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Strong functors on many-sorted sets

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Dedicated to the memory of Věra Trnková

Abstract. We show that, on a category of many-sorted sets, the only functors that admit a cartesian strength are those that are given componentwise.

Keywords: strong functor; strong monad; many-sorted set

Classification: 18A22

1. Introduction

This note is inspired by Věra Trnková’s pioneering work on set functors [4], and her later work with J. Adámek on functors on many-sorted sets [1].

Strong functors and strong monads arise in many settings. For example, in computer science they are used in denotational semantics [3], as well as programming languages such as Haskell.

On Set, the situation is extremely straightforward: every endofunctor admits a unique strength, so the notions of endofunctor and strong endofunctor coincide.

Our task is to examine the situation for the category Set^I, where I is a set. This category is known as “many-sorted sets”. It turns out that the situation here is very different. The only endofunctors on this category that are strong are those that (up to isomorphism) are of the form \( \prod_{i \in I} H_i \) for a family \((H_i)_{i \in I}\) of endofunctors on Set. For example, the monad on Set^2 generated by a unary operation from sort 0 to sort 1 sends \((X,Y)\) to \((X,Y + X)\), and this (even as a mere endofunctor) is not strong.

We first review the basic facts about strong functors in Section 2. Then in Section 3 we see how, on a product category \( \prod_{i \in I} C_i \), under suitable assumptions, the notion of strong functor reduces to that of strong functor on the categories \((C_i)_{i \in I}\). This gives the desired result for Set^I.

2. Strong functors

We begin with the basic definition of strength.

Definition 1 ([2]). Let \( \mathcal{C} \) be a monoidal category.

1. A strong endofunctor \( H \) on \( \mathcal{C} \) (more precisely: left strong) is an endofunctor equipped with a strength, i.e. a family of maps \( t_{a,b} : a \otimes Hb \longrightarrow H(a \otimes b) \)
natural in \(a, b \in \mathcal{C}\) and satisfying

\[
\begin{align*}
H_a &\quad \lambda_a \\
1 \otimes H_a &\xrightarrow{t_{1,a}} H(1 \otimes a)
\end{align*}
\]

\[
\begin{align*}
(a \otimes b) \otimes Hc &\xrightarrow{t_{a \otimes b, c}} H((a \otimes b) \otimes c) \\
\alpha_{a,b,Hc} &\quad H\alpha_{a,b,c}
\end{align*}
\]

(2) The identity strong endofunctor on \(\mathcal{C}\) is the identity endofunctor with strength given at \(a, b \in \mathcal{C}\) by \(\text{id}_{a \otimes b}\).

(3) The composite (in diagrammatic order) of strong endofunctors \((H, s)\) and \((K, t)\) on \(\mathcal{C}\) is the endofunctor \(KH\) with strength given at \(a, b \in \mathcal{C}\) by

\[
a \otimes KHb \xrightarrow{t_{a,Hb}} K(a \otimes Hb) \xrightarrow{Ks_{a,b}} KH(a \otimes b).
\]

(4) For strong endofunctors \((H, s)\) and \((K, t)\) on \(\mathcal{C}\), a natural transformation \(\gamma: H \rightarrow K\) is strong when

\[
\begin{align*}
a \otimes Hb &\xrightarrow{s_{a,b}} H(a \otimes b) \\
\alpha \otimes \gamma_b &\quad \gamma_{a \otimes b} \\
a \otimes Kb &\xrightarrow{t_{a,b}} K(a \otimes b)
\end{align*}
\]

**Definition 2.**

(1) Let \(\mathcal{C}\) be a category. The category\(^1\) of endofunctors on \(\mathcal{C}\) and natural transformations, strict monoidal via composition (diagrammatic order, let us say), is written \(\text{Endo}(\mathcal{C})\). A **monad** on \(\mathcal{C}\) is a monoid in \(\text{Endo}(\mathcal{C})\), and a **monad morphism** is a monoid morphism.

(2) Let \(\mathcal{C}\) be a monoidal category. The category of strong endofunctors and strong natural transformations, strict monoidal via composition (diagrammatic order), is written \(\text{StrEndo}(\mathcal{C})\). A **strong monad** on \(\mathcal{C}\) is a monoid in \(\text{StrEndo}(\mathcal{C})\), and a strong monad morphism is a monoid morphism.

**Definition 3.** A monoidal category \(\mathcal{C}\) is said to be **strength-compliant** when every natural transformation between strong endofunctors is strong, i.e. the forgetful

\(^{1}\text{In a suitably large sense.}\)
functor

\[ \mathcal{U}_C : \text{StrEndo}(C) \longrightarrow \text{Endo}(C) \]

is fully faithful.

**Proposition 4.** Let \( C \) be a monoidal category. If \( C \) is strength-compliant, then any endofunctor on \( C \) admits at most one strength.

**Proof:** Let \( H \) be an endofunctor with strengths \( s \) and \( t \). Then \( \text{id}_H \) is a strong natural transformation \( (H, s) \ra (H, t) \), so \( s = t \).

A category \( C \) with a terminal object is **well-pointed** when the functor \( M : C \longrightarrow \text{Set} \) sending \( X \) to \( C(1, X) \) is faithful. The following is adapted from [3, Proposition 3.4], with the same proof.

**Proposition 5.** Let \( C \) be a cartesian category that is well-pointed.

1. The category \( C \) is strength-compliant.
2. An endofunctor \( H \) on \( C \) is strong if and only if for all \( a, b \in C \) the function

\[ \tilde{t}_{a,b} : M(a \times Fb) \longrightarrow MF(a \times b) \]

\[ \langle x, y \rangle \mapsto \left( 1 \xrightarrow{y} Fb \xrightarrow{F\langle \langle \rangle; x, \text{id}_b \rangle} F(a \times b) \right) \]

is in the range of \( M \), and then \( t_{a,b} \) is the preimage of \( \tilde{t}_{a,b} \).

**Proposition 6.** \( \text{Set} \), with cartesian structure, is strength-compliant, and any endofunctor on it admits a unique strength.

3. **Product categories**

We turn now to endofunctors on product categories.

**Lemma 7.** Let \( (C_i)_{i \in I} \) be a family of categories with an initial object. The functor

\[ \prod_{i \in I} \text{Endo}(C_i) \longrightarrow \text{Endo} \left( \prod_{i \in I} C_i \right), \]

\[ (H_i)_{i \in I} \mapsto \prod_{i \in I} H_i \]

is a coreflective embedding, i.e. fully faithful with a right adjoint.

**Proof:** Recall that for any adjunction

\[ \begin{array}{ccc} \mathcal{C} & \xleftarrow{\scriptstyle F} & \mathcal{D} \\ \scriptstyle G \downarrow & & \downarrow \scriptstyle G \end{array} \]

the left adjoint \( F \) is fully faithful if and only if the unit is an isomorphism.
Let \( j \in I \). We first define the coreflection

\[
C_j \xrightarrow{\pi_j} \prod_{i \in I} C_i
\]

as follows.

- \( \pi_j \) is the projection \( b \mapsto b_j \).
- \( \text{Pad}_j \) sends \( a \in C_j \) to the tuple whose \( i \)th component is \( a \) if \( i = j \) and 0 otherwise.
- The unit sends \( a \in C_j \) to \( \text{id}_a \).
- The counit \( \varepsilon_j \) sends \( b \in \prod_{i \in I} C_i \) to the map \( \text{Pad}_j \pi_j b \xrightarrow{\cdot} b \) whose \( i \)th component is \( \text{id}_{b_i} \) if \( i = j \) and \([ \cdot ] : 0 \xrightarrow{\cdot} b_i \) otherwise.

This gives rise to the coreflection

\[
\pi_j \rightarrow \prod_{i \in I} C_i \rightarrow \varepsilon_j
\]

where the unit is identity, and the counit sends \( G \in \text{Endo}(\prod_{i \in I} C_i) \) to the natural transformation whose \( b \)th component for \( b \in \prod_{i \in I} C_i \) is

\[
(G_{i \in I, b} : G_i \text{Pad}_i b_i \xrightarrow{\cdot} G_i b)_{i \in I}
\]

writing \( G_i \equiv \pi_i G \).

In a monoidal category \( C \), an object \( 0 \) is right-distributive initial when \( 0 \otimes a \) is initial for all \( a \in C \). This implies that \( 0 \) is initial.
Proposition 8. Let \((C_i)_{i \in I}\) be a family of monoidal categories with a right-distributive initial object. The functor

\[
\prod_{i \in I} \text{StrEndo}(C_i) \rightarrow \text{StrEndo}\left(\prod_{i \in I} C_i\right),
\]

\((H_i)_{i \in I} \mapsto \prod_{i \in I} H_i\)

is an equivalence.

Proof: Firstly, for \(j \in I\) we define a lift

\[
\text{StrEndo}\left(\prod_{i \in I} C_i\right) \xrightarrow{Q_j} \text{StrEndo}(C_j)
\]

\[
\text{Endo}\left(\prod_{i \in I} C_i\right) \xrightarrow{\pi_j \ldots \text{Pad}_j} \text{Endo}(C_j)
\]

It sends \((G, t)\) to the functor \(a \mapsto G_j(\text{Pad}_j a)\) with strength given at \(a, b\) by

\[
a \otimes G_j \text{Pad}_j b = (\text{Pad}_j a \otimes G \text{Pad}_j b)_j \xrightarrow{(\text{Pad}_j a \otimes \text{Pad}_j b)_j} G_j(\text{Pad}_j a \otimes \text{Pad}_j b) \xrightarrow{G_j m^j_{a,b}} G_j \text{Pad}_j(a \otimes b)
\]

where we write

\[
m^j_{a,b} : \text{Pad}_j a \otimes \text{Pad}_j b \cong \text{Pad}_j(a \otimes b)
\]

for the isomorphism whose \(i\)th component is \(\text{id}_{a \otimes b}\) if \(i = j\) and the unique map \(0 \otimes 0 \rightarrow 0\) otherwise.

We then obtain a lift of the coreflection in Lemma 7:
It remains to prove that the counit is an isomorphism. That means we must show for \((G, t) \in \text{StrEndo}(\prod_{i \in I} C_i)\) and \(b \in \prod_{i \in I} C_i\) and \(j \in I\) that the map

\[ G_j \varepsilon_{j,b} : G_j \text{Pad}_j b_j \to G_j b \]

is an isomorphism. We write

- \(z\) for the object in \(\prod_{i \in I} C_i\) whose \(i\)th component is 1 for \(i = j\) and 0 otherwise
- \(n : z \to 1\) for the map whose \(i\)th component is identity for \(i = j\) and the unique map \(0 \to 1\) otherwise
- \(u : z \otimes b \cong \text{Pad}_j b_j\) for the map whose \(i\)th component is \(\lambda b_j\) for \(i = j\) and the unique map \(0 \otimes b_i \to 0\) otherwise.

Then the inverse of \(G_j \varepsilon_{j,b}\) is

\[ G_j b \xrightarrow{(\lambda^{-1}_{Gb})_j} (1 \otimes Gb)_j \xrightarrow{id} (z \otimes Gb)_j \xrightarrow{(t_z,b)_j} G_j(z \otimes b) \xrightarrow{G_j u} G_j \text{Pad}_j b_j \]

as shown by the following commutative diagrams:
Corollary 9. Let $(C_i)_{i \in I}$ be a family of monoidal categories that are strength-compliant and have a right-distributive initial object.

1. $\prod_{i \in I} C_i$ is strength-compliant and has a right-distributive initial object.
2. An endofunctor on $\prod_{i \in I} C_i$ is strong if and only if it is isomorphic to $\prod_{i \in I} H_i$ for some $H \in \prod_{i \in I} \text{StrEndo}(C_i)$.

Proof:

1. Consider the commutative square of strict monoidal functors:

\[
\begin{array}{ccc}
\prod_{i \in I} \text{StrEndo}(C_i) & \xrightarrow{\Pi} & \text{StrEndo}(\prod_{i \in I} C_i) \\
\downarrow \text{StrEndo} & & \downarrow \text{StrEndo} \\
\prod_{i \in I} \text{Endo}(C_i) & \xrightarrow{\Pi} & \text{Endo}(\prod_{i \in I} C_i)
\end{array}
\]

The left and lower functors are fully faithful, and the upper functor is an equivalence, so the right functor is fully faithful.

2. From Proposition 8.

Theorem 10.

1. $\text{Set}^I$ with cartesian structure is strength-compliant.
2. An endofunctor on $\text{Set}^I$ is strong if and only if it is isomorphic to $\prod_{i \in I} H_i$ for some family $(H_i)_{i \in I}$ of endofunctors on $\text{Set}$.

Proof:

1. From Proposition 6 and Corollary 9 (1).

2. From Proposition 6 and Corollary 9 (2).
REFERENCES


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