Jhon J. Bravo; Jose L. Herrera
Fermat $k$-Fibonacci and $k$-Lucas numbers


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FERMAT $k$-FIBONACCI AND $k$-LUCAS NUMBERS

Jhon J. Bravo, Jose L. Herrera, Popayán

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Abstract. Using the lower bound of linear forms in logarithms of Matveev and the theory of continued fractions by means of a variation of a result of Dujella and Pethő, we find all $k$-Fibonacci and $k$-Lucas numbers which are Fermat numbers. Some more general results are given.

Keywords: generalized Fibonacci number; Fermat number, linear form in logarithms; reduction method

MSC 2010: 11B39, 11J86

1. Introduction and preliminary results

For an integer $k \geq 2$ we consider the linear recurrence sequence $G^{(k)} := (G^{(k)}_n)_{n \geq 2-k}$ of order $k$, defined as

$$G^{(k)}_n = G^{(k)}_{n-1} + G^{(k)}_{n-2} + \ldots + G^{(k)}_{n-k} \quad \forall n \geq 2,$$

with the initial conditions $G^{(k)}_{-(k-2)} = G^{(k)}_{-(k-3)} = \ldots = G^{(k)}_{-1} = 0$, $G^{(k)}_0 = a$ and $G^{(k)}_1 = b$, where $a$ and $b$ are both integers.

If $a = 0$ and $b = 1$, then $G^{(k)}$ is known as the $k$-Fibonacci sequence $F^{(k)} := (F^{(k)}_n)_{n \geq 2-k}$. We shall refer to $F^{(k)}_n$ as the $n$th $k$-Fibonacci number. We note that this generalization is in fact a family of sequences where each new choice of $k$ produces a distinct sequence. For example, the usual Fibonacci numbers are obtained for $k = 2$. For small values of $k$, these sequences are called Tribonacci ($k = 3$), Tetranacci ($k = 4$), Pentanacci ($k = 5$), Hexanacci ($k = 6$), Heptanacci ($k = 7$) and Octanacci ($k = 8$). In a similar way, if $a = 2$ and $b = 1$, then $G^{(k)}$ is known as the $k$-Lucas sequence $L^{(k)} := (L^{(k)}_n)_{n \geq 2-k}$, which extends the usual Lucas sequence $L^{(2)}$. Other generalization for Lucas numbers can be found in [14].

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An interesting fact about the $k$-Fibonacci sequence is that the first $k + 1$ nonzero terms in $F^{(k)}$ are powers of two, namely

\[(1) \quad F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2}, \quad 2 \leq n \leq k + 1,\]

while the next term is $F_{k+2}^{(k)} = 2^k - 1$. In fact, the inequality

\[(2) \quad F_n^{(k)} < 2^{n-2} \quad \text{holds for all} \quad n \geq k + 2\]

(see [3]). Similarly, the $k$-Lucas sequence $L^{(k)}$ has the remarkable property that the first few terms are given by

$$L_n^{(k)} = 3 \cdot 2^{n-2}, \quad 2 \leq n \leq k.$$ 

Below we present the values of these numbers for the first few values of $k$ and $n$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Name</th>
<th>First nonzero terms ($n \geq 1$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Fibonacci</td>
<td>1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...</td>
</tr>
<tr>
<td>3</td>
<td>Tribonacci</td>
<td>1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...</td>
</tr>
<tr>
<td>4</td>
<td>Tetranacci</td>
<td>1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...</td>
</tr>
<tr>
<td>5</td>
<td>Pentanacci</td>
<td>1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, ...</td>
</tr>
<tr>
<td>6</td>
<td>Hexanacci</td>
<td>1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ...</td>
</tr>
<tr>
<td>7</td>
<td>Heptanacci</td>
<td>1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ...</td>
</tr>
<tr>
<td>8</td>
<td>Octanacci</td>
<td>1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ...</td>
</tr>
<tr>
<td>9</td>
<td>Nonanacci</td>
<td>1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, ...</td>
</tr>
<tr>
<td>10</td>
<td>Decanacci</td>
<td>1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1023, 2045, 4088, 8172, ...</td>
</tr>
</tbody>
</table>

Table 1. First nonzero $k$-Fibonacci numbers

<table>
<thead>
<tr>
<th>$k$</th>
<th>Name</th>
<th>First nonzero terms ($n \geq 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Lucas</td>
<td>2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, ...</td>
</tr>
<tr>
<td>3</td>
<td>3-Lucas</td>
<td>2, 1, 3, 6, 10, 19, 35, 64, 118, 217, 399, 734, 1350, 2483, 4567, ...</td>
</tr>
<tr>
<td>4</td>
<td>4-Lucas</td>
<td>2, 1, 3, 6, 12, 22, 43, 83, 160, 308, 594, 1145, 2207, 4254, 8200, ...</td>
</tr>
<tr>
<td>5</td>
<td>5-Lucas</td>
<td>2, 1, 3, 6, 12, 24, 48, 94, 187, 371, 736, 1460, 2896, 5744, 11394, ...</td>
</tr>
<tr>
<td>6</td>
<td>6-Lucas</td>
<td>2, 1, 3, 6, 12, 24, 48, 96, 190, 379, 755, 1504, 2996, 5968, 11888, ...</td>
</tr>
<tr>
<td>7</td>
<td>7-Lucas</td>
<td>2, 1, 3, 6, 12, 24, 48, 96, 192, 382, 763, 1523, 3040, 6068, 12112, ...</td>
</tr>
<tr>
<td>8</td>
<td>8-Lucas</td>
<td>2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1534, 3067, 6131, 12256, ...</td>
</tr>
<tr>
<td>9</td>
<td>9-Lucas</td>
<td>2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1534, 3067, 6131, 12256, ...</td>
</tr>
<tr>
<td>10</td>
<td>10-Lucas</td>
<td>2, 1, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1534, 3067, 6131, 12256, ...</td>
</tr>
</tbody>
</table>

Table 2. First nonzero $k$-Lucas numbers
Several authors have worked on problems involving generalized Fibonacci sequences. For instance, Luca in [11] and Marques in [12] proved that 55 and 44 are the largest repdigits in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, Marques conjectured that there are no repdigits with at least two digits belonging to $F^{(k)}$ for $k > 3$. This conjecture was confirmed in [4]. In addition, the Diophantine equation $F^{(k)}_n = 2^m$ was studied in [3]. Similar equations have been considered for $L^{(k)}$ (see, for example, [1] and [5]).

When $k = 2$, Finkelstein found that the only Fibonacci and Lucas numbers of the form $y^2 + 1, y \in \mathbb{Z}, y \geq 0$ are $F_1 = F_2 = 1, F_3 = 2, F_5 = 5, L_0 = 2$ and $L_1 = 1$ (see [8], [9]). In 2006, Bugeaud et al. generalized the problem discussed above and proved that the only nonnegative integer solutions $(n, y, m)$ of equations $F_n \pm 1 = y^m$ with $m \geq 2$ are

$$
F_0 + 1 = 0 + 1 = 1, \quad F_1 - 1 = F_2 - 1 = 1 - 1 = 0,
$$
$$
F_3 + 1 = 3 + 1 = 2^2, \quad F_3 - 1 = 2 - 1 = 1, 
$$
$$
F_6 + 1 = 8 + 1 = 3^2, \quad F_5 - 1 = 5 - 1 = 2^2.
$$

As a consequence of the above, the only nonnegative integer solutions $(n, m)$ of equation

$$
F_n = 2^m + 1
$$

are $(n, m) \in \{(3, 0), (4, 1), (5, 2)\}$.

In the present paper we aim to generalize the above equation (3) for generalized Fibonacci sequences, i.e. we consider the more general Diophantine equations

$$
F^{(k)}_n = 2^m + 1, \quad (4)
$$
$$
L^{(k)}_n = 2^m + 1 \quad (5)
$$

in nonnegative integers $n, k, m$ with $k \geq 2$. As a particular case of the above equations (4) and (5), we determine all $k$-Fibonacci and $k$-Lucas numbers which are Fermat numbers. Recall that a Fermat number is a number of the form $F_m = 2^{2^m} + 1$, where $m$ is a nonnegative integer. The first six Fermat numbers are

$$
F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537 \quad \text{and} \quad F_5 = 4294967297.
$$

It is important to mention that equation (3) can also be solved by using the well known factorization $F_n - 1 = F_{(n-\delta)/2}L_{(n+\delta)/2}$, where $\delta \in \{-2, 1, 2, -1\}$ depends on the class of $n$ modulo 4. In this case, the resulting equation can be easily solved by using prime factorization. However, similar divisibility properties for $F^{(k)}$ when $k \geq 3$ are not known and therefore it is necessary to attack the problem differently.
We begin our analysis of equations (4) and (5) by noting that $F_{3}^{(k)} = 2$, $L_{0}^{(k)} = 2$ and $L_{2}^{(k)} = 3$ are valid for all $k \geq 2$; thus, the triples

$$ (n, k, m) = (3, k, 0) \text{ are the solutions of (4) for all } k \geq 2, $$

and

$$ (n, k, m) \in \{(0, k, 0), (2, k, 1)\} \text{ are the solutions of (5) for all } k \geq 2. $$

The above solutions will be called *trivial solutions*. In this paper, we prove the following theorems.

**Theorem 1.** The only nontrivial solutions of the Diophantine equation (4) in nonnegative integers $n, k, m$ with $k \geq 2$ are $(n, k, m) \in \{(4, 2, 1), (5, 2, 2)\}$.

**Theorem 2.** The Diophantine equation (5) has no nontrivial solutions in nonnegative integers $n, k, m$ with $k \geq 2$.

As an immediate consequence of Theorem 1 and Theorem 2 we have the following corollaries.

**Corollary 1.** The only Fermat numbers in the $k$-Fibonacci family of sequences are $F_{4} = 3$ and $F_{5} = 5$.

**Corollary 2.** The only Fermat number in the $k$-Lucas family of sequences is $L_{2}^{(k)} = 3$, which holds for all $k \geq 2$.

To prove our main results we use lower bounds for linear forms in logarithms (Baker’s theory) to bound $n$ and $m$ polynomially in terms of $k$. When $k$ is small, we use the theory of continued fractions by means of a variation of a result of Dujella and Pethő to lower such bounds to cases that allow us to treat our problem computationally. For large values of $k$, Bravo, Gómez and Luca in [2], [3], [5] developed some ideas for dealing with Diophantine equations involving $k$-Fibonacci and $k$-Lucas numbers.

Before proceeding further, it may be mentioned that the characteristic polynomial of $G^{(k)}$, namely

$$ \Psi_{k}(x) = x^{k} - x^{k-1} - \ldots - x - 1, $$

is irreducible in $\mathbb{Q}[x]$ and has just one zero root outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single zero. The other roots are strictly inside the unit circle, so $\alpha(k)$ is a Pisot number of degree $k$. Moreover, it is also known that
\( \alpha(k) \) is located between \( 2(1 - 2^{-k}) \) and 2, see [10], Lemma 2.3 or [15], Lemma 3.6. To simplify the notation, we shall omit the dependence on \( k \) of \( \alpha \).

We now consider the function \( f_k(x) = (x - 1)/(2 + (k + 1)(x - 2)) \) for an integer \( k \geq 2 \) and \( x > 2(1 - 2^{-k}) \). It is easy to see that the inequalities

\[
\frac{1}{2} < f_k(\alpha) < \frac{3}{4} \quad \text{and} \quad |f_k(\alpha^{(i)})| < 1, \quad 2 \leq i \leq k
\]

hold, where \( \alpha := \alpha^{(1)}, \ldots, \alpha^{(k)} \) are all the zeros of \( \Psi_k(x) \). So, by computing norms from \( \mathbb{Q}(\alpha) \) to \( \mathbb{Q} \), for example, we see that the number \( f_k(\alpha) \) is not an algebraic integer. Proofs for this fact and for (6) can be found in [2].

With the above notation, Dresden and Du showed in [6] that

\[
F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)})\alpha^{(i)n-1} \quad \text{and} \quad |F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| < \frac{1}{2}
\]

hold for all \( n \geq 1 \) and \( k \geq 2 \).

In addition to this, Bravo and Luca proved in [4] that

\[
\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 1 \quad \text{and} \quad k \geq 2.
\]

The observations in expressions (7) and (8) lead us to call \( \alpha \) the dominant zero of \( G^{(k)} \).

Note that sequences \( G^{(k)} \) and \( F^{(k)} \) have the same recurrence relation. This makes us think that there is some relationship between them. In this sense, Bravo and Luca in [5] proved that \( G_n^{(k)} = aF_{n+1}^{(k)} + (b - a)F_n^{(k)} \). In particular,

\[
L_n^{(k)} = 2F_{n+1}^{(k)} - F_n^{(k)}.
\]

The above result supports the following lemma (see the proof in [5]).

**Lemma 1.** Let \( k \geq 2 \) be an integer. Then

(a) \( \alpha^{n-1} \leq L_n^{(k)} \leq 2\alpha^{n} \) for all \( n \geq 1 \),

(b) \( L^{(k)} \) satisfies the following “Binet-like” formula

\[
L_n^{(k)} = \sum_{i=1}^{k} (2\alpha_i - 1)f_k(\alpha_i)\alpha_i^{n-1},
\]

where \( \alpha = \alpha_1, \ldots, \alpha_n \) are the zeros of \( \Psi_k(x) = x^k - x^{k-1} - \ldots - x - 1 \),

(c) \( |L_n^{(k)} - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{3}{2} \) for all \( n \geq 2 - k \),

(d) If \( 2 \leq n \leq k \), then \( L_n^{(k)} = 3 \cdot 2^{n-2} \).
2. Linear forms in logarithms

In order to prove our main result, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers, and such a bound, which plays an important role in this paper, was given by Matveev (see [13]). We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

\[ a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}), \]

where the leading coefficient \( a_0 \) is positive and the \( \eta^{(i)} \)'s are the conjugates of \( \eta \). Then

\[ h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^{d} \log(\max\{|\eta^{(i)}|, 1\}) \right) \]

is called the logarithmic height of \( \eta \). In particular, if \( \eta = p/q \) is a rational number with \( \gcd(p, q) = 1 \) and \( q > 0 \), then \( h(\eta) = \log(\max\{|p|, q\}) \).

The following properties of the logarithmic height, which will be used in next sections without special reference, are also known:

\[ \triangleright h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2. \]
\[ \triangleright h(\eta \gamma^{\pm 1}) \leq h(\eta) + h(\gamma). \]
\[ \triangleright h(\eta^s) = |s|h(\eta). \]

Matveev in [13] proved the following deep theorem.

\textbf{Theorem 3} (Matveev’s theorem). Let \( K \) be a number field of degree \( D \) over \( \mathbb{Q} \), \( \gamma_1, \ldots, \gamma_t \) be positive real numbers of \( K \), and \( b_1, \ldots, b_t \) rational integers. Put

\[ \Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1 \quad \text{and} \quad B \geq \max\{|b_1|, \ldots, |b_t|\}. \]

Let \( A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\} \) be real numbers for \( i = 1, \ldots, t \). Then, assuming that \( \Lambda \neq 0 \), we have

\[ |\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \ldots A_t). \]

To conclude this section, we give estimates for the logarithmic heights of some algebraic numbers. Let \( K = \mathbb{Q}(\alpha) \). Knowing that \( \mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha)) \) and that \( |f_k(\alpha^{(i)})| \leq 1 \) for all \( i = 1, \ldots, k \) and \( k \geq 2 \), we obtain that \( h(\alpha) = (\log \alpha)/k \).
and \( h(f_k(\alpha)) = (\log a_0)/k \), where \( a_0 \) is the leading coefficient of minimal primitive polynomial over the integers of \( f_k(\alpha) \). Put

\[
g_k(x) = \prod_{i=1}^{k} (x - f_k(\alpha^{(i)})) \in \mathbb{Q}[x] \quad \text{and} \quad \mathcal{N} = N_{\mathbb{K}/\mathbb{Q}}(2 + (k + 1)(\alpha - 2)) \in \mathbb{Z}.
\]

We conclude that \( \mathcal{N} g_k(x) \in \mathbb{Z}[x] \) vanishes at \( f_k(\alpha) \). Thus, \( a_0 \) divides \( |\mathcal{N}| \). But for \( k \geq 2 \),

\[
|\mathcal{N}| = \left| \prod_{i=1}^{k} (2 + (k + 1)(\alpha^{(i)} - 2)) \right| = (k + 1)^k \left| \prod_{i=1}^{k} \left( 2 - \frac{2}{k + 1} - \alpha^{(i)} \right) \right|
\]

\[
= (k + 1)^k \left| \Psi_k \left( 2 - \frac{2}{k + 1} \right) \right|
\]

\[
= \frac{2^{k+1}k^k - (k + 1)^{k+1}}{k - 1} < 2^k k^k.
\]

Hence, we will use the following inequalities:

(10) \( h(\alpha) < \frac{7}{10k} \) and \( h(f_k(\alpha)) < 2 \log k, \; k \geq 2 \).

Additionally, Bravo and Luca in [5] proved that \( h(2\alpha - 1) < \log 3 \) for all \( k \geq 2 \). So,

(11) \( h((2\alpha - 1)f_k(\alpha)) < \log 3 + 2 \log k < 4 \log k, \; k \geq 2 \).

### 3. Proof of Theorem 1

Assume first that we have a nontrivial solution \((n, k, m)\) of equation (4). If \( n = 1 \), then \( 1 = 2^n + 1 \), which is impossible because \( m \geq 0 \). Now, if \( 2 \leq n \leq k + 1 \), then we obtain from (1) that \( 2^{n-2} = 2^m + 1 \). From this, we get only the trivial solutions \((n, k, m) = (3, k, 0)\) for all \( k \geq 2 \). So, from now on, we assume that \( n \geq k + 2 \) and therefore \( n \geq 4 \). In fact, after a quick inspection of the first table presented above, we can assume that \( n \geq 6 \) since the only solutions for the values \( n = 4, 5 \) are given by \( F_4 = 3 \) and \( F_5 = 5 \). By inequalities (2) and (4), we have

\[
2^m < 2^m + 1 = F_n^{(k)} < 2^{n-2}
\]

obtaining

(12) \( m \leq n - 3 \).
We shall have some use for it later. Using now (4) once again and (7) we get that

$$|f_k(\alpha)\alpha^{n-1} - 2^m| < \frac{1}{2} + 1 = \frac{3}{2},$$

giving

\begin{equation}
|1 - \frac{2^m}{\alpha^{n-1} f_k(\alpha)}| < \frac{3}{\alpha^{n-1}},
\end{equation}

where we used the fact that $f_k(\alpha) > \frac{1}{2}$ as has already been mentioned (see (6)). In order to use the result of Matveev theorem 3, we take $t := 3$ and

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := f_k(\alpha).$$

We also take $b_1 := m, b_2 := -(n - 1)$ and $b_3 := -1$. We begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\alpha)$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = k$. The left-hand side of (13) is not zero. Indeed, if this were zero, we would then get that $f_k(\alpha) = 2^m \cdot \alpha^{-(n-1)}$ and so $f_k(\alpha)$ would be an algebraic integer, contradicting something previously mentioned. Note that $\alpha^{-1}$ is an algebraic integer, because it is a root of the monic polynomial $x^k \Psi_k(1/x) \in \mathbb{Z}[x]$, and recall that the set of algebraic integers form a ring.

Since $h(\gamma_1) = \log 2$, it follows that we can take $A_1 := k \log 2$. Further, in view of (10), we can take $A_2 = \frac{7}{10}$ and $A_3 := 2k \log k$. Finally, by recalling that $m \leq n - 3$, we can take $B := n - 1$. Then Matveev’s theorem together with a straightforward calculation gives

\begin{equation}
|1 - 2^m \alpha^{-(n-1)}(f_k(\alpha))^{-1}| > \exp(-8.34 \times 10^{11} k^4 \log^2 k \log(n - 1)),
\end{equation}

where we used that $1 + \log k \leq 3 \log k$ for all $k \geq 2$ and $1 + \log(n - 1) \leq 2 \log(n - 1)$ for all $n \geq 4$. Comparing (13) and (14), taking logarithms and then performing the respective calculations, we get that

\begin{equation}
\frac{n - 1}{\log(n - 1)} < 1.76 \times 10^{12} k^4 \log^2 k.
\end{equation}

We next use the fact that the inequality $x/\log x < A$ implies $x < 2A \log A$ whenever $A \geq 3$ in order to get an upper bound for $n$ depending on $k$. Indeed, taking $x := n - 1$ and $A := 1.76 \times 10^{12} k^4 \log^2 k$, and performing the respective calculations, inequality (15) yields $n < 1.7 \times 10^{14} k^4 \log^3 k$. We record what we have proved so far as a lemma.

**Lemma 2.** If $(n, m, k)$ is a nontrivial solution in positive integers of equation (4), then $n \geq k + 2$ and

$$m + 3 \leq n < 1.7 \times 10^{14} k^4 \log^3 k.$$
3.1. The case $k > 170$. In this case the following inequalities hold:

$$m + 3 \leq n < 1.7 \times 10^{14} k^4 \log^3 k < 2^{k/2}.$$ 

We recall the following result due to Bravo, Gómez and Luca (see [2]).

**Lemma 3.** If $r < 2^k$, then the following estimate holds:

$$F^{(k)}_r = 2^{r-2} \left( 1 + \frac{k - r}{2k+1} + \zeta(k, r) \right),$$

where $\zeta = \zeta(k, r)$ is a real number such that $|\zeta| < 4r^2/2^{2k+2}$.

So, from (4) and Lemma 3 applied to $r := n < 2^{k/2}$, we get

$$|2^{n-2} - 2^m| = \left| (F^{(k)}_n - 2^m) - 2^{n-2} \left( \frac{k - n}{2k+1} + \zeta \right) \right| < 1 + 2^{n-2} \left( \frac{n - k}{2k+1} + \frac{4n^2}{2^{2k+2}} \right).$$

Factoring $2^{n-2}$ on the right-hand side of the above inequality and taking into account that $1/2^{n-2} < 1/2^{k/2}$ (because $n \geq k + 2$ by Lemma 2), $(n-k)/2^{k+1} < 1/2^{k/2}$ and $4n^2/2^{2k+2} < 1/2^{k/2}$, which are all valid for $k > 170$, we conclude that

$$|1 - 2^{m-n+2}| < \frac{3}{2^{k/2}}.$$ 

By recalling that $m \leq n - 3$ (see (12)), we have that $m - n + 2 \leq -1$. So, from (16) and the previous result we have

$$\frac{1}{2} \leq 1 - 2^{m-n+2} < \frac{3}{2^{k/2}}$$

giving $2^{k/2} < 6$, which contradicts the fact that $k > 170$. Consequently, equation (4) has no solutions for $k > 170$.

3.2. The case $2 \leq k \leq 170$. For these values of $k$, we will use the following lemma, which is an immediate variation of the result due to Dujella and Pethő from [7], and will be the key tool used in this paper to reduce the upper bounds on the variables of the Diophantine equation (4).

**Lemma 4.** Let $A$, $B$, $\gamma$, $\mu$ be positive real numbers and $M$ a positive integer. Suppose that $p/q$ is a convergent of the continued fraction expansion of the irrational $\gamma$ such that $q > 6M$. Put $\varepsilon = ||\mu q|| - M ||\gamma q||$, where $|| \cdot ||$ denotes the distance
from the nearest integer. If \( \varepsilon > 0 \), then there is no positive integer solution \((u, v, w)\) to the inequality
\[
0 < |w\gamma - v + \mu| < AB^{-w},
\]
subject to the restrictions that
\[
u \leq M \quad \text{and} \quad w \geq \log A + \log q - \log \varepsilon / \log B.
\]

In order to apply this result, we let \( z := m \log 2 - (n - 1) \log \alpha - \log f_k(\alpha) \) and we observe that (13) can be rewritten as
\[
|e^z - 1| < 3 / \alpha^{n-1}.
\]

Note that \( z \neq 0 \); thus, we distinguish the following cases. If \( z > 0 \), then \( e^z - 1 > 0 \), so from (17) we obtain
\[
0 < z < 3 / \alpha^{n-1}.
\]

Suppose now that \( z < 0 \). Since the dominant zeros of \( F^{(k)} \) are strictly increasing as \( k \) increases, we deduce that \( 3 / \alpha^{n-1} < 3 / (\alpha(2))^{n-1} < 1 / 2 \) for all \( n \geq 5 \). Here, \( \alpha(2) \) denotes the golden section as mentioned before. Then from (17) we have that \( |e^z - 1| < 1 / 2 \) and therefore \( e^{|z|} < 2 \). Since \( z < 0 \), we have
\[
0 < |z| \leq e^{|z|} - 1 = e^{|z|} |e^z - 1| < 6 / \alpha^{n-1}.
\]

In any case, we have that the inequality
\[
0 < |z| < 6 / \alpha^{n-1}
\]
holds for all \( k \geq 2 \) and \( n \geq 5 \). Replacing \( z \) in the above inequality by its formula and dividing it across by \( \log \alpha \), we conclude that
\[
0 < \left| m \frac{\log 2}{\log \alpha} - n + \left( 1 - \frac{\log f_k(\alpha)}{\log \alpha} \right) \right| < \frac{13}{\alpha^{(n-1)}},
\]
where we have used the fact that \( 1 / \log \alpha < 2.1 \). We put
\[
\tilde{\gamma} := \tilde{\gamma}(k) = \frac{\log 2}{\log \alpha}, \quad \tilde{\mu} := \tilde{\mu}(k) = 1 - \frac{\log f_k(\alpha)}{\log \alpha}, \quad A := 13 \quad \text{and} \quad B := \alpha.
\]

We also put \( M_k := [1.7 \times 10^{14} k^4 \log^3 k] \), which is an upper bound on \( m \) by Lemma 2. The fact that \( \alpha \) is a unit in \( \mathcal{O}_K \), the ring of integers of \( K \), ensures that \( \tilde{\gamma} \) is an irrational
number. Even more, \( \hat{\gamma} \) is transcendental by the Gelfond-Schneider Theorem. Then, the above inequality (18) yields

\[
0 < |m\hat{\gamma} - n + \hat{\mu}| < AB^{-(n-1)}.
\]

It then follows from Lemma 4, applied to inequality (19), that

\[
n - 1 < \frac{\log A + \log q - \log \varepsilon}{\log B},
\]

where \( q = q(k) > 6M_k \) is a denominator of a convergent of the continued fraction of \( \hat{\gamma} \) such that \( \varepsilon = \varepsilon(k) = \|\hat{\mu}q\| - M_k\|\hat{\gamma}q\| > 0 \). A computer search with Mathematica revealed that if \( k \in [2, 170] \), then the maximum value of \( (\log A + \log q - \log \varepsilon)/\log B \) is < 360. Hence, we deduce that the possible solutions \((n, k, m)\) of equation (4) for which \( k \) is in the range [2, 170] all have \( n < 360 \).

Finally, a brute force search with Mathematica in the range

\[
2 \leq k \leq 170 \quad \text{and} \quad k + 2 \leq n < 360
\]

allows us to conclude that the only nontrivial solutions of (4) are

\[
(n, k, m) \in \{(4, 2, 1), (5, 2, 2)\}.
\]

This completes the analysis in the case \( k \in [2, 170] \) and therefore the proof of Theorem 1.

\[\square\]

4. Proof of Theorem 2

Assume first that we have a nontrivial solution \((n, k, m)\) of equation (5). Thus, \( n \neq 0 \) and \( n \neq 2 \). Note that if \( 3 \leq n \leq k \), then by (5) and Lemma 1 (d) we get \( 3 \cdot 2^{n-2} = 2^m + 1 \), which is not possible. Hence, from now on, we can assume that \( m \geq 2 \) and \( n \geq k + 1 \).

On the other hand, by Lemma 1 (a) and (5) we get

\[
2^m < 2^m + 1 = L_n^{(k)} \leq 2\alpha n < 2^{n+1}
\]

implying that \( m \leq n \). However, using (2) and (9), and taking into account that \( n \geq k + 1 \), we have that

\[
F_n^{(k)} + 2^m + 1 = 2F_{n+1}^{(k)} < 2^n.
\]
From the expression above we see that $m = n$ cannot be. Hence $m < n$. Using now (5) and Lemma 1 (c), we get that

$$|2^m - (2\alpha - 1)f_k(\alpha)\alpha^{n-1}| < \frac{5}{2}. \tag{20}$$

Dividing both sides of the above inequality by the second term of the left-hand side (which is positive), we obtain

$$\left| \frac{2^m \alpha^{-(n-1)}}{(2\alpha - 1)f_k(\alpha)} - 1 \right| < \frac{3}{\alpha^{n-1}}, \tag{21}$$

where we used the facts $1/f_k(\alpha) < 2$ and $1/(2\alpha - 1) < \frac{1}{2}$. The left-hand size of (21) is not zero. Indeed, if this were zero, we would then get that

$$2^m = (2\alpha - 1)f_k(\alpha)\alpha^{n-1}.$$ 

Conjugating the above relation by some automorphism of the Galois group of the decomposition field of $\Psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i \geq 2$ we have

$$4 \leq 2^m = |(2\alpha_i - 1)f_k(\alpha_i)\alpha_i^{n-1}| < 3,$$

which is a contradiction.

In order to use Theorem 3, we take $t := 3$,

$$\gamma_1 := 2, \quad \gamma_2 := \alpha, \quad \gamma_3 := (2\alpha - 1)f_k(\alpha)$$

and

$$b_1 := m, \quad b_2 := -(n - 1), \quad b_3 := -1.$$ 

For this choice we have $D = k$ (because $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{K} := \mathbb{Q}(\alpha)$) and $B = n - 1$. Thus, we can take $A_1 := k \log 2$, $A_2 := \frac{7}{10}$ (see (10)) and $A_3 := 4k \log k$ (see (11)).

By Matveev’s theorem and proceeding as in the proof of Lemma 2 we obtain the following lemma.

**Lemma 5.** If $(n, m, k)$ is a nontrivial solution in positive integers of equation (5), then $n \geq k + 1$ and

$$m < n < 1.68 \times 10^{14}k^4 \log^3 k.$$
4.1. The case $k > 170$. For these values of $k$, from Lemma 5 we deduce that $n < 2^{k/2}$. Bravo and Luca in [5] established that if $r > 1$ is an integer satisfying $r - 1 < 2^{k/2}$, then

$$ (2\alpha - 1)f_k(\alpha)\alpha^{r-1} = 3 \cdot 2^{r-2} + 3 \cdot 2^{r-1} \eta + \frac{\delta}{2} + \eta \delta, $$

where $\delta$ and $\eta$ are real numbers such that $|\delta| < 2^{r+2}/2^{k/2}$ and $|\eta| < 2k/2$. Consequently, from (22) (with $r := n$) and (20) we obtain

$$ |3 \cdot 2^{n-2} - 2^m| \leq |(2\alpha - 1)f_k(\alpha)\alpha^{n-1} - 2^m| + 3|\eta|2^{n-1} + \frac{|\delta|}{2} + |\eta \delta| $$

$$ < 3 \cdot 2^{n-2} \left(\frac{5}{3 \cdot 2^{n-1}} + \frac{4k}{2^k} + \frac{8}{3 \cdot 2^{k/2}} + \frac{32k}{3 \cdot 2^{3k/2}}\right). $$

Dividing the above inequality across by $2^{n-2}$ we conclude that

$$ |3 - 2^{m-n+2}| < 3 \left(\frac{1}{2^{k/2}} + \frac{4k}{2^k} + \frac{8}{3 \cdot 2^{k/2}} + \frac{32k}{3 \cdot 2^{3k/2}}\right) < 18 \frac{2^{k/2}}{k/2}. $$

In the last inequality we have used that $5/(3 \cdot 2^{n-1}) < 1/2^{k/2}$ (because $n \geq k + 1$), $4k/2^k < 1/2^{k/2}$, $8/(3 \cdot 2^{k/2}) < 3/2^{k/2}$ and $32k/(3 \cdot 2^{3k/2}) < 1/2^{k/2}$, which are all valid for $k > 170$. By recalling that $m < n$, we have $m - n + 2 \leq 1$ and so, from (23), we get

$$ 1 \leq 3 - 2^{m-n+2} < 18 \frac{2^{k/2}}{k/2}. $$

That is, $2^{k/2} < 18$ which is impossible since $k > 170$. Then (5) has no solutions for $k > 170$.

4.2. The case $2 \leq k \leq 170$. If we take $z = m \log 2 - (n - 1) \log \alpha - \log \mu$, where $\mu = (2\alpha - 1)f_k(\alpha)$, and proceeding as in Section 3.2, we deduce that the possible solutions $(n, k, m)$ of equation (5) for which $k$ is in the range $[2, 170]$ all have $n < 340$.

Finally, we conclude by a brute force search in Mathematica that equation (5) has no solutions in the range

$$ 2 \leq k \leq 170 \quad \text{and} \quad k + 1 \leq n < 340. $$

This proves Theorem 2.

Finally, Corollary 1 and Corollary 2 are immediate consequences of Theorem 1 and Theorem 2, respectively.

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