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Kybernetika, Vol. 55 (2019), No. 5, 879–895

Persistent URL: `http://dml.cz/dmlcz/147957`

KYBERNETIKA — VOLUME *55* (2019), NUMBER 5, PAGES 879–895

QUOTIENT STRUCTURES IN LATTICE EFFECT ALGEBRAS

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In this paper, we define some types of filters in lattice effect algebras, investigate some relations between them and introduce some new examples of lattice effect algebras. Then by using the strong filter, we find a CI-lattice congruence on lattice effect algebras, such that the induced quotient structure of it is a lattice effect algebra, too. Finally, under some suitable conditions, we get a quotient MV-effect algebra and a quotient orthomodular lattice, by this congruence relation.

Keywords: Lattice effect algebra, CI-lattice, Sasaki arrow, (strong, fantastic, implicative, positive implicative) filter, Riesz ideal, D-ideal, MV-effect algebra, orthomodular lattice

Classification: 06B10, 81R05

1. INTRODUCTION

Foulis and Pulmannová in [10] commenced an approach to the study of lattice effect algebras that emphasizes the structure as algebraic model for the semantics of (possibly) non-standard symbolic logics considering Sasaki product, Sasaki arrow and orthosuplement as the basic logical connectives and they named this kind of structures as CI-lattices. In [3] we investigated on material implications in lattice effect algebras and showed that Sasaki arrow is the best implication on lattice effect algebras. Moreover, in [16], it is introduced a complete axiomatisation for lattice effect algebra with Sasaki arrow, orthosuplement and falsum as the basic connectives.

In section 3, since the defined CI-lattice structure on lattice effect algebras contains Sasaki arrow as its implication, we shall adopt the types of filters in BL-algebras from [11] for lattice effect algebras, as well. Furthermore, we will define a strong filter as a kind of filter on lattice effect algebras which helps us to find a CI-lattice congruence and some quotient structures on lattice effect algebras. Then, we will try to investigate on the relations between all kinds of these filters and for this reason, build some new examples of lattice effect algebras. In section 4, we shall find a CI-lattice congruence by strong filters on lattice effect algebras, using the fact that they are the dual filters of Reisz ideals of lattice effect algebras. Some additional studies about ideals and congruences on (lattice) effect algebras, can be found in papers [1, 13, 15].

DOI: 10.14736/kyb-2019-5-0879

2. PRELIMINARIES

In this section, we collect the relevant definitions and results, that we use in the next sections.

A lattice $(L;\leq)$ is called *involutive*, if there exists a unitary operation "$'$" on L such that, for any $a,b\in L$, $a''=a$ and if $a\leq b$ then $b'\leq a'$. An *orthomodular lattice* is a bounded involutive lattice L such that, every element of L is *sharp*, that is for any $a\in L$, $a\wedge a'=0$, and it satisfies the *orthomodular law*: $a\leq b \;\Rightarrow\; a\vee(a'\wedge b)=b$.

A *CI-lattice* is a system $(L;\leq,\cdot,\rightarrow,0,1)$ in which $(L;\leq,0,1)$ is a bounded lattice and, for any $a,b,c\in L$,
(CI1): $1\cdot a=a\cdot 1=a$;
(CI2): $a\cdot c\leq b$ if and only if $c\leq a\rightarrow b$.

If the operation "$'$" on CI-lattice L is defined by $a'=a\rightarrow 0$, then the following properties are valid in L:
(CI3): $1'=0$;
(CI4): $b\leq(a\rightarrow(a.b))$;
(CI5): $a=1\rightarrow a$;
(CI6): $a\leq b \;\Leftrightarrow\; a\rightarrow b=1$;
(CI7): $a.b\leq a$;
(CI8): $a\cdot(a\rightarrow b)\leq b$.

An *effect algebra* $(E;\oplus,0,1)$ is a structure consisting of a set E, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $E\times E$ that satisfy the following conditions, for every $a,b,c\in E$:
(E1): If $a\oplus b$ is defined, then $b\oplus a$ is defined and $a\oplus b=b\oplus a$,
(E2): If $b\oplus c$ and $a\oplus(b\oplus c)$ are defined, then $a\oplus b$ and $(a\oplus b)\oplus c$ are defined and $a\oplus(b\oplus c)=(a\oplus b)\oplus c$,
(E3): For every $a\in E$, there exists a unique $a'\in E$ such that $a\oplus a'$ is defined and $a\oplus a'=1$.
(a' is called the *orthosupplement* of a and "$'$" is called the *orthosupplementation*).
(E4): If $a\oplus 1$ is defined then $a=0$.

If E is an effect algebra and for $a,b\in E$, $a\oplus b$ exists, then it is called "a is orthogonal to b" and denoted by $a\perp b$. The binary relation $\leq$ on an effect algebra E which is defined by

$$a\leq b \text{ if and only if there exists } c\in E \text{ such that } a\oplus c=b$$

is a partial order which is called the *effect partial order*. If the above partial ordered set $(E;\leq)$ is a lattice, then E is called a *lattice effect algebra*. A subset A of an effect algebra E is called a *sub-effect algebra* of E if (i): $0,1\in A$, (ii): if $a\in A$, then $a'\in A$, (iii): for all $a,b\in A$, if $a\perp b$ then $a\oplus b\in A$. A *sub-lattice effect algebra* of a lattice effect algebra E, is a subset of E which is a sub-effect algebra and a sub-lattice of E.

An effect algebra E is called *a horizontal sum* of the family $\{E_\chi\}_{\chi\in H}$ of its sub-effect algebras such that: (i) $E=\bigcup_{\chi\in H}E_\chi$, (ii) if $x\in E_{\chi_1}\backslash\{0,1\}$, $y\in E_{\chi_2}\backslash\{0,1\}$ and $\chi_1\neq\chi_2$, $\chi_1,\chi_2\in H$, then $x\wedge y=0$, $x\vee y=1$.

If $a\leq b$, then the element c for which $a\oplus c=b$ is uniquely determined and we denote it by $c=b\ominus a$ and we have $b\ominus a=(b'\oplus a)'$ (see [9]). For lattice effect algebra E and $a,b,c\in E$, if $a,b\leq c'$, then the *distributivity laws* $c\oplus(a\vee b)=(c\oplus a)\vee(c\oplus b)$ and

$c \oplus (a \wedge b) = (c \oplus a) \wedge (c \oplus b)$ are valid (see [6]). Moreover, two operations $a \otimes b$ and $a \rightarrow_s b$ on lattice effect algebra E are defined as follows:
(i): $a \otimes b = (a' \oplus (a \wedge b'))' = a \ominus (a \wedge b')$: *Sasaki product* (see [2, 10] and [3]),
(ii): $a \rightarrow_s b = a' \oplus (a \wedge b) = (a \ominus (a \wedge b))'$: *Sasaki arrow* (see [10] and [3]).

Proposition 2.1. (Foulis and Pulmannová [10]) If E is a lattice effect algebra, then $(E; \leq, \otimes, \rightarrow_s, 0, 1)$ is an involutive CI-lattice which satisfies the following properties:
(i) $c \leq a,\ c \leq b\ \Rightarrow c \leq a \otimes (a \rightarrow_s b)$ (divisibility law),
(ii) $[(a \rightarrow_s b) \rightarrow_s c] \otimes [(b \rightarrow_s a) \rightarrow_s c] \leq c$ (strong prelinearity law),
(iii) $a \otimes b \leq c' \ \Rightarrow\ a \otimes c \leq b'$ (self-adjointness law).

An *MV-algebra* is an algebra $(M; +, ^*, 0, 1)$ of type $(2, 1, 0, 0)$ satisfies the following properties:
(MV1): $(a + b) + c = a + (b + c)$,
(MV2): $a + b = b + a$,
(MV3): $a + 0 = a$,
(MV4): $a^{**} = a$,
(MV5): $a + 1 = 1$,
(MV6): $(a^* + b)^* + b = (b^* + a)^* + a$.

Proposition 2.2. (Foulis [8]) If $(M; +, ^*, 0, 1)$ is an MV-algebra, then $(M; \oplus, 0, 1)$ is a lattice effect algebra where the partial operation $\oplus$ is defined and $a \oplus b = a + b$ if and only if $a \leq b^*$.

A lattice effect algebra in Proposition 2.2, is called *the derived lattice effect algebra* from an MV-algebra. An effect algebra E is called to be an *MV-effect algebra* if there exists an MV-algebra $(M; +, ^*, 0, 1)$ such that E is isomorphic to its derived lattice effect algebra.

Proposition 2.3. (Foulis [8]) Every MV-effect algebra $(E; \oplus, 0, 1)$ is an MV-algebra $(E; +, ', 0, 1)$ where "$'$" is the orthosuplement operation of E and $a + b = a \oplus (a' \wedge b)$, for any $a, b \in E$.

Let E be a lattice effect algebra and $a, b \in E$. Then we say that a and b are *compatible* and write $a \leftrightarrow b$ if and only if $a \vee b = a \oplus (b \ominus (a \wedge b))$ or equivalently $a \oplus (b \ominus (a \wedge b))$ exists. Moreover, a maximal subset M of mutually compatible elements of E is called a *block* of E (see [17]).

Theorem 2.4. (Dvurečenskij [6]) For a lattice effect algebra E, the following statements are equivalent:
(i) for all $a, b \in E$, $a \otimes b = b \otimes a$,
(ii) all elements of E are mutually compatible,
(iii) for all $a, b \in E$, if $a \wedge b = 0$ then $a \perp b$,
(iv) E is an MV-effect algebra.

Theorem 2.5. (Riečanová [17]) Let E be a lattice effect algebra. Then
(i) every block of E is a sub-lattice effect algebra of E,
(ii) E is a set-theoretic union of its blocks,
(iii) every block of E is an MV-effect algebra.

Let E be a lattice effect algebra and $\leq$ be the effect partial order on E. Then a non-empty subset I of E is called an *ideal* of E if (I1): $a, b \in I$ and $a \perp b$ implies $a \oplus b \in I$; (I2): $a \in I$ and $b \leq a$ implies $b \in I$. It is clear that $\{0\}$ and E are ideals of E. We say that an ideal I of E is *proper*, if $I \neq E$, I is called *prime*, if $a \wedge b \in I$ implies that $a \in I$ or $b \in I$. An ideal I of E is called *Riesz ideal*, if for all $a, b, c \in E$, if $a \in I$ and $a \leq b \oplus c$, then there exist $b_1, c_1 \in I$ such that $b_1 \leq b$, $c_1 \leq c$ and $a \leq b_1 \oplus c_1$ [13]. Moreover, from [1], another equivalent form of a Riesz ideal is a *D-ideal* which is defined as an ideal I such that, if $a \in I$, then $(a \vee b) \ominus b \in I$, for every $b \in E$. A *D-congruence* on E is a lattice congruence $\sim$ on E such that for all $a_1, a_2, b_1, b_2 \in E$, if $a_1 \sim a_2$, $b_1 \sim b_2$, $b_1 \leq a_1$ and $b_2 \leq a_2$, then $a_1 \ominus b_1 \sim a_2 \ominus b_2$.

Notation. From now on, in this paper, we let E be a lattice effect algebra, unless otherwise stated.

3. SOME KINDS OF FILTERS IN LATTICE EFFECT ALGEBRAS

In this section, we define some kinds of filters in lattice effect algebras and, considering the definitions of various types of filters from the theory of BL-algebras in [11], we investigate their relations among them. Then we give some examples for this results.

Proposition 3.1. Let F be a subset of E containing 1. We consider the following conditions:
(F1): if $a \in F$ and $a \to_s b \in F$, then $b \in F$,
(FF): if $c \to_s (b \to_s a) \in F$ and $c \in F$, then $((a \to_s b) \to_s b) \to_s a \in F$,
(IF): if $a \to_s (b \to_s c) \in F$ and $a \to_s b \in F$, then $a \to_s c \in F$,
(PI): if $a \to_s ((b \to_s c) \to_s b) \in F$ and $a \in F$, then $b \in F$.

Now, if F satisfies in the one of conditions (FF), (IF) or (PI), then F satisfies in the condition (F1).

Proof. Let F satisfy (FF), $a \in F$ and $a \to_s b \in F$. Since $a \to_s (1 \to_s b) = a \to_s b$, we get $a \to_s (1 \to_s b) \in F$. Now, since F satisfies (FF), we obtain $((b \to_s 1) \to_s 1) \to_s b \in F$. Hence $b \in F$. Now let F satisfy (IF), $a \in F$ and $a \to_s b \in F$. Since for every $c \in F$, $1 \to_s c = c$ we have $1 \to_s (a \to_s b) \in F$ and $1 \to_s a \in F$ and so $b = 1 \to_s b \in F$. Finally, let F satisfy (PI), $a \in F$ and $a \to_s b \in F$. Then $a \to_s ((b \to_s 1) \to_s b) = a \to_s (1 \to_s b) = a \to_s b$ implies $a \to_s ((b \to_s 1) \to_s b) \in F$. Since F satisfies (PI) and $a \in F$, we have $b \in F$. $\square$

Definition 3.2. Let F be a subset of E, $1 \in F$ and $a, b, c \in E$. Then F is called
(i) a *filter* of E, if it satisfies (F1),
(ii) a *fantastic filter* of E, if it satisfies (FF),
(iii) an *implicative filter* of E, if it satisfies (IF),
(iv) a *positive implicative filter* of E, if it satisfies (PI),
(v) a *strong filter* of E, if F is a filter and for any $a \in F$ and $b \in E$, $b \to_s a \in F$.

Note. By Proposition 3.1, any fantastic (implicative, positive implicative) filter of E is a filter.

Definition 3.3. A lattice effect algebra E is called *simple* if E has only two trivial filters, namely $\{1\}$ and E. Clearly, trivial filters are strong filters.

Proposition 3.4. Let F be a non-empty subset of E. Then the following statements are equivalent:
(F1): F is a filter of E,
(F2): (i) $a, b \in F$ implies $a \otimes b \in F$; (ii) $a \in F$ and $a \leq b$ imply $b \in F$, for any $a, b \in E$,
(F3): for any $a, b, c \in E$, if $a, b \in F$ and $a \otimes b \leq c$, then $c \in F$.

Proof. (F1) $\Leftrightarrow$ (F2): Let F be a filter of E and $a, b \in F$. By (CI4), $b \leq a \rightarrow_s (a \otimes b)$ and so by (CI6) $b \rightarrow_s (a \rightarrow_s (a \otimes b)) = 1 \in F$. Hence, by using two times of (F1), we get $a \otimes b \in F$. Now, let $a \in F$, $b \in E$ and $a \leq b$. Then $a \rightarrow_s b = 1 \in F$ and so by (F1), $b \in F$.

Conversely, suppose that F is a non-empty subset of E satisfying (F2). Since F is a non-empty subset of E, by (ii), $1 \in F$. Now, let $a, a \rightarrow_s b \in F$. By (F2), $a \otimes (a \rightarrow_s b) \in F$ and by (CI8), $a \otimes (a \rightarrow_s b) \leq b$. Then by (ii), $b \in F$. Hence F is a filter of E.

(F1) $\Leftrightarrow$ (F3): Let F be a filter of E and $a, b, c \in E$. If $a, b \in F$ and $a \otimes b \leq c$, then by (F2) $a \otimes b \in F$ and so $c \in F$.

Conversely, suppose that F is a non-empty subset of E satisfying (F3). Since F is a non-empt subset, we can easily see that $1 \in F$. Now, let $a, a \rightarrow_s b \in F$. Then by (CI8), $a \otimes (a \rightarrow_s b) \leq b$. Hence by (F3), $b \in F$ and so F is a filter of E. $\square$

Lemma 3.5. Let $a, b \in E$. Then $(a \rightarrow_s b) \rightarrow_s b \geq a$ or equivalently $a \rightarrow_s ((a \rightarrow_s b) \rightarrow_s b) = 1$.

Proof. First we should notice that $(a \rightarrow_s b) \wedge b \geq a \wedge b$ since:

$$(a \rightarrow_s b) \wedge b = (a' \oplus (a \wedge b)) \wedge b \geq (a' \vee (a \wedge b)) \wedge b \geq (a \wedge b) \wedge b = a \wedge b.$$

Thus, $(a \rightarrow_s b) \ominus ((a \rightarrow_s b) \wedge b) \leq (a \rightarrow_s b) \ominus (a \wedge b) = a'$. Now it follows that $((a \rightarrow_s b) \ominus ((a \rightarrow_s b) \wedge b))' \geq a$ which means $(a \rightarrow_s b) \rightarrow_s b \geq a$ and equivalently $a \rightarrow_s ((a \rightarrow_s b) \rightarrow_s b) = 1$, by (CI6). $\square$

Proposition 3.6. Let F be a filter of E. Then F is a fantastic filter if and only if $c \rightarrow_s (b \rightarrow_s a) \in F$ and $c \in F$ implies that $(((a \rightarrow_s b) \rightarrow_s b) \ominus a)' \in F$.

Proof. It is clear by Lemma 3.5. $\square$

Proposition 3.7. Let F be a filter of E. Then F is a positive implicative filter if and only if $(a \rightarrow_s b) \rightarrow_s a \in F$ implies that $a \in F$, for all $a, b \in E$.

Proof. ($\Rightarrow$) Let F be a positive implicative filter of E and $(a \rightarrow_s b) \rightarrow_s a \in F$, for $a, b \in E$. Since by (CI5), $1 \rightarrow_s ((a \rightarrow_s b) \rightarrow_s a) = (a \rightarrow_s b) \rightarrow_s a$, we have $1 \rightarrow_s ((a \rightarrow_s b) \rightarrow_s a) \in F$. Now, since $1 \in F$ and F is a positive implicative filter, we get $a \in F$.

($\Leftarrow$) Let $a \in F$ and $a \rightarrow_s ((b \rightarrow_s c) \rightarrow_s b) \in F$, for $a, b, c \in E$. Since F is a filter, $(b \rightarrow_s c) \rightarrow_s b \in F$ and so by hypothesis $b \in F$. $\square$

By Theorem 2.5, we can assume that E is the set-theoretic union of MV-effect algebras $\{M_i\}_{i\in I}$ and we can write $E = \bigcup_{i\in I} M_i$, where every M_i is a maximal subset of mutually compatible elements of E and is called a block of E. As we mentioned in preliminary, if we define the operations $+$ and * on M_i as $a + b = a \oplus (a' \wedge b)$ and $a^* = a'$ then each M_i can be viewed as an MV-algebra. Hence, if we consider $\rightarrow_m$ as the implication of MV-algebras, then $a \rightarrow_m b = a^* + b$ and so $a \rightarrow_m b = a \rightarrow_s b$, in every M_i. Moreover, replacing $\rightarrow_s$ by $\rightarrow_m$ in Definition 3.2, we get the same kinds of filters which are defined for MV-algebras in [11] for every M_i as an MV-algebra.

Corollary 3.8. Let E be an MV-effect algebra. Then
(i) every filter of E is a fantastic filter;
(ii) implicative filters and positive implicative filters of E are equivalent;
(iii) every filter of E is a strong filter.

Proof. (i) This follows from [11] since, in MV-algebras, every filter is a fantastic filter.

(ii) This is true, since from [7] we know that implicative filters and positive implicative filters are equivalent in MV-algebras.

(iii) Assume that F is a filter of E, $a \in F$ and $b \in E$. It is clear that $a \leq b' + a$ and so $a \leq b \rightarrow_m a$. Hence, $b \rightarrow_m a \in F$ which means $b \rightarrow_s a \in F$. $\square$

Lemma 3.9. Let F be a filter of E and $a \in E$. Then $a, a' \in F$ if and only if $F = E$.

Proof. Let F be a filter of E and $a, a \rightarrow_s 0 = a' \in F$. Then $0 \in F$ and so $F = E$. The converse is clear. $\square$

Proposition 3.10. Let $I \neq \emptyset$ be an arbitrary index set, $\{E_i\}_{i\in I}$ be a set of lattice effect algebras such that $E_i \cap E_j = \{0, 1\}$ for any $i, j \in I$ and $i \neq j$, and each F_k in $\{F_i\}_{i\in I}$ be a proper subset of E_k.

(i) If $\{F_i\}_{i\in I}$ are filters of $\{E_i\}_{i\in I}$, respectively, then $\bigcup_{k\in I} F_k$ is a filter of the horizontal sum of $\{E_i\}_{i\in I}$.
(ii) If $\{F_i\}_{i\in I}$ are fantastic filters of $\{E_i\}_{i\in I}$, respectively, then $\bigcup_{k\in I} F_k$ is a fantastic filter of the horizontal sum of $\{E_i\}_{i\in I}$.
(iii) Let for all $a \in E_i$, a or a' be an element of F_i $(i \in I)$. If $\{F_i\}_{i\in I}$ are implicative filters of $\{E_i\}_{i\in I}$, respectively, then $\bigcup_{k\in I} F_k$ is an implicative filter of the horizontal sum of $\{E_i\}_{i\in I}$.
(iv) If $\{F_i\}_{i\in I}$ are positive implicative filters of $\{E_i\}_{i\in I}$, respectively, then $\bigcup_{k\in I} F_k$ is a positive implicative filter of the horizontal sum of $\{E_i\}_{i\in I}$.

Proof. (i) If $a \in F_i$ and $b \in E_i \backslash F_i$ $(i \in I)$, then $a \rightarrow_s b \in E_i \backslash F_i$, since otherwise, b would be in F_i.

Let $a \in F_i$ and $b \in E_j \backslash F_j$ $(i, j \in I$ and $i \neq j)$. Then $a \rightarrow_s b = a' \in E_i$, since $a \wedge b = 0$ as we are calculating in the horizontal sum of E_is. Moreover, $a' \notin F_i$, since otherwise, by Lemma 3.9, we obtain $F_i = E_i$ which is a contradiction with our hypothesis and so $a \to_s b \in E_i \backslash F_i$.

From the above cases, we conclude that if $a \in \bigcup_{k \in I} F_k$ and $b \notin \bigcup_{k\in I} F_k$, then $a \to_s b \notin \bigcup_{k\in I} F_k$. Therefore, $\bigcup_{k\in I} F_k$ is a filter of the horizontal sum of $\{E_i\}_{i\in I}$.

(ii) Let $c, c \to_s (b \to_s a) \in \bigcup_{k\in I} F_k$. For some E_i, we have $c \in E_i$. Thus $c, c \to_s (b \to_s a) \in E_i$ and so $c, c \to_s (b \to_s a) \in F_i$. Hence $((a \to_s b) \to_s b) \to_s a \in F_i$, since for each i, F_i is a fantastic filter of E_i. Then it follows that $((a \to_s b) \to_s b) \to_s a \in \bigcup_{k\in I} F_k$ which means that $\bigcup_{k\in I} F_k$ is a fantastic filter of the horizontal sum of $\{E_i\}_{i\in I}$.

(iii) Let $a \to_s (b \to_s c), a \to_s b \in \bigcup_{k\in I} F_k$. If $a \in E_i$ then clearly $a \to_s (b \to_s c), a \to_s b \in E_i$ and so $a \to_s (b \to_s c), a \to_s b \in F_i$.

If $b \notin E_i$, we have also $b \to_s c \notin E_i$ and so $a' \in F_i$, since $a \to_s (b \to_s c) = a \to_s b = a'$. Thus from $a' \leq a \to_s c$ we conclude that $a \to_s c \in F_i$ which means $a \to_s c \in \bigcup_{k\in I} F_k$.

If $b, c \in E_i$, by the fact that F_i is an implicative filter of E_i, we have $a \to_s c \in F_i$ which means $a \to_s c \in \bigcup_{k\in I} F_k$.

If $b \in E_i$ and $c \notin E_i$, then $a \to_s (b \to_s c) = a \to_s b' \in F_i$. From $a \to_s b, a \to_s b' \in F_i$ we conclude that $a \notin F_i$, since otherwise $b, b' \in F_i$ and so $F_i = E_i$, by Lemma 3.9, which is a contradiction with the assumption that F_i is a proper subset of E_i. Thus, using the hypothesis, from $a \notin F_i$ we get $a' \in F_i$ and so $a \to_s c \in F_i$, since $a \to c = a'$. Hence, $a \to_s c \in \bigcup_{k\in I} F_k$.

Therefore, from the above three cases, $\bigcup_{k\in I} F_k$ is an implicative filter of the horizontal sum of $\{E_i\}_{i\in I}$.

(iv) Let $a \in F_i$ and $b \in E_i \backslash F_i$ $(i \in I)$. If $c \in E_i$, then $a \to_s ((b \to_s c) \to_s b) \in E_i \backslash F_i$ since E_i is closed under operations and F_i is a positive implicative filter of E_i. If $c \notin E_i$, then $a \to_s ((b \to_s c) \to_s b) = a \to_s (b' \to_s b)$, since $b \wedge c = 0$. It is clear that $a \to_s (b' \to b) \in E_i$. Suppose that $a \to_s (b' \to_s b) \in F_i$. Then we can rewrite it as $a \to_s ((b \to_s 0) \to b) = a \to_s (b' \to_s b)$ and conclude $0 \in F_i$ which means $E_i = F_i$, since F_i is a positive implicative filter of E_i. But it is a contradiction with the hypothesis and thus $a \to_s ((b \to_s c) \to_s b) \in E_i \backslash F_i$.

Let $a \in F_i$ and $b \in E_j \backslash F_j$ $(i, j \in I$ and $i \neq j)$. If $c \in E_i$ then as before $a \to_s ((b \to_s c) \to_s b) = a \to_s (b' \to b)$. Now, from $a \in E_i$ and $b' \to_s b \in E_j$, we conclude $a \to_s ((b \to_s c) \to_s b) = a'$ and then, by Lemma 3.9, we get $a \to_s ((b \to_s c) \to_s b) \in E_i \backslash F_i$. If $c \notin E_i$, then easily we can see that $(b \to_s c) \to_s b \in E_j$. Thus, we get again $a \to_s ((b \to_s c) \to_s b) = a'$ and as before we have $a \to_s ((b \to_s c) \to_s b) \in E_i \backslash F_i$.

From the above cases we conclude that if $a \in \bigcup_{k\in I} F_k$ and $b \notin \bigcup_{k\in I} F_k$ then $a \to_s ((b \to_s c) \to_s b) \notin \bigcup_{k\in I} F_k$, for all c in the horizontal sum of $\{E_i\}_{i\in I}$. Therefore, $\bigcup_{k\in I} F_k$ is a positive implicative filter of the horizontal sum of $\{E_i\}_{i\in I}$.

$\square$

Now, in the following we give some examples of lattice effect algebras, their blocks and filters.

Example 3.11. Let $t \in \mathbb{R}$ and $E = ([0, t]; +, 0, t)$ be a lattice effect algebra, in which $a + b$ is defined if and only if $a + b \in [0, t]$. Then E has only two filters $\{t\}$ and $[0, t]$. Moreover $[0, t]$ is the only implicative and positive implicative filter of it. For the proof, it is clear that $\{t\}$ is a filter of E. Now, let F be a filter of E and $t \neq b \in F$. Then there is $a \in F$ such that $t > a \geq b$ and $2a - t \geq 0$. Now, it is easy to see that $2a - t < a$ and from

$$a \to_s (2a - t) = (t - a) + (a \wedge (2a - t)) = t - a + 2a - t = a,$$

we get $a \to_s (2a - t) \in F$. Hence $(2a - t) \in F$. If we continue on a similar way, we will find an element less than $\frac{t}{2}$ which is a member of F and so $\frac{t}{2} \in F$. Now, since $\frac{t}{2} \to_s 0 = \frac{t}{2}$ we get $0 \in F$, which means that $F = [0, t]$. For the second part of example, let F be an implicative filter of E. Then it is enough to show that $\frac{t}{2} \in F$. We should notice that E is an MV-effect algebra and F also is a positive implicative filter of E. By a simple calculation, we get $\frac{t}{2} \to_s \frac{t}{2} = t$ and $\frac{t}{2} \to_s (\frac{t}{2} \to_s 0) = t$ and so $\frac{t}{2} \in F$.

Example 3.12. Let $E = \{0, a, b, c, d, e, f, g, h, 1\}$ and the operations "$'$" and $\oplus$ are defined on E as follows:

x	0	a	b	e	f
x'	1	d	c	h	g

$\oplus$	0	a	b	c	d	e	f	g	h	1
0	0	a	b	c	d	e	f	g	h	1
a	a	c	d	$-$	1	$-$	$-$	$-$	$-$	$-$
b	b	d	$-$	1	$-$	f	$-$	h	$-$	$-$
c	c	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$
d	d	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
e	e	$-$	f	$-$	$-$	$-$	$-$	c	1	$-$
f	f	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$
g	g	$-$	h	$-$	$-$	c	1	$-$	$-$	$-$
h	h	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Then $(E; \oplus, 0, 1)$ is a lattice effect algebra. Now, the blocks are $\{0, a, b, c, d, 1\}$ and $\{0, b, c, e, f, g, h, 1\}$ and the induced order is visualized in Figure 1.

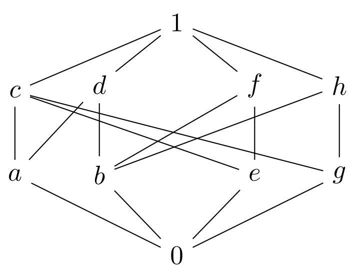

Fig. 1.

It is easy to check that every filter of E is fantastic filter. The filter $\{c, 1\}$ is strong but the filters $\{f, 1\}$, $\{h, 1\}$, $\{c, e, f, 1\}$ and $\{c, g, h, 1\}$ are not strong. The only non-trivial implicative and positive implicative filter which is simultaneously a strong filter is $\{b, d, f, h, 1\}$.

Example 3.13. Let $E = \{0, a, b, c, d, e, f, g, h, i, j, k, l, 1\}$ and the operations "$'$" and $\oplus$ are defined on E as follows:

x	0	a	b	e	f	i	j
x'	1	d	c	h	g	l	k

$\oplus$	0	a	b	c	d	e	f	g	h	i	j	k	l	1
0	0	a	b	c	d	e	f	g	h	i	j	k	l	1
a	a	b	$-$	d	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
b	b	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
c	c	d	1	$-$	$-$	f	$-$	h	$-$	$-$	$-$	$-$	$-$	$-$
d	d	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
e	e	$-$	$-$	f	$-$	$-$	$-$	b	1	$-$	$-$	$-$	$-$	$-$
f	f	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$
g	g	$-$	$-$	h	$-$	b	1	$-$	$-$	j	$-$	l	$-$	$-$
h	h	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
i	i	$-$	$-$	$-$	$-$	$-$	$-$	j	$-$	$-$	$-$	f	1	$-$
j	j	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$
k	k	$-$	$-$	$-$	$-$	$-$	$-$	l	$-$	f	1	$-$	$-$	$-$
l	l	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Then $(E; \oplus, 0, 1)$ is a lattice effect algebra. Moreover, the blocks are $\{0, a, b, c, d, 1\}$, $\{0, b, c, e, f, g, h, 1\}$ and $\{0, i, f, g, j, k, l, 1\}$ and the induced order is visualized in Figure 2.

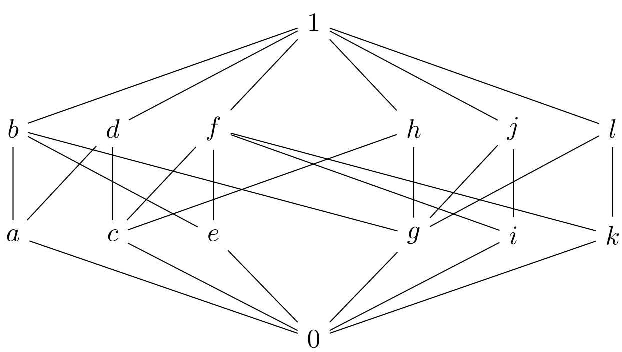

Fig. 2.

Every filter of E is fantastic. $\{c, d, f, h, 1\}$, $\{c, d, f, h, i, j, 1\}$ and $\{c, d, f, h, l, k, 1\}$ are implicative and positive implicative filters of E, which are not strong filters. Moreover, we can claim that there is no non-trivial strong filter, in this example.

Example 3.14. Let $E = \{0, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, 1\}$ and the operations "$'$" and $\oplus$ are defined on E as follows:

x	0	a	b	e	f	i	j	m	n
x'	1	d	c	h	g	l	k	p	o

$\oplus$	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	1
0	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	1
a	a	b	$-$	d	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
b	b	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
c	c	d	1	$-$	$-$	f	$-$	h	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
d	d	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
e	e	$-$	$-$	f	$-$	$-$	$-$	b	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
f	f	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
g	g	$-$	$-$	h	$-$	b	1	$-$	$-$	j	$-$	l	$-$	o	p	$-$	$-$	$-$
h	h	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
i	i	$-$	$-$	$-$	$-$	$-$	$-$	j	$-$	$-$	$-$	f	1	$-$	$-$	$-$	$-$	$-$
j	j	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$
k	k	$-$	$-$	$-$	$-$	$-$	$-$	l	$-$	f	1	$-$	$-$	n	$-$	p	$-$	$-$
l	l	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	p	$-$	$-$	$-$	$-$
m	m	$-$	$-$	$-$	$-$	$-$	$-$	o	$-$	$-$	$-$	n	p	i	f	j	1	$-$
n	n	$-$	$-$	$-$	$-$	$-$	$-$	p	$-$	$-$	$-$	$-$	$-$	f	$-$	1	$-$	$-$
o	o	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	p	$-$	j	1	$-$	$-$	$-$
p	p	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	1	$-$	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Then $(E; \oplus, 0, 1)$ is a lattice effect algebra. Moreover, the blocks are $\{0, a, b, c, d, 1\}$, $\{0, b, c, e, f, g, h, 1\}$ and $\{0, f, g, i, j, k, l, m, n, o, p, 1\}$ and the induced order is visualized in Figure 3.

It is easy to see that every filter of E is fantastic and the only non-trivial implicative and positive implicative filter of E is $\{1, l, f, c, d, h, k, p, n\}$ which is not a strong filter. Moreover, we can claim that there is no non-trivial strong filter, in this example.

Example 3.15. Let the MV-algebra $M = [0, \cdots, n] \times [0, \cdots, m]$ be the direct product of two finite MV-chains. Then any (implicative) filters are a direct product of (implicative) filters on these chains. Hence, from the fact that every filter in M is a strong filter and there are some filters in M which are not implicative filters, we can conclude that there exist lattice effect algebras in which some strong filters are neither implicative filters nor positive implicative filters.

Open problems: It seems that, in lattice effect algebras, every filter is not necessarily a fantastic filter or implicative filters and positive implicative filters are not necessarily the same. However, in all instances of lattice effect algebras we could make, there are not any examples of none-fantastic filters or any examples which shows that implicative filters and positive implicative filters are distinct. The question is how can we construct instances of lattice effect algebras to find such kinds of filters?

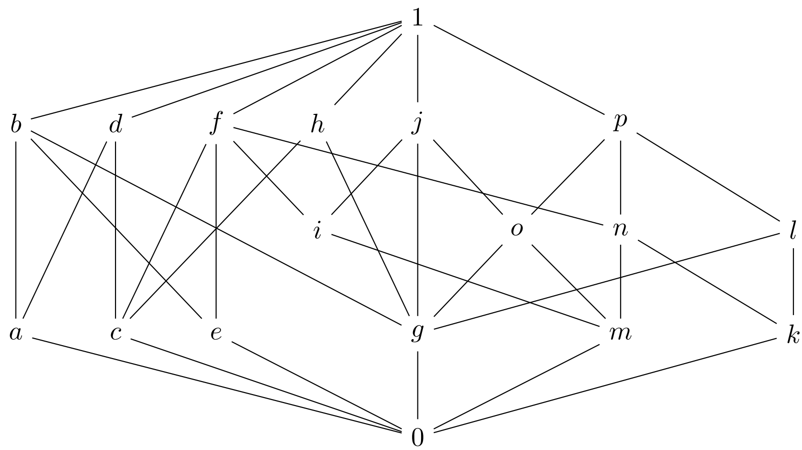

Fig. 3.

On the other hand, are there any proofs to show that every filter is a fantastic filter or every implicative filter is a positive implicative filter or vice versa? Of course in all presented examples we have $b \to_s a \leq ((a \to_s b) \to_s b) \to_s a$ so if we could prove that this holds in any lattice effect algebras then every filter would be a fantastic filter. Moreover, in every lattice effect algebra E, if we could show that

$$a \to_s (b \to_s c) \leq (a \to_s b) \to_s (((a \to_s c) \to_s 0) \to_s (a \to_s c))$$

which is equivalent to $a \to_s (b \to_s c) \leq (a \to_s b) \to_s ((a \to_s c)' \to_s (a \to_s c))$ or $a \to_s (b \to_s c) \leq (a \to_s b) \to_s ((a \to_s c) \oplus ((a \to_s c)' \wedge (a \to_s c)))$, then, supposing that F is a positive implicative filter on E and $a \to_s b, a \to_s (b \to_s c) \in F$, we could prove that F is an implicative filter on E. Since, as F is a filter, from $a \to_s (b \to_s c) \in F$ we would have $(a \to_s b) \to_s (((a \to_s c) \to_s 0) \to_s (a \to_s c)) \in F$. Then from $a \to_s b \in F$ and the fact that F is a positive implicative filter, we would obtain $a \to_s c \in F$ which concludes the claim.

4. QUOTIENT STRUCTURES IN LATTICE EFFECT ALGEBRAS

In [1] the authors proved that by using a D-ideal of a lattice effect algebra E, we can derive a D-congruence on E and vice versa. Now, in this section, we obtain a CI-lattice congruence on lattice effect algebras by using strong filter, which is the dual of a D-ideal (Riesz ideal). Then we investigate the quotient structures of lattice effect algebras, by this congruence.

Lemma 4.1. Let E be a lattice effect algebra and $a, b \in E$. Then

$$a \wedge b = a \otimes (a \to_s b) = b \otimes (b \to_s a) = (a \to_s (a \to_s b)')' = (b \to_s (b \to_s a)')'$$

P r o o f. From (CI7), we have $a\otimes(a \to_s b) \le a$ and from (CI8), we have $a\otimes(a \to_s b) \le b$. Hence $a \otimes (a \to_s b) \le a \wedge b$. Now, by using the divisibility laws in Proposition 2.1, we obtain $a \wedge b = a \otimes (a \to_s b) = b \otimes (b \to_s a)$. Then, by using the translation of $\otimes$ to $\to_s$, we get that $a \wedge b = (a \to_s (a \to_s b)')' = (b \to_s (b \to_s a)')'$. □

Proposition 4.2. If F is a strong filter of E and $a, b \in F$, then $a \wedge b \in F$.

P r o o f. Let F be a strong filter of E and $a, b \in F$. Then $a \to_s b \in F$ and so $a \otimes (a \to_s b) \in F$. Now, by Lemma 4.1, $a \wedge b \in F$. □

Proposition 4.3. Let F be a strong filter of E. Then for any $a, b \in E$, the following statements are equivalent:
(i) $(a \wedge b) \oplus (a' \wedge b') \in F$,
(ii) $(a \to_s b) \wedge (b \to_s a) \in F$,
(iii) $(a \to_s b), (b \to_s a) \in F$.

P r o o f. (i)⇔(ii) By using the distributivity laws, for any $a, b \in E$, we have

$$(a \to_s b) \wedge (b \to_s a) = (a' \oplus (a \wedge b)) \wedge (b' \oplus (a \wedge b)) = (a \wedge b) \oplus (a' \wedge b')$$

(ii)⇒(iii) Let $(a \to_s b) \wedge (b \to_s a) \in F$, for $a, b \in E$. Since $(a \to_s b) \wedge (b \to_s a) \le (a \to_s b), (b \to_s a)$ and F is a filter, we get $(a \to_s b), (b \to_s a) \in F$.

(iii)⇒(ii) It follows from Proposition 4.2. □

Definition 4.4. An equivalence relation $\sim$ on E is called a *CI-lattice congruence*, if $a \sim b$ and $c \sim d$ imply that $(a \wedge c) \sim (b \wedge d)$, $(a \vee c) \sim (b \vee d)$, $(a \otimes c) \sim (b \otimes d)$ and $(a \to_s c) \sim (b \to_s d)$, for any $a, b, c, d \in E$.

Proposition 4.5. An equivalence relation $\sim$ on E is a CI-lattice congruence if and only if $a \sim b$ implies $a' \sim b'$ and $a \sim b, c \sim d$ imply that $(a \otimes c) \sim (b \otimes d)$ and $(a \to_s c) \sim (b \to_s d)$, for any $a, b, c, d \in E$.

P r o o f. (⇒) Let $\sim$ be a CI-lattice congruence. It is enough to show that $a \sim b$ implies $a' \sim b'$, for any $a, b \in E$. Since $0 \sim 0$, from $a \sim b$ we get $(a \to_s 0) \sim (b \to_s 0)$ and so $a' \sim b'$.

(⇐) Using the assumptions and straightforward application of Lemma 4.1, if $a \sim b$ and $c \sim d$, then $(a \wedge c) \sim (b \wedge d)$, for any $a, b, c, d \in E$. Moreover, using the fact that E is an involutive lattice we have $a \vee b = (a' \wedge b')'$, then $a \sim b, c \sim d$ imply $(a \vee c) \sim (b \vee d)$, for any $a, b, c, d \in E$. □

Proposition 4.6. An equivalence relation $\sim$ on E is a CI-lattice congruence if and only if it is a D-congruence.

P r o o f. Let $\sim$ be a CI-lattice congruence, $a_1 \sim a_2$, $b_1 \sim b_2$, $b_1 \le a_1$ and $b_2 \le a_2$, for any $a_1, a_2, b_1, b_2 \in E$. Then $a_1 \to_s b_1 \sim a_2 \to_s b_2$ and so $a_1' \oplus (a_1 \wedge b_1) \sim a_2' \oplus (a_2 \wedge b_2)$. Using the facts $a_1 \wedge b_1 = b_1$ and $a_2 \wedge b_2 = b_2$, we get $a_1' \oplus b_1 \sim a_2' \oplus b_2$. Then $(a_1' \oplus b_1)' \sim (a_2' \oplus b_2)'$, which means that $a_1 \ominus b_1 \sim a_2 \ominus b_2$. Therefore, $\sim$ is a D-congruence.

Conversely, suppose that $\sim$ is a D-congruence. If for $a, b \in E$, $a \sim b$, then by using $1 \sim 1$, we conclude $1 \ominus a \sim 1 \ominus b$ and so $a' \sim b'$. Furthermore, if $a_1 \sim a_2$ and $b_1 \sim b_2$, then $(a_1 \ominus (a_1 \wedge b_1))' \sim (a_2 \ominus (a_2 \wedge b_2))'$ and $a_1 \ominus (a_1 \wedge b_1') \sim a_2 \ominus (a_2 \wedge b_2')$. Hence, $a_1' \oplus (a_1 \wedge b_1) \sim a_2' \oplus (a_2 \wedge b_2)$ and $(a_1' \oplus (a_1 \wedge b_1'))' \sim (a_2' \oplus (a_2 \wedge b_2'))'$ which mean $a_1 \to_s b_1 \sim a_2 \to_s b_2$ and $a_1 \otimes b_1 \sim a_2 \otimes b_2$. Therefore, by Proposition 4.5, $\sim$ is a CI-lattice congruence. □

Proposition 4.7. Let I be a D-ideal of E and for any $a, b \in E$, $a \Delta b = (a \vee b) \ominus (a \wedge b)$. Then the relation $\sim_I$ on E which is defined by $a \sim_I b$ if and only if $a \Delta b \in I$, is a D-congruence on E.

Proof. It follows from [1, Theorem 4.5]. □

Definition 4.8. Let F be a strong filter of E. Then we define the relation $\sim_F$ on E as follow:

$$a \sim_F b \Longleftrightarrow (a \to_s b), (b \to_s a) \in F.$$

Proposition 4.9.
(i) If F is a strong filter of E, then $I_F = \{a' \mid a \in F\}$ is a D-ideal of E.
(ii) If I is a D-ideal of E, then $F_I = \{a' \mid a \in I\}$ is a strong filter of E.

Proof. (i) Since $0' = 1 \in F$, we get $0 \in I_F$ and so I_F is a non-empty subset of E. Let $a, b \in I_F$ and $a \perp b$. Then $a', b' \in F$ and $b \leq a'$. Thus $(a \oplus b)' = (a \oplus (a' \wedge b))' = a' \otimes b' \in F$ and so $a \oplus b \in I_F$. Let $a \in I_F$ and $b \leq a$. Then $a' \in F$ and $a' \leq b'$ and so $b' \in F$. Thus $b \in I_F$. Now, let $a \in I_F$ and $b \in E$. Then $a' \in F$ and $b' \in E$. Hence

$$((a \vee b) \ominus b)' = (a \vee b)' \oplus b = b \oplus (b' \wedge a') = b' \to_s a' \in F$$

which means $(a \vee b) \ominus b \in I_F$.

(ii) Since $1' = 0 \in I$, we get $1 \in F_I$ and so F_I is a non-empty subset of E. Let $a, b \in F_I$. Then $a', b' \in I$ and $a \wedge b' \in I$. Hence $a' \perp (a \wedge b')$. Thus $(a \otimes b)' = a' \oplus (a \wedge b') \in I$ and so $a \otimes b \in F_I$. Let $a \in F_I$ and $a \leq b$. Then $a' \in I$ and $b' \leq a'$ and so $b' \in I$. Thus $b \in F_I$. Now, let $a \in F_I$ and $b \in E$. Then $a' \in I$ and $b' \in E$. Thus

$$(b \to_s a)' = (b' \oplus (b \wedge a))' = (b' \vee a') - b' \in I$$

and so $b \to_s a \in F_I$. □

Corollary 4.10. Let F be a strong filter of E. Then the relation $\sim_F$ is a CI-lattice congruence on E.

Proof. As $\sim_{I_F}$ is a D-congruence and so by Proposition 4.6, a CI-lattice congruence on E, it is enough to show that $a \sim_F b$ if and only if $a \sim_{I_F} b$. Indeed, using Proposition 4.3, we have $a \sim_F b$ if and only if $(a \wedge b) \oplus (a' \wedge b') \in F$. But

$$(a \vee b) \ominus (a \wedge b) = ((a \vee b)' \oplus (a \wedge b))' = ((a \wedge b) \oplus (a' \wedge b'))'$$

and so $(a \vee b) \ominus (a \wedge b) \in I_F$ which means that $a \sim_{I_F} b$. □

Corollary 4.11. Let I be a D-ideal of E. Then $a \sim_I b$ if and only if $a \sim_{F_I} b$.

Proof. It is easy to see that $I_{F_I} = I$ and so $a \sim_{F_I} b$ if and only if $a \sim_{I_{F_I}} b$ if and only if $a \sim_I b$. □

Lemma 4.12. Let $\sim$ be a CI-lattice congruence on E. Then $F_\sim = [1]_\sim$ is a strong filter of E.

Proof. Clearly, we have $1 \in F_\sim$. Let $a, a \to_s b \in F_\sim$. Then $a \sim 1$ and $a \to_s b \sim 1$. From $a \sim 1$ and $b \sim b$, we get $a \to_s b \sim 1 \to_s b = b$. Then by using $a \to_s b \sim 1$, we get $b \sim 1$ which means that $b \in F_\sim$. Let $a \in F_\sim$ and $b \in E$. Then $a \sim 1$ and by reflexivity of $\sim$, we have $b \sim b$. Hence $b \to_s a \sim b \to_s 1 = 1$ and so $b \to_s a \in F_\sim$. □

Proposition 4.13.
(i) If F is a strong filter of E, then $F = F_{\sim_F}$,
(ii) If $\sim$ is a CI-lattice congruence on E then $\sim = \sim_{F_\sim}$.

Proof. (i) We can simply see that $a \in F$ if and only if $(a \to_s 1) \wedge (1 \to_s a) \in F$ if and only if $a \sim_F 1$ if and only if $a \in F_{\sim_F}$.

(ii) Let $\sim$ be a CI-lattice congruence on E. If $a \sim b$, using $b \sim b$, we can get $a \to_s b \sim b \to_s b = 1$. As $F_\sim = [1]_\sim$, we have $a \to_s b \in F_\sim$ and similarly $b \to_s a \in F_\sim$. Since $F_\sim$ is a strong filter, using Corollary 4.2 we conclude $(a \to_s b) \wedge (b \to_s a) \in F_\sim$ and so $a \sim_{F_\sim} b$. On the other hand let $a \sim_{F_\sim} b$. Then $(a \to_s b) \wedge (b \to_s a) \in F_\sim$ and so $(a \to_s b), (b \to_s a) \in F_\sim$ which means $a \to_s b \sim 1$ and, by Proposition 4.5, $(a \to_s b)' \sim 0$. Hence $a \to_s (a \to_s b)' \sim a \to_s 0 = a'$ and so $(a \to_s (a \to_s b)')' \sim a$. Thus $a \wedge b \sim a$ and by a similar way, $a \wedge b \sim b$. Therefore, $a \sim b$. □

The following proposition is necessary to understand which conditions make a CI-lattice into a lattice effect algebra.

Proposition 4.14. Let $(L; \leq, \cdot, \to, 0, 1)$ be an involutive CI-lattice, in which the involution is defined as $a' = a \to 0$ for $a \in L$, and let $a \oplus b = a' \to b$ if and only if $a \leq b'$. Then $(L; \oplus, 0, 1)$ is a lattice effect algebra in which $a \to b = a \to_s b$ and $a \cdot b = a \otimes b$ if and only if the following conditions hold:
(cw1): $a \cdot b = (a \to b')'$;
(cw2): If $a \leq b'$ then $a' \to b = b' \to a$ and $a \leq a' \to b$;
(cw3): If $a \leq b'$ and $a \leq c'$ then $a' \to b \leq c'$ implies $a' \to c \leq b'$;
(cw4): If $b' \leq c$ and $a' \leq b \cdot c$ then $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Proof. $(\Rightarrow)$ Let $(L; \oplus, 0, 1)$ be a lattice effect algebra in which $a \to b = a \to_s b$ and $a \cdot b = a \otimes b$. It is easy to check that (cw1) and (cw2) are hold and (cw3) is (self-adjointness law) in Proposition 2.1. For condition (cw4), we must notice that if $b' \leq c$ and $a' \leq b \otimes c$ then $b \oplus c = (b' \otimes c')'$ and $a \oplus (b \oplus c) = (a' \otimes (b' \otimes c')')'$ and so (cw4) is an equivalent form of the condition (E2) of effect algebra's definition.

$(\Leftarrow)$ It follows from [16, Corollary 1]. □

Now we try to change the conditions in Proposition 4.14, into some equations. Then using the quotient of E on a CI-lattice congruence, we can obtain a lattice effect algebra.

Proposition 4.15. Let $(L; \leq, \cdot, \rightarrow, 0, 1)$ be an involutive CI-lattice, in which the involution is defined as $a' = a \rightarrow 0$ for any $a \in L$, and let $a \oplus b = a' \rightarrow b$ if and only if $a \leq b'$. Then $(L; \oplus, 0, 1)$ is a lattice effect algebra in which $a \rightarrow b = a \rightarrow_s b$ and $a \cdot b = a \otimes b$ if and only if the following conditions hold:

(cw1): $a \cdot b = (a \rightarrow b')'$;
(cw2'): $a' \rightarrow (b \wedge a') = (a \vee b') \rightarrow a$ and $a \wedge (a' \rightarrow (b \wedge a')) = a$;
(cw3'): $(a' \rightarrow (c \wedge (a' \rightarrow b)')) \vee (a \vee b') = (a \vee b')$;
(cw4'): $(a' \wedge (b \cdot c))' \cdot (b \cdot (b' \vee c)) = ((a' \wedge (b \cdot c))' \cdot b) \cdot (b' \vee c)$.

Proof. From Proposition 4.14, we need to show that (cw2')$\Leftrightarrow$(cw2), (cw3')$\Leftrightarrow$(cw3) and (cw4')$\Leftrightarrow$(cw4).
(cw2')$\Leftrightarrow$(cw2): For right to left, from $a \leq a \vee b'$ it follows that $a' \rightarrow (b \wedge a') = (a \vee b') \rightarrow a$ and $a \leq a' \rightarrow (a' \wedge b)$. The converse is also true since from $a \leq b'$ we have $b \wedge a' = b$ and $a \vee b' = b'$.
(cw3')$\Leftrightarrow$(cw3): For right to left, we know $a \leq (a' \wedge b)'$ and we can easily show that $a \leq (c \wedge (a' \rightarrow b)')'$ and $a' \rightarrow (a' \wedge b) \leq (c \wedge (a' \rightarrow b)')'$. Then by (cw3) we conclude $a' \rightarrow (c \wedge (a' \rightarrow b)') \leq a \vee b'$ and so (cw3').
(cw4')$\Leftrightarrow$(cw4): Right to left is clear, since $b' \leq b' \vee c$ and $(a' \wedge (b \cdot c)) \leq (b \cdot (b' \vee c))$. Conversely, if $b' \leq c$ and $a' \leq b.c$, then $b' \vee c = c$ and $(a' \wedge (b \cdot c))' = a$. So (cw4) follows from (cw4'). □

Theorem 4.16. Let F be a strong filter of E. Then E/F is a lattice effect algebra.

Proof. Since $\sim_F$ is a CI-lattice congruence on E, obviously, E/F is a CI-lattice. Since $a'' = a$ and $a \wedge b = a$ iff $b' \wedge a' = b'$, we conclude that E/F is an involutive CI-lattice. Moreover, by Proposition 4.15, E satisfies (cw1), (cw2'), (cw3') and (cw4') and using the properties of CI-lattice congruence $\sim_F$, we conclude that E/F also satisfies (cw1), (cw2'), (cw3') and (cw4'). Therefore, by Proposition 4.15, E/F is a lattice effect algebra. □

Definition 4.17. A strong filter F of E is called *prime*, if $a \vee b \in F$ implies that $a \in F$ or $b \in F$.

The non-trivial strong filter in Example 3.12, is a prime strong filter.

Proposition 4.18. Let F be a strong filter of E. Then F is a prime strong filter of E if and only if E/F is linearly ordered.

Proof. Let F be a prime strong filter of E. We show that E/F is a linearly ordered. By distributivity laws, we have:

$$(a \oplus (a' \wedge b')) \vee (b \oplus (a' \wedge b')) = (a \vee b) \oplus (a' \wedge b') = (a \vee b) \oplus (a \vee b)' = 1 \in F.$$

Since F is a prime strong filter of E, we have $a \oplus (a' \wedge b') \in F$ or $b \oplus (a' \wedge b') \in F$. But $(a \wedge (a \vee b)) \oplus (a' \wedge (a' \wedge b')) = a \oplus (a' \wedge b')$ and so $a \sim_F a \vee b$ or $b \sim_F a \vee b$. Hence, $[a \vee b] = [a]$ or $[a \vee b] = [b]$ which means $[b] \leq [a]$ or $[a] \leq [b]$. On the other hand, let for strong filter F of E, E/F be linearly ordered and $a \vee b \in F$. By the linearly of E/F, we have $[a \vee b] = [a]$ or $[a \vee b] = [b]$, which means either $a \in F$ or $b \in F$.

□

Theorem 4.19. Let F be a strong filter of E. Then,
(i) E/F is an MV-effect algebra if and only if $a \otimes b \sim_F b \otimes a$, for any $a, b \in E$,
(ii) E/F is an orthomodular lattice if and only if $a \wedge a' \sim_F 0$, for any $a, b \in E$.

Proof. (i) From Theorem 4.16, E/F is a lattice effect algebra. Now, by Theorem 2.4, E/F is an MV-effect algebra if and only if $[a] \otimes [b] = [b] \otimes [a]$ if and only if $[a \otimes b] = [b \otimes a]$ if and only if $a \otimes b \sim_F b \otimes a$.

(ii) Since a lattice effect algebra is an orthomodular lattice if and only if every element is sharp [12], hence E/F is an orthomodular lattice if and only if $a \wedge a' \sim_F 0$.

□

Corollary 4.20. Let F be a prime strong filter of E. Then E/F is an MV-effect algebra.

Proof. By Proposition 4.18, we know that E/F is linearly ordered. Now, let $[a]_{\sim_F} \wedge [b]_{\sim_F} = 0$. Then $[a]_{\sim_F} = 0$ or $[b]_{\sim_F} = 0$ and so $[a]_{\sim_F} \leq [b]'_{\sim_F}$. Hence, $[a]_{\sim_F} \perp [b]_{\sim_F}$ and so by Theorem 2.4, E/F is an MV-effect algebra. □

Theorem 4.21. If a strong filter F of E is a (positive) implicative filter, then E/F is an orthomodular lattice.

Proof. In both cases, to apply Theorem 4.19 (ii), we show, for $a \in E$, that $a \wedge a' \to_s 0 \in F$ then we conclude $a \wedge a' \sim_F 0$, since $0 \to_s a \wedge a' = 1 \in F$.

Case 1: Let F be an implicative filter of E. Then from $(a \wedge a') \to_s (a \to_s 0) = (a \wedge a') \to_s a' = 1 \in F$ and $(a \wedge a') \to_s a = 1 \in F$ we have $a \wedge a' \to_s 0 \in F$.

Case 2: Let F be a positive implicative filter of E. Then from $((a' \vee a) \to_s 0) \to_s (a' \vee a) = 1 \in F$ and Proposition 3.7, we have $a' \vee a \in F$ and so $a \wedge a' \to_s 0 \in F$. □

Open problem: Let E be a lattice effect algebra and F be a strong filter. For which classes of quotient structures $E/F$are the equivalent conditions that F is a fantastic or an implicative or a positive implicative filter?

5. CONCLUSION

In this paper, we defined some types of filters on lattice effect algebras adapted from theory of BL-algebras and investigated among their relations, using some propositions and theorems and constructing some new examples. Then we defined CI-lattice congruences which can be obtained by strong filters on lattice effect algebras and we showed that using this kind of congruence the quotient structure of a lattice effect algebra would be again a lattice effect algebra. Finally we introduced some constraints on the derived quotient algebras to obtain an MV-effect algebra or orthomodular lattice.

(Received October 20, 2018)

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