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## Insertion of a Contra-Baire-1 (Baire-.5) Function

*Majid Mirmiran*

**Abstract.** Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that  $F_\sigma$ -kernel of sets are  $F_\sigma$ -sets.

### 1 Introduction

A generalized class of closed sets was considered by Maki in 1986 [17]. He investigated the sets that can be represented as union of closed sets and called them  $V$ -sets. Complements of  $V$ -sets, i.e., sets that are intersection of open sets are called  $\Lambda$ -sets [17].

Recall that a real-valued function  $f$  defined on a topological space  $X$  is called  $A$ -continuous [20] if the preimage of every open subset of  $\mathbb{R}$  belongs to  $A$ , where  $A$  is a collection of subsets of  $X$ . Most of the definitions of function used throughout this paper are consequences of the definition of  $A$ -continuity. However, for unknown concepts the reader may refer to [4], [10]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1], [3], [7], [8], [9], [11], [12], [19].

Results of Katětov [13], [14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that  $F_\sigma$ -kernel of sets are  $F_\sigma$ -sets.

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A real-valued function  $f$  defined on a topological space  $X$  is called *contra-Baire-.5* (*Baire-.5*) if the preimage of every open subset of  $\mathbb{R}$  is a  $G_\delta$ -set in  $X$  [21].

If  $g$  and  $f$  are real-valued functions defined on a space  $X$ , we write  $g \leq f$  (resp.  $g < f$ ) in case  $g(x) \leq f(x)$  (resp.  $g(x) < f(x)$ ) for all  $x$  in  $X$ .

The following definitions are modifications of conditions considered in [15].

A property  $P$  defined relative to a real-valued function on a topological space is a *B-.5-property* provided that any constant function has property  $P$  and provided that the sum of a function with property  $P$  and any Baire-.5 function also has property  $P$ . If  $P_1$  and  $P_2$  are *B-.5-properties*, the following terminology is used:

- (i) A space  $X$  has the *weak B-.5-insertion property* for  $(P_1, P_2)$  if and only if for any functions  $g$  and  $f$  on  $X$  such that  $g \leq f$ ,  $g$  has property  $P_1$  and  $f$  has property  $P_2$ , then there exists a Baire-.5 function  $h$  such that  $g \leq h \leq f$ .
- (ii) A space  $X$  has the *B-.5-insertion property* for  $(P_1, P_2)$  if and only if for any functions  $g$  and  $f$  on  $X$  such that  $g < f$ ,  $g$  has property  $P_1$  and  $f$  has property  $P_2$ , then there exists a Baire-.5 function  $h$  such that  $g < h < f$ .

In this paper, for a topological space that  $F_\sigma$ -kernel of sets are  $F_\sigma$ -sets, is given a sufficient condition for the weak *B-.5-insertion property*. Also for a space with the weak *B-.5-insertion property*, we give a necessary and sufficient condition for the space to have the *B-.5-insertion property*. Several insertion theorems are obtained as corollaries of these results.

## 2 The Main Result

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

**Definition 1.** Let  $A$  be a subset of a topological space  $(X, \tau)$ . We define the subsets  $A^\Lambda$  and  $A^V$  as follows:

$$A^\Lambda = \bigcap \{O : O \supseteq A, O \in (X, \tau)\}$$

and

$$A^V = \bigcup \{F : F \subseteq A, F^c \in (X, \tau)\}.$$

In [6], [16], [18],  $A^\Lambda$  is called the *kernel* of  $A$ .

**Definition 2.** We define the subsets  $G_\delta(A)$  and  $F_\sigma(A)$  as follows:

$$G_\delta(A) = \bigcup \{O : O \subseteq A, O \text{ is } G_\delta\text{-set}\}$$

and

$$F_\sigma(A) = \bigcap \{F : F \supseteq A, F \text{ is } F_\sigma\text{-set}\}.$$

$F_\sigma(A)$  is called the  *$F_\sigma$ -kernel* of  $A$ .

The following first two definitions are modifications of conditions considered in [13], [14].

**Definition 3.** If  $\rho$  is a binary relation in a set  $S$  then  $\bar{\rho}$  is defined as follows:  $x \bar{\rho} y$  if and only if  $y \rho v$  implies  $x \rho v$  and  $u \rho x$  implies  $u \rho y$  for any  $u$  and  $v$  in  $S$ .

**Definition 4.** A binary relation  $\rho$  in the power set  $P(X)$  of a topological space  $X$  is called a *strong binary relation* in  $P(X)$  in case  $\rho$  satisfies each of the following conditions:

1. If  $A_i \rho B_j$  for any  $i \in \{1, \dots, m\}$  and for any  $j \in \{1, \dots, n\}$ , then there exists a set  $C$  in  $P(X)$  such that  $A_i \rho C$  and  $C \rho B_j$  for any  $i \in \{1, \dots, m\}$  and any  $j \in \{1, \dots, n\}$ .
2. If  $A \subseteq B$ , then  $A \bar{\rho} B$ .
3. If  $A \rho B$ , then  $F_\sigma(A) \subseteq B$  and  $A \subseteq G_\delta(B)$ .

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 5.** If  $f$  is a real-valued function defined on a space  $X$  and if

$$\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$$

for a real number  $\ell$ , then  $A(f, \ell)$  is a *lower indefinite cut set* in the domain of  $f$  at the level  $\ell$ .

We now give the following main results:

**Theorem 1.** Let  $g$  and  $f$  be real-valued functions on the topological space  $X$ , that  $F_\sigma$ -kernel of sets in  $X$  are  $F_\sigma$ -sets, with  $g \leq f$ . If there exists a strong binary relation  $\rho$  on the power set of  $X$  and if there exist lower indefinite cut sets  $A(f, t)$  and  $A(g, t)$  in the domain of  $f$  and  $g$  at the level  $t$  for each rational number  $t$  such that if  $t_1 < t_2$  then  $A(f, t_1) \rho A(g, t_2)$ , then there exists a Baire-.5 function  $h$  defined on  $X$  such that  $g \leq h \leq f$ .

*Proof.* Let  $g$  and  $f$  be real-valued functions defined on the  $X$  such that  $g \leq f$ . By hypothesis there exists a strong binary relation  $\rho$  on the power set of  $X$  and there exist lower indefinite cut sets  $A(f, t)$  and  $A(g, t)$  in the domain of  $f$  and  $g$  at the level  $t$  for each rational number  $t$  such that if  $t_1 < t_2$  then  $A(f, t_1) \rho A(g, t_2)$ .

Define functions  $F$  and  $G$  mapping the rational numbers  $\mathbb{Q}$  into the power set of  $X$  by  $F(t) = A(f, t)$  and  $G(t) = A(g, t)$ . If  $t_1$  and  $t_2$  are any elements of  $\mathbb{Q}$  with  $t_1 < t_2$ , then  $F(t_1) \bar{\rho} F(t_2), G(t_1) \bar{\rho} G(t_2)$ , and  $F(t_1) \rho G(t_2)$ . By Lemmas 1 and 2 of [14] it follows that there exists a function  $H$  mapping  $\mathbb{Q}$  into the power set of  $X$  such that if  $t_1$  and  $t_2$  are any rational numbers with  $t_1 < t_2$ , then  $F(t_1) \rho H(t_2), H(t_1) \rho H(t_2)$  and  $H(t_1) \rho G(t_2)$ .

For any  $x$  in  $X$ , let

$$h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}.$$

We first verify that  $g \leq h \leq f$ : If  $x$  is in  $H(t)$  then  $x$  is in  $G(t')$  for any  $t' > t$ ; since  $x$  in  $G(t') = A(g, t')$  implies that  $g(x) \leq t'$ , it follows that  $g(x) \leq t$ . Hence

$g \leq h$ . If  $x$  is not in  $H(t)$ , then  $x$  is not in  $F(t')$  for any  $t' < t$ ; since  $x$  is not in  $F(t') = A(f, t')$  implies that  $f(x) > t'$ , it follows that  $f(x) \geq t$ . Hence  $h \leq f$ .

Also, for any rational numbers  $t_1$  and  $t_2$  with  $t_1 < t_2$ , we have

$$h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\sigma(H(t_1)).$$

Hence  $h^{-1}(t_1, t_2)$  is a  $G_\delta$ -set in  $X$ , i.e.,  $h$  is a Baire-.5 function on  $X$ .  $\square$

The above proof used the technique of Theorem 1 of [13].

**Theorem 2.** *Let  $P_1$  and  $P_2$  be  $B$ -.5-property and  $X$  be a space that satisfies the weak  $B$ -.5-insertion property for  $(P_1, P_2)$ . Also assume that  $g$  and  $f$  are functions on  $X$  such that  $g < f$ ,  $g$  has property  $P_1$  and  $f$  has property  $P_2$ . The space  $X$  has the  $B$ -.5-insertion property for  $(P_1, P_2)$  if and only if there exist lower cut sets  $A(f - g, 3^{-n+1})$  and there exists a decreasing sequence  $\{D_n\}$  of subsets of  $X$  with empty intersection and such that for each  $n$ ,  $X \setminus D_n$  and  $A(f - g, 3^{-n+1})$  are completely separated by Baire-.5 functions.*

*Proof.* Assume that  $X$  has the weak  $B$ -.5-insertion property for  $(P_1, P_2)$ . Let  $g$  and  $f$  be functions such that  $g < f$ ,  $g$  has property  $P_1$  and  $f$  has property  $P_2$ . By hypothesis there exist lower cut sets  $A(f - g, 3^{-n+1})$  and there exists a sequence  $(D_n)$  such that  $\bigcap_{n=1}^{\infty} D_n = \emptyset$  and such that for each  $n$ ,  $X \setminus D_n$  and  $A(f - g, 3^{-n+1})$  are completely separated by Baire-.5 functions. Let  $k_n$  be a Baire-.5 function such that  $k_n = 0$  on  $A(f - g, 3^{-n+1})$  and  $k_n = 1$  on  $X \setminus D_n$ . Let a function  $k$  on  $X$  be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the  $B$ -.5-properties, the function  $k$  is a Baire-.5 function. Since  $\bigcap_{n=1}^{\infty} D_n = \emptyset$  and since  $k_n = 1$  on  $X \setminus D_n$ , it follows that  $0 < k$ . Also  $2k < f - g$ : In order to see this, observe first that if  $x$  is in  $A(f - g, 3^{-n+1})$ , then  $k(x) \leq 1/4(3^{-n})$ . If  $x$  is any point in  $X$ , then  $x \notin A(f - g, 1)$  or for some  $n$ ,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case  $2k(x) < 1$ , and in the latter  $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$ . Thus if  $f_1 = f - k$  and if  $g_1 = g + k$ , then  $g < g_1 < f_1 < f$ . Since  $P_1$  and  $P_2$  are  $B$ -.5-properties, then  $g_1$  has property  $P_1$  and  $f_1$  has property  $P_2$ . Since  $X$  has the weak  $B$ -.5-insertion property for  $(P_1, P_2)$ , then there exists a Baire-.5 function such that  $g_1 \leq h \leq f_1$ . Thus  $g < h < f$ , it follows that  $X$  satisfies the  $B$ -.5-insertion property for  $(P_1, P_2)$ . (The technique of this proof is by Katětov [13].)

Conversely, let  $g$  and  $f$  be functions on  $X$  such that  $g$  has property  $P_1$ ,  $f$  has property  $P_2$  and  $g < f$ . By hypothesis, there exists a Baire-.5 function such that  $g < h < f$ . We follow an idea contained in Lane [15]. Since the constant function 0 has property  $P_1$ , since  $f - h$  has property  $P_2$ , and since  $X$  has the  $B$ -.5-insertion property for  $(P_1, P_2)$ , then there exists a Baire-.5 function  $k$  such that  $0 < k < f - h$ . Let  $A(f - g, 3^{-n+1})$  be any lower cut set for  $f - g$  and let

$$D_n = \{x \in X : k(x) < 3^{-n+2}\}.$$

Since  $k > 0$  it follows that  $\bigcap_{n=1}^{\infty} D_n = \emptyset$ . Since

$$A(f - g, 3^{-n+1}) \subseteq \{x \in X : (f - g)(x) \leq 3^{-n+1}\} \subseteq \{x \in X : k(x) \leq 3^{-n+1}\}$$

and since

$$\{x \in X : k(x) \leq 3^{-n+1}\}$$

and

$$\{x \in X : k(x) \geq 3^{-n+2}\} = X \setminus D_n$$

are completely separated by Baire-.5 function  $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$ , it follows that for each  $n$ ,  $A(f - g, 3^{-n+1})$  and  $X \setminus D_n$  are completely separated by Baire-.5 functions.  $\square$

### 3 Applications

**Definition 6.** A real-valued function  $f$  defined on a space  $X$  is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if  $f^{-1}(-\infty, t)$  (resp.  $f^{-1}(t, +\infty)$ ) is a  $G_\delta$ -set for any real number  $t$ .

The abbreviations usc, lsc, cusB-.5 and clsB-.5 are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

**Remark 1.** [13], [14]. A space  $X$  has the weak  $c$ -insertion property for (usc, lsc) if and only if  $X$  is normal.

Before stating the consequences of Theorem 1, and Theorem 2 we suppose that  $X$  is a topological space that  $F_\sigma$ -kernel of sets are  $F_\sigma$ -sets.

**Corollary 1.** For each pair of disjoint  $F_\sigma$ -sets  $F_1, F_2$ , there are two  $G_\delta$ -sets  $G_1$  and  $G_2$  such that  $F_1 \subseteq G_1$ ,  $F_2 \subseteq G_2$  and  $G_1 \cap G_2 = \emptyset$  if and only if  $X$  has the weak B-.5-insertion property for (cusB-.5, clsB-.5).

*Proof.* Let  $g$  and  $f$  be real-valued functions defined on the  $X$ , such that  $f$  is lsc  $B_1$ ,  $g$  is usc  $B_1$ , and  $g \leq f$ . If a binary relation  $\rho$  is defined by  $A \rho B$  in case  $F_\sigma(A) \subseteq G_\delta(B)$ , then by hypothesis  $\rho$  is a strong binary relation in the power set of  $X$ . If  $t_1$  and  $t_2$  are any elements of  $\mathbb{Q}$  with  $t_1 < t_2$ , then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since  $\{x \in X : f(x) \leq t_1\}$  is a  $F_\sigma$ -set and since  $\{x \in X : g(x) < t_2\}$  is a  $G_\delta$ -set, it follows that

$$F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2)).$$

Hence  $t_1 < t_2$  implies that  $A(f, t_1) \rho A(g, t_2)$ . The proof follows from Theorem 1.

On the other hand, let  $F_1$  and  $F_2$  are disjoint  $F_\sigma$ -sets. Set  $f = \chi_{F_1^c}$  and  $g = \chi_{F_2}$ , then  $f$  is clsB-.5,  $g$  is cusB-.5, and  $g \leq f$ . Thus there exists Baire-.5 function  $h$  such that  $g \leq h \leq f$ . Set

$$G_1 = \left\{x \in X : h(x) < \frac{1}{2}\right\}$$

and

$$G_2 = \left\{ x \in X : h(x) > \frac{1}{2} \right\},$$

then  $G_1$  and  $G_2$  are disjoint  $G_\delta$ -sets such that  $F_1 \subseteq G_1$  and  $F_2 \subseteq G_2$ .  $\square$

**Remark 2.** [22] A space  $X$  has the weak  $c$ -insertion property for (lsc, usc) if and only if  $X$  is extremally disconnected.

**Corollary 2.** For every  $G$  of  $G_\delta$ -set,  $F_\sigma(G)$  is a  $G_\delta$ -set if and only if  $X$  has the weak  $B$ -5-insertion property for (clsB-.5, cusB-.5).

Before giving the proof of this corollary, the necessary lemma is stated.

**Lemma 1.** The following conditions on the space  $X$  are equivalent:

- (i) For every  $G$  of  $G_\delta$ -set we have  $F_\sigma(G)$  is a  $G_\delta$ -set.
- (ii) For each pair of disjoint  $G_\delta$ -sets as  $G_1$  and  $G_2$  we have  $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$ .

The proof of Lemma 1 is a direct consequence of the definition  $F_\sigma$ -kernel of sets.

We now give the proof of Corollary 2.

*Proof of Corollary 2.* Let  $g$  and  $f$  be real-valued functions defined on the  $X$ , such that  $f$  is clsB-.5,  $g$  is cusB-.5, and  $f \leq g$ . If a binary relation  $\rho$  is defined by  $A \rho B$  in case

$$F_\sigma(A) \subseteq G \subseteq F_\sigma(G) \subseteq G_\delta(B)$$

for some  $G_\delta$ -set  $g$  in  $X$ , then by hypothesis and Lemma 1  $\rho$  is a strong binary relation in the power set of  $X$ . If  $t_1$  and  $t_2$  are any elements of  $\mathbb{Q}$  with  $t_1 < t_2$ , then

$$A(g, t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\} = A(f, t_2);$$

since  $\{x \in X : g(x) < t_1\}$  is a  $G_\delta$ -set and since  $\{x \in X : f(x) \leq t_2\}$  is a  $F_\sigma$ -set, by hypothesis it follows that  $A(g, t_1) \rho A(f, t_2)$ . The proof follows from Theorem 1.

On the other hand, Let  $G_1$  and  $G_2$  are disjoint  $G_\delta$ -sets. Set  $f = \chi_{G_2}$  and  $g = \chi_{G_1^c}$ , then  $f$  is clsB-.5,  $g$  is cusB-.5, and  $f \leq g$ .

Thus there exists Baire-.5 function  $h$  such that  $f \leq h \leq g$ . Set

$$F_1 = \left\{ x \in X : h(x) \leq \frac{1}{3} \right\}$$

and

$$F_2 = \left\{ x \in X : h(x) \geq \frac{2}{3} \right\}$$

then  $F_1$  and  $F_2$  are disjoint  $F_\sigma$ -sets such that  $G_1 \subseteq F_1$  and  $G_2 \subseteq F_2$ . Hence

$$F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset. \quad \square$$

Before stating the consequences of Theorem 2, we state and prove the necessary lemmas.

**Lemma 2.** *The following conditions on the space  $X$  are equivalent:*

- (i) *Every two disjoint  $F_\sigma$ -sets of  $X$  can be separated by  $G_\delta$ -sets of  $X$ .*
- (ii) *If  $F$  is a  $F_\sigma$ -set of  $X$  which is contained in a  $G_\delta$ -set  $G$ , then there exists a  $G_\delta$ -set  $H$  such that  $F \subseteq H \subseteq F_\sigma(H) \subseteq G$ .*

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that  $F \subseteq G$ , where  $F$  and  $G$  are  $F_\sigma$ -set and  $G_\delta$ -set of  $X$ , respectively. Hence,  $G^c$  is a  $F_\sigma$ -set and  $F \cap G^c = \emptyset$ .

By (i) there exists two disjoint  $G_\delta$ -sets  $G_1, G_2$  such that  $F \subseteq G_1$  and  $G^c \subseteq G_2$ . But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since  $G_2^c$  is a  $F_\sigma$ -set containing  $G_1$  we conclude that  $F_\sigma(G_1) \subseteq G_2^c$ , i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting  $H = G_1$ , condition (ii) holds.

(ii)  $\Rightarrow$  (i) Suppose that  $F_1, F_2$  are two disjoint  $F_\sigma$ -sets of  $X$ .

This implies that  $F_1 \subseteq F_2^c$  and  $F_2^c$  is a  $G_\delta$ -set. Hence by (ii) there exists a  $G_\delta$ -set  $H$  such that  $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$ .

But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore,  $(F_\sigma(H))^c$  is a  $G_\delta$ -set of  $X$ . Hence  $F_1 \subseteq H$ ,  $F_2 \subseteq (F_\sigma(H))^c$  and  $H \cap (F_\sigma(H))^c = \emptyset$ . This means that condition (i) holds.  $\square$

**Lemma 3.** *Suppose that  $X$  is the topological space such that we can separate every two disjoint  $F_\sigma$ -sets by  $G_\delta$ -sets. If  $F_1$  and  $F_2$  are two disjoint  $F_\sigma$ -sets of  $X$ , then there exists a Baire-.5 function  $h: X \rightarrow [0, 1]$  such that*

$$h(F_1) = \{0\} \quad \text{and} \quad h(F_2) = \{1\}.$$

*Proof.* Suppose  $F_1$  and  $F_2$  are two disjoint  $F_\sigma$ -sets of  $X$ . Since  $F_1 \cap F_2 = \emptyset$ , hence  $F_1 \subseteq F_2^c$ . In particular, since  $F_2^c$  is a  $G_\delta$ -set of  $X$  containing  $F_1$ , by Lemma 2, there exists a  $G_\delta$ -set  $H_{1/2}$  such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that  $H_{1/2}$  is a  $G_\delta$ -set and contains  $F_1$ , and  $F_2^c$  is a  $G_\delta$ -set and contains  $F_\sigma(H_{1/2})$ . Hence, by Lemma 2, there exists  $G_\delta$ -sets  $H_{1/4}$  and  $H_{3/4}$  such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every  $t \in D$ , where  $D \subseteq [0, 1]$  is the set of rational numbers that their denominators are exponents of 2, we obtain  $G_\delta$ -sets  $H_t$  with the property that if  $t_1, t_2 \in D$  and  $t_1 < t_2$ , then  $H_{t_1} \subseteq H_{t_2}$ . We define the function  $h$  on  $X$  by

$$h(x) = \inf\{t : x \in H_t\}$$

for  $x \notin F_2$  and  $h(x) = 1$  for  $x \in F_2$ .

Note that for every  $x \in X$ ,  $0 \leq h(x) \leq 1$ , i.e.,  $h$  maps  $X$  into  $[0, 1]$ . Also, we note that for any  $t \in D$ ,  $F_1 \subseteq H_t$ ; hence  $h(F_1) = \{0\}$ . Furthermore, by definition,  $h(F_2) = \{1\}$ . It remains only to prove that  $h$  is a Baire-.5 function on  $X$ . For every  $\alpha \in \mathbb{R}$ , we have if  $\alpha \leq 0$  then  $\{x \in X : h(x) < \alpha\} = \emptyset$  and if  $0 < \alpha$  then

$$\{x \in X : h(x) < \alpha\} = \bigcup\{H_t : t < \alpha\},$$

hence, they are  $G_\delta$ -sets of  $X$ . Similarly, if  $\alpha < 0$  then

$$\{x \in X : h(x) > \alpha\} = X$$

and if  $0 \leq \alpha$  then

$$\{x \in X : h(x) > \alpha\} = \bigcup\{(F_\sigma(H_t))^c : t > \alpha\}$$

hence, every of them is a  $G_\delta$ -set. Consequently  $h$  is a Baire-.5 function.  $\square$

**Lemma 4.** Suppose that  $X$  is the topological space such that every two disjoint  $F_\sigma$ -sets can be separated by  $G_\delta$ -sets. The following conditions are equivalent:

- (i) Every countable covering of  $G_\delta$ -sets of  $X$  has a refinement consisting of  $G_\delta$ -sets such that, for every  $x \in X$ , there exists a  $G_\delta$ -set containing  $x$  such that it intersects only finitely many members of the refinement.
- (ii) Corresponding to every decreasing sequence  $\{F_n\}$  of  $F_\sigma$ -sets with empty intersection there exists a decreasing sequence  $\{G_n\}$  of  $G_\delta$ -sets such that,  $\bigcap_{n=1}^{\infty} G_n = \emptyset$  and for every  $n \in \mathbb{N}$ ,  $F_n \subseteq G_n$ .

*Proof.* (i)  $\Rightarrow$  (ii). suppose that  $\{F_n\}$  be a decreasing sequence of  $F_\sigma$ -sets with empty intersection. Then  $\{F_n^c : n \in \mathbb{N}\}$  is a countable covering of  $G_\delta$ -sets. By hypothesis (i) and Lemma 2, this covering has a refinement  $\{V_n : n \in \mathbb{N}\}$  such that every  $V_n$  is a  $G_\delta$ -set and  $F_\sigma(V_n) \subseteq F_n^c$ . By setting  $F_n = (F_\sigma(V_n))^c$ , we obtain a decreasing sequence of  $G_\delta$ -sets with the required properties.

(ii)  $\Rightarrow$  (i). Now if  $\{H_n : n \in \mathbb{N}\}$  is a countable covering of  $G_\delta$ -sets, we set for  $n \in \mathbb{N}$ ,  $F_n = (\bigcup_{i=1}^n H_i)^c$ . Then  $\{F_n\}$  is a decreasing sequence of  $F_\sigma$ -sets with empty intersection. By (ii) there exists a decreasing sequence  $\{G_n\}$  consisting of  $G_\delta$ -sets such that,  $\bigcap_{n=1}^{\infty} G_n = \emptyset$  and for every  $n \in \mathbb{N}$ ,  $F_n \subseteq G_n$ . Now we define the subsets  $W_n$  of  $X$  in the following manner:

$W_1$  is a  $G_\delta$ -set of  $X$  such that  $G_1^c \subseteq W_1$  and  $F_\sigma(W_1) \cap F_1 = \emptyset$ .

$W_2$  is a  $G_\delta$ -set of  $X$  such that  $F_\sigma(W_1) \cup G_2^c \subseteq W_2$  and  $F_\sigma(W_2) \cap F_2 = \emptyset$ , and so on. (By Lemma 2,  $W_n$  exists).

Then since  $\{G_n^c : n \in \mathbb{N}\}$  is a covering for  $X$ , hence  $\{W_n : n \in \mathbb{N}\}$  is a covering for  $X$  consisting of  $G_\delta$ -sets. Moreover, we have

(i)  $F_\sigma(W_n) \subseteq W_{n+1}$

(ii)  $G_n^c \subseteq W_n$

(iii)  $W_n \subseteq \bigcup_{i=1}^n H_i$ .

Now suppose that  $S_1 = W_1$  and for  $n \geq 2$ , we set  $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$ .

Then since  $F_\sigma(W_{n-1}) \subseteq W_n$  and  $S_n \supseteq W_{n+1} \setminus W_n$ , it follows that  $\{S_n : n \in \mathbb{N}\}$  consists of  $G_\delta$ -sets and covers  $X$ . Furthermore,  $S_i \cap S_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ . Finally, consider the following sets:

$$\begin{array}{ccccccc} S_1 \cap H_1, & S_1 \cap H_2 & & & & & \\ S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & & \\ S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & & \end{array}$$

and continue ad infinitum. These sets are  $G_\delta$ -sets, cover  $X$  and refine  $\{H_n : n \in \mathbb{N}\}$ . In addition,  $S_i \cap H_j$  can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if  $x \in X$  and  $x \in S_n \cap H_m$ , then  $S_n \cap H_m$  is a  $G_\delta$ -set containing  $x$  that intersects at most finitely many of sets  $S_i \cap H_j$ . Consequently,

$$\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \dots, i + 1\}$$

refines  $\{H_n : n \in \mathbb{N}\}$  such that its elements are  $G_\delta$ -sets, and for every point in  $X$  we can find a  $G_\delta$ -set containing the point that intersects only finitely many elements of that refinement. □

**Remark 3.** [13], [14] A space  $X$  has the  $c$ -insertion property for (usc, lsc) if and only if  $X$  is normal and countably paracompact.

**Corollary 3.**  $X$  has the  $B$ -.5-insertion property for (cusB-.5, clsB-.5) if and only if every two disjoint  $F_\sigma$ -sets of  $X$  can be separated by  $G_\delta$ -sets, and in addition, every countable covering of  $G_\delta$ -sets has a refinement that consists of  $G_\delta$ -sets such that, for every point of  $X$  we can find a  $G_\delta$ -set containing that point such that, it intersects only a finite number of refining members.

*Proof.* Suppose that  $F_1$  and  $F_2$  are disjoint  $F_\sigma$ -sets. Since  $F_1 \cap F_2 = \emptyset$ , it follows that  $F_2 \subseteq F_1^c$ . We set  $f(x) = 2$  for  $x \in F_1^c$ ,  $f(x) = \frac{1}{2}$  for  $x \notin F_1^c$ , and  $g = \chi_{F_2}$ .

Since  $F_2$  is a  $F_\sigma$ -set, and  $F_1^c$  is a  $G_\delta$ -set, therefore  $g$  is cusB-.5,  $f$  is clsB-.5 and furthermore  $g < f$ . Hence by hypothesis there exists a Baire-.5 function  $h$  such that,  $g < h < f$ . Now by setting

$$G_1 = \{x \in X : h(x) < 1\}$$

and

$$G_2 = \{x \in X : h(x) > 1\}.$$

We can say that  $G_1$  and  $G_2$  are disjoint  $G_\delta$ -sets that contain  $F_1$  and  $F_2$ , respectively. Now suppose that  $\{F_n\}$  is a decreasing sequence of  $F_\sigma$ -sets with empty intersection. Set  $F_0 = X$  and define for every  $x \in F_n \setminus F_{n+1}$ ,

$$f(x) = \frac{1}{n+1}.$$

Since

$$\bigcap_{n=0}^{\infty} F_n = \emptyset$$

and for every  $x \in X$ , there exists  $n \in \mathbb{N}$ , such that,  $x \in F_n \setminus F_{n+1}$ ,  $f$  is well defined. Furthermore, for every  $r \in \mathbb{R}$ , if  $r \leq 0$  then

$$\{x \in X : f(x) > r\} = X$$

is a  $G_\delta$ -set and if  $r > 0$  then by Archimedean property of  $\mathbb{R}$ , we can find  $i \in \mathbb{N}$  such that

$$\frac{1}{i+1} \leq r.$$

Now suppose that  $k$  is the least natural number such that  $\frac{1}{k+1} \leq r$ . Hence  $\frac{1}{k} > r$  and consequently,

$$\{x \in X : f(x) > r\} = X \setminus F_k$$

is a  $G_\delta$ -set. Therefore,  $f$  is clsB-.5. By setting  $g = 0$ , we have  $g$  is cusB-.5 and  $g < f$ . Hence by hypothesis there exists a Baire-.5 function  $h$  on  $X$  such that,  $g < h < f$ .

By setting

$$G_n = \left\{x \in X : h(x) < \frac{1}{n+1}\right\},$$

we have  $G_n$  is a  $G_\delta$ -set. But for every  $x \in F_n$ , we have

$$f(x) \leq \frac{1}{n+1}$$

and since  $g < h < f$  therefore

$$0 < h(x) < \frac{1}{n+1},$$

i.e.,  $x \in G_n$  therefore  $F_n \subseteq G_n$  and since  $h > 0$  it follows that

$$\bigcap_{n=1}^{\infty} G_n = \emptyset.$$

Hence by Lemma 4, the conditions holds.

On the other hand, since every two disjoint  $F_\sigma$ -sets can be separated by  $G_\delta$ -sets, therefore by Corollary 1,  $X$  has the weak  $B$ -.5-insertion property for

(cusB-.5, clsB-.5). Now suppose that  $f$  and  $g$  are real-valued functions on  $X$  with  $g < f$ , such that,  $g$  is cusB-.5 and  $f$  is clsB-.5. For every  $n \in \mathbb{N}$ , set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$

Since  $g$  is cusB-.5, and  $f$  is clsB-.5, therefore  $f - g$  is clsB-.5. Hence  $A(f - g, 3^{-n+1})$  is a  $F_\sigma$ -set of  $X$ . Consequently,  $\{A(f - g, 3^{-n+1})\}$  is a decreasing sequence of  $F_\sigma$ -sets and furthermore since  $0 < f - g$ , it follows that

$$\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset.$$

Now by Lemma 4, there exists a decreasing sequence  $\{D_n\}$  of  $G_\delta$ -sets such that

$$A(f - g, 3^{-n+1}) \subseteq D_n$$

and

$$\bigcap_{n=1}^{\infty} D_n = \emptyset.$$

But by Lemma 3,  $A(f - g, 3^{-n+1})$  and  $X \setminus D_n$  of  $F_\sigma$ -sets can be completely separated by Baire-.5 functions. Hence by Theorem 2, there exists a Baire-.5 function  $h$  defined on  $X$  such that,  $g < h < f$ , i.e.,  $X$  has the  $B$ -.5-insertion property for (cusB-.5, clsB-.5).  $\square$

**Remark 4.** [15] A space  $X$  has the  $c$ -insertion property for (lsc, usc) iff  $X$  is extremally disconnected and if for any decreasing sequence  $\{G_n\}$  of open subsets of  $X$  with empty intersection there exists a decreasing sequence  $\{F_n\}$  of closed subsets of  $X$  with empty intersection such that  $G_n \subseteq F_n$  for each  $n$ .

**Corollary 4.** For every  $G$  of  $G_\delta$ -set,  $F_\sigma(G)$  is a  $G_\delta$ -set and in addition for every decreasing sequence  $\{G_n\}$  of  $G_\delta$ -sets with empty intersection, there exists a decreasing sequence  $\{F_n\}$  of  $F_\sigma$ -sets with empty intersection such that for every  $n \in \mathbb{N}$ ,  $G_n \subseteq F_n$  if and only if  $X$  has the  $B$ -.5-insertion property for (clsB-.5, cusB-.5).

*Proof.* Since for every  $G$  of  $G_\delta$ -set,  $F_\sigma(G)$  is a  $G_\delta$ -set, therefore by Corollary 2,  $X$  has the weak  $B$ -.5-insertion property for (clsB-.5, cusB-.5). Now suppose that  $f$  and  $g$  are real-valued functions defined on  $X$  with  $g < f$ ,  $g$  is clsB-.5, and  $f$  is cusB-.5. Set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) < 3^{-n+1}\}.$$

Then since  $f - g$  is cusB-.5, hence  $\{A(f - g, 3^{-n+1})\}$  is a decreasing sequence of  $G_\delta$ -sets with empty intersection. By hypothesis, there exists a decreasing sequence  $\{D_n\}$  of  $F_\sigma$ -sets with empty intersection such that, for every  $n \in \mathbb{N}$ ,  $A(f - g, 3^{-n+1}) \subseteq D_n$ . Hence  $X \setminus D_n$  and  $A(f - g, 3^{-n+1})$  are two disjoint  $G_\delta$ -sets and therefore by Lemma 1, we have

$$F_\sigma(A(f - g, 3^{-n+1})) \cap F_\sigma((X \setminus D_n)) = \emptyset$$

and therefore by Lemma 3,  $X \setminus D_n$  and  $A(f-g, 3^{-n+1})$  are completely separable by Baire-.5 functions. Therefore by Theorem 2, there exists a Baire-.5 function  $h$  on  $X$  such that,  $g < h < f$ , i.e.,  $X$  has the  $B$ -.5-insertion property for (clsB-.5, cusB-.5).

On the other hand, suppose that  $G_1$  and  $G_2$  be two disjoint  $G_\delta$ -sets. Since  $G_1 \cap G_2 = \emptyset$ . We have  $G_2 \subseteq G_1^c$ . We set  $f(x) = 2$  for  $x \in G_1^c$ ,  $f(x) = \frac{1}{2}$  for  $x \notin G_1^c$  and  $g = \chi_{G_2}$ .

Then since  $G_2$  is a  $G_\delta$ -set and  $G_1^c$  is a  $F_\sigma$ -set, we conclude that  $g$  is clsB-.5 and  $f$  is cusB-.5 and furthermore  $g < f$ . By hypothesis, there exists a Baire-.5 function  $h$  on  $X$  such that,  $g < h < f$ . Now we set

$$F_1 = \left\{ x \in X : h(x) \leq \frac{3}{4} \right\}$$

and

$$F_2 = \{x \in X : h(x) \geq 1\}.$$

Then  $F_1$  and  $F_2$  are two disjoint  $F_\sigma$ -sets contain  $G_1$  and  $G_2$ , respectively. Hence  $F_\sigma(G_1) \subseteq F_1$  and  $F_\sigma(G_2) \subseteq F_2$  and consequently  $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$ . By Lemma 1, for every  $G$  of  $G_\delta$ -set, the set  $F_\sigma(G)$  is a  $G_\delta$ -set.

Now suppose that  $\{G_n\}$  is a decreasing sequence of  $G_\delta$ -sets with empty intersection.

We set  $G_0 = X$  and  $f(x) = \frac{1}{n+1}$  for  $x \in G_n \setminus G_{n+1}$ . Since  $\bigcap_{n=0}^{\infty} G_n = \emptyset$  and for every  $n \in \mathbb{N}$  there exists  $x \in G_n \setminus G_{n+1}$ ,  $f$  is well-defined. Furthermore, for every  $r \in \mathbb{R}$ , if  $r \leq 0$  then

$$\{x \in X : f(x) < r\} = \emptyset$$

is a  $G_\delta$ -set and if  $r > 0$  then by Archimedean property of  $\mathbb{R}$ , there exists  $i \in \mathbb{N}$  such that  $\frac{1}{i+1} \leq r$ . Suppose that  $k$  is the least natural number with this property. Hence  $\frac{1}{k} > r$ . Now if  $\frac{1}{k+1} < r$  then

$$\{x \in X : f(x) < r\} = G_k$$

is a  $G_\delta$ -set and if  $\frac{1}{k+1} = r$  then

$$\{x \in X : f(x) < r\} = G_{k+1}$$

is a  $G_\delta$ -set. Hence  $f$  is a cusB-.5 on  $X$ . By setting  $g = 0$ , we have conclude that  $g$  is clsB-.5 on  $X$  and in addition  $g < f$ . By hypothesis there exists a Baire-.5 function  $h$  on  $X$  such that,  $g < h < f$ .

Set

$$F_n = \left\{ x \in X : h(x) \leq \frac{1}{n+1} \right\}.$$

This set is a  $F_\sigma$ -set. But for every  $x \in G_n$ , we have  $f(x) \leq \frac{1}{n+1}$  and since  $g < h < f$  thus  $h(x) < \frac{1}{n+1}$ , this means that  $x \in F_n$  and consequently  $G_n \subseteq F_n$ .

By definition of  $F_n$ ,  $\{F_n\}$  is a decreasing sequence of  $F_\sigma$ -sets and since  $h > 0$ ,  $\bigcap_{n=1}^{\infty} F_n = \emptyset$ . Thus the conditions holds.  $\square$

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