}_{9,4})$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>gen.</th>
<th>$j$</th>
<th>gen.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\tilde{w}_0$</td>
<td>11</td>
<td>$a_8 \tilde{w}_3$</td>
</tr>
<tr>
<td>1</td>
<td>$\tilde{w}_2$</td>
<td>12</td>
<td>$a_8 \tilde{w}_2^2, a_8 \tilde{w}_4, \tilde{w}_2^4 \tilde{w}_4$</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{w}_3$</td>
<td>13</td>
<td>$a_8 \tilde{w}_2 \tilde{w}_3$</td>
</tr>
<tr>
<td>3</td>
<td>$\tilde{w}_2^2, \tilde{w}_4$</td>
<td>14</td>
<td>$a_8 \tilde{w}_2^3, a_8 \tilde{w}_2 \tilde{w}_4$</td>
</tr>
<tr>
<td>4</td>
<td>$\tilde{w}_2 \tilde{w}_3$</td>
<td>15</td>
<td>$a_8 \tilde{w}_3 \tilde{w}_4$</td>
</tr>
<tr>
<td>5</td>
<td>$\tilde{w}_3 \tilde{w}_4$</td>
<td>16</td>
<td>$a_8 \tilde{w}_4^2, a_8 \tilde{w}_2 \tilde{w}_4$</td>
</tr>
<tr>
<td>6</td>
<td>$\tilde{w}_2^2 \tilde{w}_4$</td>
<td>17</td>
<td>$a_8 \tilde{w}_2 \tilde{w}_3 \tilde{w}_4$</td>
</tr>
<tr>
<td>7</td>
<td>$a_8, w_2, \tilde{w}_2^2 \tilde{w}_4$</td>
<td>18</td>
<td>$a_8 \tilde{w}_3 \tilde{w}_4$</td>
</tr>
<tr>
<td>8</td>
<td>$\tilde{w}_2^3 \tilde{w}_4$</td>
<td>19</td>
<td>$a_8 \tilde{w}_4^2$</td>
</tr>
<tr>
<td>9</td>
<td>$\tilde{w}_2^4 \tilde{w}_4$</td>
<td>20</td>
<td>$a_8 \tilde{w}_4 \tilde{w}_4$</td>
</tr>
</tbody>
</table>

where $a_8$ is an element in $H^8(\tilde{G}_{9,4}) \setminus C(8; 9, 4)$.

\textbf{Proof.} First, as $J_{9,4} = J_{8,4}$, we have $C(j; 9, 4) = C(j; 8, 4)$ for all $j$.

We have charrank$(\tilde{G}_{9,4}) = 7$ so $H^j(\tilde{G}_{9,4}) = C(j; 9, 4) = C(j; 8, 4)$ for $j \leq 7$ and $\alpha_8(\tilde{G}_{9,4}) \geq 1$. To obtain an upper bound for $\alpha_8(\tilde{G}_{9,4})$ we consider $N_3(G_{9,4}) =$...
\{w_3 g_6, w_2 g_7, g_9\}. Any two elements from this set are linearly independent, which means \(\alpha_8(\tilde{G}_{9,4}) = 1\). Denote \(a_8\) an element in \(H^8(\tilde{G}_{9,4}) \setminus C(8; 9, 4)\).

Clearly, the Poincaré dual to \(\tilde{w}_3^2 \tilde{w}_4\) is \(a_8\) and similarly as before we have \(a_8 \tilde{w}_2 \tilde{w}_3^3 \tilde{w}_4 \neq 0\).

Next, \(N_4(G_{9,4}) = \{w_2^2 g_6, w_4 g_6, w_3 g_7, w_2 g_8\}\). We will show that these four elements are linearly independent. Suppose that for some \(c_i \in \mathbb{Z}_2, 1 \leq i \leq 4\) we have

\[c_1 w_2^2 g_6 + c_2 w_4 g_6 + c_3 w_3 g_7 + c_4 w_2 g_8 = 0\]

Since every element in \(\mathbb{Z}_2[w_2, w_3, w_4]\) is of order 2, the equation implies

\[(c_1 + c_3 + c_4) w_2^2 g_6 + c_2 w_4 g_6 + c_3 (w_3 g_7 + w_2^2 g_6) + c_4 (w_2 g_8 + w_2^2 g_6) = 0,\]

\[(c_1 + c_3 + c_4) (w_2^5 + w_2^2 w_3^5) + c_2 w_4 g_6 + c_3 w_2^5 + c_4 (w_2^3 w_4 + w_2 w_4^2) = 0.\]

If \(c_1 + c_3 + c_4\) or \(c_3\) or both are nonzero, the LHS is not divisible by \(w_4\), which is a contradiction. Thus \(c_1 + c_3 + c_4 = c_3 = 0\) and the equation simplifies so much, we immediately deduce \(c_2 = c_4 = 0\). Which in turn implies \(c_1 = 0\). We have proved independence and so \(\alpha_9(\tilde{G}_{9,4}) = 0\).

Now that Betti numbers \(b_0(\tilde{G}_{9,4}), \ldots, b_9(\tilde{G}_{9,4})\) are known, from calculating \(\chi(G_{9,4}) = 6\) and \(\chi(\tilde{G}_{9,4}) = 12\) we obtain \(b_{10}(\tilde{G}_{9,4}) = 2\).

This gives enough information to quickly determine all \(\alpha_j(\tilde{G}_{9,4})\) and all are once again covered by the obvious generators derived from \(a_8 \tilde{w}_2 \tilde{w}_3^3 \tilde{w}_4 = a_8 \tilde{w}_2 \tilde{w}_3^3 \tilde{w}_4 \neq 0\).

**Theorem 3.5.** We have the following generators of \(H^3(\tilde{G}_{10,4})\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>(\text{gen.})</th>
<th>(j)</th>
<th>(\text{gen.})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\tilde{w}_0)</td>
<td>13</td>
<td>(-)</td>
</tr>
<tr>
<td>1</td>
<td>(-)</td>
<td>14</td>
<td>(a_{12}\tilde{w}<em>2, b</em>{12}\tilde{w}_2)</td>
</tr>
<tr>
<td>2</td>
<td>(\tilde{w}_2)</td>
<td>15</td>
<td>(a_{12}\tilde{w}_3)</td>
</tr>
<tr>
<td>3</td>
<td>(\tilde{w}_3)</td>
<td>16</td>
<td>(a_{12}\tilde{w}<em>2^2, a</em>{12}\tilde{w}<em>4, b</em>{12}\tilde{w}_2^2)</td>
</tr>
<tr>
<td>4</td>
<td>(\tilde{w}_2^2, \tilde{w}_4)</td>
<td>17</td>
<td>(a_{12}\tilde{w}_2 \tilde{w}_3)</td>
</tr>
<tr>
<td>5</td>
<td>(\tilde{w}_2 \tilde{w}_3)</td>
<td>18</td>
<td>(a_{12}\tilde{w}<em>2^3, a</em>{12}\tilde{w}_2 \tilde{w}<em>4, b</em>{12}\tilde{w}_2^3)</td>
</tr>
<tr>
<td>6</td>
<td>(\tilde{w}_2^3, \tilde{w}_2 \tilde{w}_4, \tilde{w}_2 \tilde{w}_4)</td>
<td>19</td>
<td>(a_{12}\tilde{w}_2 \tilde{w}_3 \tilde{w}_4)</td>
</tr>
<tr>
<td>7</td>
<td>(\tilde{w}_3 \tilde{w}_4)</td>
<td>20</td>
<td>(a_{12}\tilde{w}<em>2^4, b</em>{12}\tilde{w}_2^4)</td>
</tr>
<tr>
<td>8</td>
<td>(\tilde{w}_2^4, \tilde{w}_2 \tilde{w}_3^3, \tilde{w}_2^2 \tilde{w}_4)</td>
<td>21</td>
<td>(a_{12}\tilde{w}_2 \tilde{w}_3 \tilde{w}_4)</td>
</tr>
<tr>
<td>9</td>
<td>(\tilde{w}_2 \tilde{w}_3 \tilde{w}_4)</td>
<td>22</td>
<td>(a_{12}\tilde{w}_2^3 \tilde{w}<em>4 = b</em>{12}\tilde{w}_2^5)</td>
</tr>
<tr>
<td>10</td>
<td>(\tilde{w}_3^3, \tilde{w}_2 \tilde{w}_4)</td>
<td>23</td>
<td>(-)</td>
</tr>
<tr>
<td>11</td>
<td>(-)</td>
<td>24</td>
<td>(a_{12}\tilde{w}_2^4 \tilde{w}_4)</td>
</tr>
<tr>
<td>12</td>
<td>(a_{12}, b_{12}, \tilde{w}_2^6, \tilde{w}_2^4 \tilde{w}_4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \(a_{12}, b_{12}\) are linearly independent elements in \(H^{12}(\tilde{G}_{9,4}) \setminus C(12; 9, 4)\) such that 
\(a_{12}\tilde{w}_2^3 \tilde{w}_4 \neq 0, a_{12}\tilde{w}_2^5 = 0, b_{12}\tilde{w}_2^6 \neq 0\) and \(b_{12}\tilde{w}_2^4 \tilde{w}_4 = 0\).
The result is $c$ yields therefore linearly independent.

$H$ of are the three generators.

As before, we begin with determining the generators of $C(j; 10, 4)$. For $j \leq 6$ we have $C(j; 10, 4)$ as stated since there are no relations.

In $C(7; 10, 4)$ we have $\tilde{w}_2^2 \tilde{w}_3 = 0$ and thus $\tilde{w}_3 \tilde{w}_4$ is the only generator.

In $C(8; 10, 4)$ we have $\tilde{w}_2^4 + \tilde{w}_2 \tilde{w}_3^2 + \tilde{w}_2^2 \tilde{w}_4^2 = \tilde{w}_4^2$ as the only relation, hence there are the three generators.

In $C(9; 10, 4)$ we have $\tilde{w}_3^2 \tilde{w}_3 = 0$ and $\tilde{w}_3^3 = 0$, so $\tilde{w}_2 \tilde{w}_3 \tilde{w}_4$ is the only generator.

In $C(10; 10, 4)$ we have $\tilde{w}_2^3 \tilde{w}_3^2 = 0$, $\tilde{w}_3^2 \tilde{w}_4 = \tilde{w}_3^2 \tilde{w}_4$, and among $\tilde{w}_2^2$, $\tilde{w}_2^3 \tilde{w}_4$, $\tilde{w}_2^2 \tilde{w}_4^2$ either is equal to the sum of the other two since $g_{10} \in J_{10, 4}$. Thus there are two generators.

In $C(11; 10, 4)$ we have $\tilde{w}_2^3 \tilde{w}_3 \tilde{w}_4$, $\tilde{w}_2^2 \tilde{w}_3$ both multiples of $\tilde{w}_2^2 \tilde{w}_3 = 0$. Then $\tilde{w}_2 \tilde{w}_3^3$ is a multiple of $\tilde{w}_3^3 = 0$ and $\tilde{w}_3 \tilde{w}_4^2 = 0$ by Lemma 3.2.

In $C(12; 10, 4)$ we have $\tilde{w}_2^4 \tilde{w}_4 = \tilde{w}_2 \tilde{w}_3^2 \tilde{w}_4$ and $\tilde{w}_2^2 + \tilde{w}_2^4 \tilde{w}_4 = \tilde{w}_2^2 \tilde{w}_4^2 = \tilde{w}_4^3$ by Lemma 3.2. Thus there are two generators, e.g. $\tilde{w}_2^5$ and $\tilde{w}_2^3 \tilde{w}_4$.

We know that $\text{charrank}(\tilde{G}_{10, 4}) = 11$, so by Poincaré duality and Lemma 2.1 we have $C(j; 10, 4) = 0$ for $j \geq 13$.

Also $\alpha_j(\tilde{G}_{10, 4}) = 0$ for $j \leq 11$, so by now we have determined all Betti numbers except for $b_{12}(\tilde{G}_{10, 4})$. We calculate it from the Euler characteristic $\chi(\tilde{G}_{10, 4}) = 20$.

The result is $b_{12}(\tilde{G}_{10, 4}) = 4$.

So $\alpha_{12}(\tilde{G}_{10, 4}) = 2$ and there are two linearly independent elements in $H^{12}(\tilde{G}_{9, 4}) \setminus C(12; 9, 4)$. By Poincaré duality we can start by arbitrarily picking a pair $(a_{12}', b_{12}')$ such that $a_{12}' \tilde{w}_2^2 \tilde{w}_4 \neq 0$ and $b_{12}' \tilde{w}_2^6 \neq 0$. Then we adjust $b_{12}$ based on the fact that $H^{20}(\tilde{G}_{10, 4}) \cong \mathbb{Z}_2$. If $b_{12} \tilde{w}_2^4 \tilde{w}_4 \neq 0$, we define $b_{12} = a_{12}' + b_{12}'$, so that $b_{12} \tilde{w}_2^4 \tilde{w}_4 = 0$. Otherwise let $b_{12} = b_{12}'$.

Similarly, since $H^{12}(\tilde{G}_{10, 4}) \cong H^2(\tilde{G}_{10, 4}) \cong \mathbb{Z}_2$, if $a_{12}' \tilde{w}_2^5 \neq 0$, we define $a_{12} = a_{12}' + b_{12}$, else $a_{12} = a_{12}'$, so that $a_{12} \tilde{w}_2^5 = 0$. We have $a_{12}, b_{12}, \tilde{w}_2^6, \tilde{w}_2^4 \tilde{w}_4$ as generators of $H^{12}(\tilde{G}_{10, 4})$.

Next, we have $H^{13}(\tilde{G}_{10, 4}) = H^{11}(\tilde{G}_{10, 4}) = 0$.

Since $a_{12} \tilde{w}_2$ and $b_{12} \tilde{w}_2$ have different products with $\tilde{w}_2^3 \tilde{w}_4$ they are distinct and therefore linearly independent.

We have $a_{12} \tilde{w}_3 \neq 0$, since $a_{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 = a_{12} \tilde{w}_2^4 \tilde{w}_4 \neq 0$.

By considering the products of nonzero elements $a_{12} \tilde{w}_2^2, a_{12} \tilde{w}_4, b_{12} \tilde{w}_2^2$ with $\tilde{w}_2^4, \tilde{w}_2 \tilde{w}_3 \tilde{w}_4$, we see that they are independent. Indeed, suppose that for some $c_1, c_2, c_3 \in \mathbb{Z}_2$ we have $c_1 a_{12} \tilde{w}_2^2 + c_2 a_{12} \tilde{w}_4 + c_3 b_{12} \tilde{w}_2^3 = 0$. Multiplying both sides by $\tilde{w}_2 \tilde{w}_3$ and recalling that $\tilde{w}_2 \tilde{w}_3 = 0$, we obtain $c_2 = 0$. Then multiplying by $\tilde{w}_2^4$ yields $c_3 = 0$ and multiplying by $\tilde{w}_2^5 \tilde{w}_4$ gives $c_1 = 0$.

We have $a_{12} \tilde{w}_2 \tilde{w}_3 \neq 0$, since $a_{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 = a_{12} \tilde{w}_2^5 \tilde{w}_4 \neq 0$.

Elements $a_{12} \tilde{w}_2^3, a_{12} \tilde{w}_2 \tilde{w}_4, b_{12} \tilde{w}_2^2$ are nonzero. They prove to be independent upon considering their products with $\tilde{w}_2^3, \tilde{w}_2^3, \tilde{w}_2 \tilde{w}_4$.

We have $a_{12} \tilde{w}_3 \tilde{w}_4 \neq 0$, since $a_{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 = a_{12} \tilde{w}_2^5 \tilde{w}_4 \neq 0$.

Elements $a_{12} \tilde{w}_2^4, b_{12} \tilde{w}_2^4$ are nonzero. They prove to be independent upon considering their products with $\tilde{w}_2^3, \tilde{w}_4$.

We have $a_{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 \neq 0$, since $a_{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 = a_{12} \tilde{w}_2^5 \tilde{w}_4 \neq 0$.

We have $a_{12} \tilde{w}_2^3 \tilde{w}_4 \neq 0$ and $b_{12} \tilde{w}_2^5 \neq 0$, therefore they are equal. □
Before we examine the last case, we separately prove one important piece of information.

**Lemma 3.6.** The set $N_6(G_{11,4})$ is linearly independent.

**Proof.** We partition the set $N_6(G_{11,4})$ into two disjoint sets $N_6^+(G_{11,4})$ and $N_6^-(G_{11,4})$, where $N_6^+(G_{11,4}) = \{w_2^3g_8, w_2^2g_9, w_2w_4g_8, w_3g_{11}\}$ and $N_6^-(G_{11,4}) = \{w_2w_3g_9, w_2^2g_{10}, w_4g_{10}\}$. To prove linear independence of $N_6(G_{11,4})$ we have to prove that no nontrivial linear combination of elements of $N_6^+(G_{11,4})$ is equal to any linear combination of elements of $N_6^-(G_{11,4})$ and vice versa. All elements of $N_6^+(G_{11,4})$ are polynomials with an even number of terms, thus any linear combination of them will have an even number of terms, since each term is of order 2. All elements of $N_6^-(G_{11,4})$ are polynomials with an odd number of terms, so the only nontrivial linear combinations worth considering are $w_2w_3g_9 + w_2^2g_{10}$, $w_2w_3g_9 + w_4g_{10}$ and $w_2^2g_{10} + w_4g_{10}$, that is the polynomials

$$R_1 = w_2^5 + w_2w_3^4 + w_2^2w_3^2w_4 + w_2^3w_4^2,$$
\[ R_2 = w_2w_3^4 + w_2^5w_4 + w_2^2w_3^2w_4 + w_2^3w_4^3, \]
\[ R_3 = w_2^5 + w_2^3w_4 + w_2^2w_3^2w_4 + w_2^3w_4^2 + w_3w_4^2 + w_2w_4^3. \]

Since none of these are zero polynomials, the set $N_6^-(G_{11,4})$ is linearly independent. It remains to show that there is no nontrivial linear combination of elements of $N_6^+(G_{11,4})$ equal to $R_1, R_2, R_3$ or zero.

First, let us consider combinations without $w_2^3g_8$, that is for any $c_1, c_2, c_3 \in \mathbb{Z}_2$ the expression

$$L_{0,c_1,c_2,c_3} = c_1w_2^2g_8 + c_2w_2w_4g_8 + c_3w_3g_{11},$$

where

$$w_2^2g_8 = w_2^4w_3^2 + w_2w_3^4 + w_2^2w_3^2w_4 + w_2^3w_4^2,$$
$$w_2w_4g_8 = w_2^5w_4 + w_2^2w_3^2w_4 + w_2^3w_4^2 + w_2w_4^3,$$
$$w_3g_{11} = w_2^4w_3^2 + w_2^3w_4^2.$$

Since none of them contain $w_2^7$ we have that $L_{0,c_1,c_2,c_3} \neq R_1, R_3$. For $L_{0,c_1,c_2,c_3}$ to be equal to zero, first $c_2$ must be zero and then $c_1$ and $c_3$ also. To have $L_{0,c_1,c_2,c_3} = R_2$, we need $c_1 = 1$ to obtain $w_2w_3^4$ and $c_2 = 1$ to obtain $w_2^5w_4$. But then no choice of $c_3$ will make $L_{0,1,1,c_3} = R_2$ true. In the end, $L_{0,1,1,c_3} \neq R_1, R_2, R_3$ and $L_{0,c_1,c_2,c_3} = 0$ only for $L_{0,0,0,0}$.

Now let us consider combinations containing $w_2^3g_8$, that is for any $c_1, c_2, c_3 \in \mathbb{Z}_2$ the expression

$$L_{1,c_1,c_2,c_3} = w_2^3g_8 + c_1w_2^2g_8 + c_2w_2w_4g_8 + c_3w_3g_{11}.$$

Since $w_2^7$ is always a term in $L_{1,c_1,c_2,c_3}$, we only need to consider if it is possible for $L_{1,c_1,c_2,c_3}$ to be equal to $R_1$ or $R_3$. Subtracting $w_2^3g_8$ from both sides of the considered equations, it is the same as considering whether $L_{0,c_1,c_2,c_3}$ is equal to
either of the following
\[ R_1 + w_2^3 g_8 = w_2^4 w_3^3 + w_2 w_4^3 + w_2^5 w_4 + w_2^2 w_3 w_4, \]
\[ R_3 + w_3^3 g_8 = w_2^4 w_3^3 + w_2^2 w_3^3 w_4 + w_3^2 w_4^3 + w_2 w_3^4. \]
Each of \( w_2^3 g_8, w_2^4 g_8, w_3^3 g_11 \) contains an even number of terms from the set \( \{ w_2^5 w_4, w_2 w_3^4 \} \), hence every \( L_{0,c_1,c_2,c_3} \) will too. However both \( R_1 + w_2^3 g_8, R_3 + w_3^3 g_8 \) contain exactly one such term. Thus for any choice of indices \( c_1, c_2, c_3 \in \mathbb{Z}_2, i \in \{1, 3\} \) the equality \( L_{0,c_1,c_2,c_3} = R_i + w_i^3 g_8 \) and equivalently \( L_{1,c_1,c_2,c_3} = R_i \) is impossible.

In conclusion, the only case when a linear combination of elements of \( N_6^+(G_{11,4}) \) is equal to some linear combination of elements of \( N_6^-(G_{11,4}) \) is when they are both trivial. With that we have proved \( N_6(G_{11,4}) \) is a linearly independent set. \( \square \)

**Theorem 3.7.** We have \( H^j(\tilde{G}_{11,4}) \cong H^j(\tilde{G}_{10,4}) \) for \( j \leq 6 \) and there are following generators of the remaining \( H^j(\tilde{G}_{11,4}) \).

<table>
<thead>
<tr>
<th>( j )</th>
<th>gen.</th>
<th>( j )</th>
<th>gen.</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>( \tilde{w}_2^3 \tilde{w}_3, \tilde{w}_3 \tilde{w}_4 )</td>
<td>18</td>
<td>( a_{12} \tilde{w}<em>2^3, a</em>{12} \tilde{w}_2 \tilde{w}<em>4, a</em>{12} \tilde{w}_3^2 )</td>
</tr>
<tr>
<td>8</td>
<td>( \tilde{w}_2^4, \tilde{w}_2 \tilde{w}_3^2, \tilde{w}_2^3 \tilde{w}_4 )</td>
<td>19</td>
<td>( a_{12} \tilde{w}_2^2 \tilde{w}<em>3, a</em>{12} \tilde{w}_3 \tilde{w}_4 )</td>
</tr>
<tr>
<td>9</td>
<td>( \tilde{w}_3^3 \tilde{w}_3, \tilde{w}_2 \tilde{w}_3^3 \tilde{w}_4 )</td>
<td>20</td>
<td>( a_{12} \tilde{w}<em>2^4, a</em>{12} \tilde{w}_2^2 \tilde{w}<em>4, a</em>{12} \tilde{w}_2 \tilde{w}_3^2 )</td>
</tr>
<tr>
<td>10</td>
<td>( \tilde{w}_2^5, \tilde{w}_2^2 \tilde{w}_3^2, \tilde{w}_2^3 \tilde{w}_4 )</td>
<td>21</td>
<td>( a_{12} \tilde{w}_2^3 \tilde{w}<em>3, a</em>{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 )</td>
</tr>
<tr>
<td>11</td>
<td>( \tilde{w}_2^4 \tilde{w}_3 )</td>
<td>22</td>
<td>( a_{12} \tilde{w}<em>2^5, a</em>{12} \tilde{w}_2^3 \tilde{w}<em>4, a</em>{12} \tilde{w}_2^2 \tilde{w}_3^2 )</td>
</tr>
<tr>
<td>12</td>
<td>( a_{12}, \tilde{w}_2^0, \tilde{w}_2^3 \tilde{w}_3^3 \tilde{w}_4 )</td>
<td>23</td>
<td>( a_{12} \tilde{w}_2^0 \tilde{w}_3 )</td>
</tr>
<tr>
<td>13</td>
<td>( \tilde{w}_2^5 \tilde{w}_3 )</td>
<td>24</td>
<td>( a_{12} \tilde{w}<em>2^6, a</em>{12} \tilde{w}_2^3 \tilde{w}_4 )</td>
</tr>
<tr>
<td>14</td>
<td>( a_{12} \tilde{w}_2, \tilde{w}_2^7 )</td>
<td>25</td>
<td>( a_{12} \tilde{w}_2^5 \tilde{w}_3 )</td>
</tr>
<tr>
<td>15</td>
<td>( a_{12} \tilde{w}_3 )</td>
<td>26</td>
<td>( a_{12} \tilde{w}_2^7 )</td>
</tr>
<tr>
<td>16</td>
<td>( a_{12} \tilde{w}<em>2^0, a</em>{12} \tilde{w}_4, \tilde{w}_2^8 )</td>
<td>27</td>
<td>( - )</td>
</tr>
<tr>
<td>17</td>
<td>( a_{12} \tilde{w}_2 \tilde{w}_3 )</td>
<td>28</td>
<td>( a_{12} \tilde{w}<em>2^8 = a</em>{16} \tilde{w}_2^4 \tilde{w}_4 )</td>
</tr>
</tbody>
</table>

where \( a_{12} \) is an element in \( H^{12}(\tilde{G}_{11,4}) \setminus C(12; 11, 4) \) such that \( a_{12} \tilde{w}_2^5 \neq 0 \) and \( a_{16} \) is an element in \( H^{16}(\tilde{G}_{11,4}) \setminus C(16; 11, 4) \) such that \( a_{16} \tilde{w}_2^4 \tilde{w}_4 \neq 0, a_{16} \tilde{w}_2^6 = 0 \) and \( a_{16} \tilde{w}_2^3 \tilde{w}_3^2 = 0 \).

**Proof.** We have \( \text{charrank}(\tilde{G}_{11,4}) = 11 \), so \( H^j(\tilde{G}_{11,4}) = C(j; 11, 4) \) for \( j \leq 11 \) and \( C(j; 11, 4) \cong C(j; 10, 4) = H^j(\tilde{G}_{10,4}) \) for \( j \leq 6 \), since neither \( J_{11,4} \) or \( J_{10,4} \) produce relations in cohomology in dimensions lower than seven.

In \( C(7; 11, 4) \) we have \( \tilde{w}_2^2 \tilde{w}_3 \) and \( \tilde{w}_3 \tilde{w}_4 \) as generators.

In \( C(8; 11, 4) \) we have \( \tilde{w}_2^4 + \tilde{w}_2 \tilde{w}_3^2 + \tilde{w}_2^2 \tilde{w}_4 = \tilde{w}_4^2 \) as the only relation, hence there are the three generators.

In \( C(9; 11, 4) \) we have \( \tilde{w}_3^3 = 0 \) and \( \tilde{w}_2^3 \tilde{w}_3, \tilde{w}_2 \tilde{w}_3 \tilde{w}_4 \) are generators.
In $C(10; 11, 4)$ we have only $\tilde{w}_2^5, \tilde{w}_2^2\tilde{w}_3^2, \tilde{w}_2^3\tilde{w}_4$ as generators, since $\tilde{w}_3\tilde{w}_4 = \tilde{w}_2^2\tilde{w}_3^2 + \tilde{w}_2\tilde{w}_4$ and $\tilde{w}_2^2\tilde{w}_4$ is the sum of all three generators, because $w_{2g8} \in J_{11,4}$.

In $C(11; 11, 4)$ we have $\tilde{w}_2\tilde{w}_3^3 = 0$ and $\tilde{w}_2^2\tilde{w}_3\tilde{w}_4 = 0$, so $\tilde{w}_2^3\tilde{w}_3 = \tilde{w}_3^2\tilde{w}_4$ is the generator.

In $C(12; 11, 4)$ we have $\tilde{w}_2\tilde{w}_3^2\tilde{w}_4 = \tilde{w}_2^2 + \tilde{w}_2^2\tilde{w}_4$, since $w_{2g10} \in J_{11,4}$, with $\tilde{w}_2^2\tilde{w}_4 = \tilde{w}_2^2 + \tilde{w}_2^2\tilde{w}_3 + \tilde{w}_2^2\tilde{w}_4$, since $w_{2g8} \in J_{11,4}$, so only $\tilde{w}_2^2, \tilde{w}_2^3\tilde{w}_3, \tilde{w}_2^3\tilde{w}_4$ are generators.

In $C(13; 11, 4)$ we have $\tilde{w}_2\tilde{w}_3^3 = 0, \tilde{w}_2^3\tilde{w}_3\tilde{w}_4 = 0$ and $\tilde{w}_2\tilde{w}_3^2\tilde{w}_4 = \tilde{w}_2\tilde{w}_3^3$, since $w_{2g11} \in J_{11,4}$, so $\tilde{w}_2\tilde{w}_3$ is the only generator.

In $C(14; 11, 4)$ we have $\tilde{w}_2^2\tilde{w}_3^2\tilde{w}_4 = 0$. Thus also $\tilde{w}_2^2 = \tilde{w}_2^3\tilde{w}_3^2$, since $w_{2g10} \in J_{11,4}$.

And we already know $\tilde{w}_2^3 = \tilde{w}_2^3\tilde{w}_3^3$ and $\tilde{w}_2^3 = \tilde{w}_2^2\tilde{w}_4$.

In $C(15; 11, 4)$ we have $\tilde{w}_2^3\tilde{w}_3 = \tilde{w}_2^3\tilde{w}_3\tilde{w}_4$, since $w_{2g11} \in J_{11,4}$, but the latter is zero. Also $\tilde{w}_2^3\tilde{w}_3^3 = 0$ and $\tilde{w}_2\tilde{w}_3\tilde{w}_4 = 0$.

In $C(16; 11, 4)$ we have $\tilde{w}_2^2\tilde{w}_3^2\tilde{w}_4 = 0, \tilde{w}_2^3\tilde{w}_3^3 = \tilde{w}_2^3\tilde{w}_4 = \tilde{w}_2\tilde{w}_4$ and $\tilde{w}_2 = \tilde{w}_2^3\tilde{w}_3$ as it is a $\tilde{w}_2$-multiple of a known equality.

From char(rank)$\tilde{G}_{11,4}$ = 11, Poincaré duality and Lemma 2.1 we have $C(j; 11, 4) = 0$ for $j \geq 17$.

Also, we have $\alpha_j(\tilde{G}_{11,4}) = 0$ for $j \leq 11$ and $\alpha_{12}(\tilde{G}_{11,4}) \geq 1$. Let us consider $N_5(G_{11,4}) = \{w_2w_3g_8, w_2^2g_9, w_4g_9, w_3g_{10}, w_2g_{11}\}$. We will show that $\{w_2w_3g_8, w_2^2g_9, w_4g_9, w_2g_{11}\}$ is a linearly independent subset and therefore $\alpha_{12}(\tilde{G}_{11,4}) = 1$. Suppose that for some $c_i \in \mathbb{Z}, 1 \leq i \leq 4$ we have

$$c_1w_2w_3g_8 + c_2w_2^2g_9 + c_3w_4g_9 + c_4w_2g_{11} = 0.$$ 

Considering that $w_4g_9 = w_3^2w_4$ is not divisible by $w_2$, we immediately see that $c_3 = 0$ and $c_1w_3g_8 + c_2w_2g_9 + c_4g_{11} = 0$.

Since both $g_8$ and $g_{11}$ have an even number of terms, the same is true for $c_1w_3g_8$ and $c_4g_{11}$, thus the parity of number of terms in LHS is the same as parity of number of terms in $c_2w_2g_9$. But $g_9 = w_3^2$, therefore $c_2 = 0$. Finally, we deduce $c_1 = c_4 = 0$ as well. In conclusion, there is one generator in $H^{12}(\tilde{G}_{11,4}) \setminus C(12; 11, 4)$, some $a_{12}$. By Poincaré duality we have $a_{12}\tilde{w}_2^3 \neq 0$.

By Lemma 3.6 and Proposition 2.4 we have $\alpha_{13}(\tilde{G}_{11,4}) = 0$.

We have $\chi(\tilde{G}_{11,4}) = 10$, so from a simple calculation we obtain $b_{14}(\tilde{G}_{11,4}) = 2$. Hence $\alpha_{14}(\tilde{G}_{11,4}) = 1$ and $a_{12}\tilde{w}_2$ is the obvious generator.

To finish the proof, recall that $\tilde{w}_2^8 = \tilde{w}_2^5\tilde{w}_3^2 = \tilde{w}_2^4\tilde{w}_4^2 = \tilde{w}_2\tilde{w}_4$ and the second term is equal to $\tilde{w}_3\tilde{w}_3^2\tilde{w}_4^2$ by $w_2w_3g_{11} \in J_{11,4}$. Also $\tilde{w}_3^3 = 0, \tilde{w}_2\tilde{w}_3\tilde{w}_4 = 0$ and $\tilde{w}_2\tilde{w}_3^2 = 0$. So $a_{12}\tilde{w}_3$ is nonzero. Next, $a_{12}\tilde{w}_2^3$ and $a_{12}\tilde{w}_4$ are nonzero and distinct, since their $\tilde{w}_3^3\tilde{w}_3$ multiples are not equal. But there is one more generator in $H^{16}(\tilde{G}_{11,4}) \setminus C(16; 11, 4)$, some $a_{16}$.

Next, $a_{12}\tilde{w}_5 \tilde{w}_3 \neq 0$. Nonzero elements $a_{12}\tilde{w}_3^2, a_{12}\tilde{w}_5 \tilde{w}_4$ and $a_{12}\tilde{w}_5^2$ are found out to be independent after considering their multiples with $\tilde{w}_2^2, \tilde{w}_2^2\tilde{w}_3^2$ and $\tilde{w}_2\tilde{w}_4$. By obvious adjustment of this argument triples $a_{12}\tilde{w}_4^2, a_{12}\tilde{w}_2\tilde{w}_4^2, a_{12}\tilde{w}_2\tilde{w}_3^3$ and $a_{12}\tilde{w}_2^5, a_{12}\tilde{w}_2^3\tilde{w}_4, a_{12}\tilde{w}_2^3\tilde{w}_3$ prove to be independent as well.
Nonzero elements $a_{12} \tilde{w}_2^2 \tilde{w}_3$ and $a_{12} \tilde{w}_3 \tilde{w}_4$ prove to be independent after considering their $\tilde{w}_2^3 \tilde{w}_3$-multiples. Similarly, elements $a_{12} \tilde{w}_2^3 \tilde{w}_3$ and $a_{12} \tilde{w}_2 \tilde{w}_3 \tilde{w}_4$ are independent. Also $a_{12} \tilde{w}_2^3$ and $a_{12} \tilde{w}_2^2 \tilde{w}_4^2$, after considering $\tilde{w}_4$-multiples. The rest is obvious.

Lastly, we will show that it is possible to choose $a_{16}$ in such a way, that $a_{16} \tilde{w}_2^3 \tilde{w}_4 \neq 0$, $a_{16} \tilde{w}_2^3 = 0$ and $a_{16} \tilde{w}_2^3 \tilde{w}_2^3 = 0$ simultaneously. Start with picking $a_{16}$ as any element, such that $(a'_{16}, a_{12} \tilde{w}_2^2, a_{12} \tilde{w}_4, \tilde{w}_4^8)$ is a basis for $H^{16}(\tilde{G}_{11,4})$. The matrix of the cup product bilinear pairing $H^{16}(\tilde{G}_{11,4}) \times H^{12}(\tilde{G}_{11,4}) \to \mathbb{Z}_2$ with respect to bases $(a'_{16}, a_{12} \tilde{w}_2^2, a_{12} \tilde{w}_4, \tilde{w}_4^8)$ and $(a_{12}, \tilde{w}_2^2, \tilde{w}_2^3 \tilde{w}_3^2, \tilde{w}_2^4 \tilde{w}_4)$ is

$$
\begin{pmatrix}
\ast & \ast & \ast & \ast \\
\ast & 1 & 1 & 1 \\
\ast & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

where the stars represent unknown values. By Poincaré duality, the rows of this matrix are linearly independent, so there are following options for the last three values in the first row.

- If we have $(1 \ 0 \ 0)$, then we define $a_{16} = a'_{16} + a_{12} \tilde{w}_4$.
- If we have $(0 \ 0 \ 1)$, then we define $a_{16} = a'_{16}$.
- If we have $(1 \ 1 \ 0)$, then we define $a_{16} = a'_{16} + a_{12} \tilde{w}_2$.
- If we have $(0 \ 1 \ 1)$, then we define $a_{16} = a'_{16} + a_{12} \tilde{w}_2^2 + a_{12} \tilde{w}_4$.

Now that we are done with the examples, we are ready to discuss some patterns. Similar to the case $k = 3$ studied in [1] we predict there will be indecomposable element $a_{2t}$ in $H^{2t}(\tilde{G}_{2t+1,4})$ reflecting the case for $H^*(\tilde{G}_{2t,3})$.

It appears that for $2^t + 1 < n \leq 2^{t+1} - 4$ there are apart from Stiefel-Whitney classes $\tilde{w}_2, \tilde{w}_3, \tilde{w}_4$ at least two aditional indecomposable elements $a_{4n-3.2^t-4} \in H^{4n-3.2^t-4}(\tilde{G}_{n,4})$ and $a_{2t+1-4} \in H^{2t+1-4}(\tilde{G}_{n,4})$. Note that previously mentioned $a_{2t}$ can be thought of as also being of the form $a_{4n-3.2^t-4}$ for $n = 2^t + 1$.

From observing that the Poincaré dual to these $a_{4n-3.2^t-1-4}$ in our examples for $n = 9, 10, 11$ was always of the form $\tilde{w}_2^3 \tilde{w}_4$, we may reasonably anticipate these duals will exhibit some kind of stability in general.

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**References**


Faculty of Mathematics, Physics and Informatics, Comenius University Bratislava, Mlynská Dolina, 842 48 Bratislava, Slovakia

E-mail: tomas.rusin@fmph.uniba.sk