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AN ABSTRACT AND GENERALIZED APPROACH TO THE VITALI
THEOREM ON NONMEASURABLE SETS

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Abstract. Here we present abstract formulations of two theorems of Solecki which deal with some generalizations of the classical Vitali theorem on nonmeasurable sets in spaces with transformation groups.

Keywords: spaces with transformation groups; k -additive measurable structure; k -small system; upper semicontinuous k -small system; k -additive algebra admissible with respect to a k -small system

MSC 2010: 28A05, 28D05

1. INTRODUCTION

Selectors for countable subgroups of arbitrary infinite groups are extremely useful in constructing nonmeasurable sets with respect to some σ -finite invariant measure. The first example of such a nonmeasurable set is a Vitali set (see [14]) which is any \mathbb{Q} -selector in \mathbb{R} , where \mathbb{Q} is the subgroup of rationals. The nonmeasurability of the classical Vitali construction depends on the invariance of Lebesgue measure and this phenomenon can be successfully applied to the study of nonmeasurability (with respect to some σ -finite, invariant measure) of any generalized Vitali set which is any selector corresponding to any countable subgroup of any abstract infinite group.

The situation is different when the subgroup is uncountable. Kharazishvili in [4], Erdős and Mauldin in [1] constructed nonmeasurable sets for σ -finite, invariant measures with respect to such subgroups. They are the union of H -selectors when H is a subgroup of cardinality ω_1 . Kharazishvili in [3] observed that any set of positive measure contains such a nonmeasurable set, and Pelc in [6] enquired whether given any σ -finite, invariant measure μ , every set of positive measure contains such a subset which is nonmeasurable with respect to every invariant extension of μ . He gave

in [6] an affirmative answer to this question when μ is an extension of a regular, left Haar measure on a topological group. Finally, Solecki (see [13], [12]) arrived at an answer with full generality.

In the following two paragraphs, we add some preliminaries before introducing Solecki's results. Here the notations and definitions are borrowed from [4].

By a transformation group we mean a pair (X, G) , where X is a nonempty set and G is a group acting on X . This means that there exists a mapping $(g, x) \mapsto gx$ of $G \times X$ into X such that

- (i) for each $g \in G$, $x \mapsto gx$ is a bijection (or permutation) of X ;
- (ii) for all $x \in X$ and $g_1, g_2 \in G$, $g_1(g_2x) = g_1g_2x$.

For any $E \subseteq X$ and $g \in G$ we write $gE = \{gx : x \in E\}$ and call a nonempty family (or class) \mathcal{A} of sets G -invariant provided $gE \in \mathcal{A}$ for every $g \in G$ and $E \in \mathcal{A}$. A measure μ defined over a σ -algebra of subsets of X is said to be G -invariant if \mathcal{A} is a G -invariant class and $\mu(gE) = \mu(E)$ for every $g \in G$ and $E \in \mathcal{A}$. The group G acts freely if $\{x \in X : gx = x\} = \emptyset$ for all $g \in G \setminus \{e\}$ (where e is the identity element of G). More generally, G acts freely with respect to μ (in short, μ -freely) on X if $\mu^*\{x \in X : gx = x\} = 0$ for all $g \in G \setminus \{e\}$, where μ^* is the outer measure induced by μ . For any subgroup H of G and $x \in X$, the set $Hx = \{hx : h \in H\}$ is called an H -orbit in X . The collection of all H -orbits generates a partition of X and a set E in X is called a partial selector for H (or a partial H -selector) if $E \cap Hx$ has at most one point and a complete H -selector (or simply an H -selector) if $E \cap Hx$ has exactly one point for every $x \in X$. In fact, every partial selector is a complete selector with respect to some subcollection of H -orbits.

The following results are due to Solecki.

Theorem 1.1. *Let (X, G) be a space with transformation group G , μ be a nonzero, σ -finite, G -invariant measure on X and E be a μ -measurable subset of X with $\mu(E) > 0$. Further, suppose G is uncountable and acts μ -freely on X . Then there exists a subset F of E which is nonmeasurable with respect to every G -invariant extension of μ .*

Using the above theorem, Solecki in [13] also gave an analogue of the classical Vitali theorem for nonzero, σ -finite, G -invariant measures in spaces with transformation groups (see also [12]).

Theorem 1.2. *Let (X, G) be a space with transformation group G and μ be a nonzero, σ -finite, G -invariant measure on X . Further, suppose G is uncountable and acts μ -freely on X . Then there exists a countable subgroup H of G such that every H -selector is nonmeasurable with respect to any G -invariant extension of μ .*

In this paper, we give abstract formulations of the above two theorems using certain classes of sets such as a G -invariant, k -additive algebra (or ideal) and a G -invariant k -small system instead of G -invariant measures in spaces with transformation groups. It is worthwhile to mention here that the notion of “small system” was introduced by Riečan (see [7]), Riečan and Neubrunn (see [10]) to give abstract formulations of several well known theorems in classical measure and integration (see also [2], [5], [7], [8], [9], [10], [11]).

2. PRELIMINARIES AND RESULTS

Definition 2.1. A k -additive measurable structure on (X, G) is a pair (Σ, \mathcal{I}) consisting of two nonempty classes Σ and \mathcal{I} of subsets of X such that

- (i) Σ is an algebra and $\mathcal{I} (\subseteq \Sigma)$ a proper ideal in X ;
- (ii) both Σ and \mathcal{I} are k -additive; this means that these classes are closed with respect to the union of at most k number of sets;
- (iii) Σ and \mathcal{I} are G -invariant.

A k -additive measurable structure (Σ, \mathcal{I}) on (X, G) is called k^+ -saturated if the cardinality of any arbitrary collection of mutually disjoint sets from $\Sigma \setminus \mathcal{I}$ is at most k .

Henceforth, a k -additive algebra (or ideal) on (X, G) means that it is a k -additive algebra (or ideal) on X which is also G -invariant.

Definition 2.2. We define a transfinite k -sequence $\{\mathcal{N}_\alpha\}_{\alpha < k}$, where \mathcal{N}_α is a nonempty class of sets in G as a k -small system on (X, G) if

- (i) $\emptyset \in \mathcal{N}_\alpha$ for all $\alpha < k$;
- (ii) each \mathcal{N}_α is a G -invariant class;
- (iii) $E \in \mathcal{N}_\alpha$ and $F \subseteq E$ implies $F \in \mathcal{N}_\alpha$; i.e. \mathcal{N}_α is a hereditary class;
- (iv) $E \in \mathcal{N}_\alpha$ and $F \in \bigcap_{\alpha < k} \mathcal{N}_\alpha$ implies $E \cup F \in \mathcal{N}_\alpha$;
- (v) for any $\alpha < k$ there exists $\alpha^* > \alpha$ such that for any one-to-one correspondence $\beta \mapsto \mathcal{N}_\beta$ with $\beta > \alpha^*$, $\bigcup_{\beta} E_\beta \in \mathcal{N}_\alpha$ whenever $E_\beta \in \mathcal{N}_\beta$;
- (vi) for any $\alpha, \beta < k$ there exists $\gamma > \alpha, \beta$ such that $\mathcal{N}_\gamma \subseteq \mathcal{N}_\alpha$ and $\mathcal{N}_\gamma \subseteq \mathcal{N}_\beta$; i.e. $\{\mathcal{N}_\alpha\}_{\alpha < k}$ is directed.

Definition 2.3. A k -additive algebra \mathcal{S} is admissible with respect to a k -small system $\{\mathcal{N}_\alpha\}_{\alpha < k}$ on (X, G) if for every $\alpha < k$

- (i) $\mathcal{S} \setminus \mathcal{N}_\alpha \neq \emptyset \neq \mathcal{S} \cap \mathcal{N}_\alpha$;
- (ii) \mathcal{N}_α has an \mathcal{S} -base, i.e. $E \in \mathcal{N}_\alpha$ is contained in some $F \in \mathcal{N}_\alpha \cap \mathcal{S}$;
- (iii) $\mathcal{S} \setminus \mathcal{N}_\alpha$ satisfies the k -chain condition, i.e. the cardinality of any arbitrary collection of mutually disjoint sets from $\mathcal{S} \setminus \mathcal{N}_\alpha$ is at most k .

Thus, a k -additive algebra \mathcal{S} on (X, G) is called admissible if with respect to some k -small $\{\mathcal{N}_\alpha\}_{\alpha < k}$ on (X, G) , \mathcal{S} is compatible, constitutes an \mathcal{S} -base and satisfies k -chain condition.

We set $\mathcal{N}_\infty = \bigcap_{\alpha < k} \mathcal{N}_\alpha$. It follows from (ii), (iii) and (v) that \mathcal{N}_∞ is a k -additive ideal on (X, G) . We denote by $\tilde{\mathcal{S}}$ the k -additive algebra on (X, G) generated by \mathcal{S} and \mathcal{N}_∞ whose elements are of the form $X\Delta Y$, where $X \in \mathcal{S}$ and $Y \in \mathcal{N}_\infty$. Thus, $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$ forms a k -additive measurable structure on (X, G) . By virtue of condition (iv) of Definition 2.2 and conditions (ii) and (iii) of Definition 2.3 it follows that the measurable structure $(\tilde{\mathcal{S}}, \mathcal{N}_\infty)$ is k^+ -saturated.

Definition 2.4. A k -small system $\{\mathcal{N}_\alpha\}_{\alpha < k}$ is upper semi-continuous relative to a k -additive algebra \mathcal{S} on (X, G) if for every nested k -sequence $\{E_\xi: \xi < k\}$ of sets from \mathcal{S} satisfying $E_\xi \notin \mathcal{N}_{\alpha_0}$ for some $\alpha_0 < k$ and all $\xi < k$, we have $\bigcap_{\xi} E_\xi \notin \mathcal{N}_\infty$.

A k -additive algebra Ω on (X, G) satisfies the $(*)$ -property if there does not exist any covering $\{Y_\alpha\}_{\alpha < k}$ of X by sets from Ω such that for some $\alpha_0 < k$ a collection $\{E_\beta: \beta \in \mathcal{D}\}$ (\mathcal{D} is an index set) of disjoint sets $E_\beta \in \Omega \setminus \mathcal{N}_{\alpha_0}$ with $\text{card}(\mathcal{D}) = k$ which are all contained in some given member of the covering can be found.

Theorem 2.5. Let \mathcal{S} be a k -additive algebra on (X, G) such that $\text{card}(G) = k^+ \leq \text{card}(X)$, where k is a regular infinite cardinal. Assume also that

- (i) G acts freely on X ,
- (ii) \mathcal{S} is admissible with respect to a k -small system $\{\mathcal{N}_\alpha\}_{\alpha < k}$ on (X, G) which is upper semi-continuous relative to \mathcal{S} , and
- (iii) $X \notin \mathcal{N}_\infty$ and $X = \bigcup_{\alpha < k} Y_\alpha$, where $Y_\alpha \in \mathcal{S}$.

Then every set E in $\tilde{\mathcal{S}} \setminus \mathcal{N}_\infty$ contains a set F that does not belong to any k -additive algebra on (X, G) which contains \mathcal{S} and satisfies the $(*)$ -property.

Proof. Since $X \notin \mathcal{N}_\infty$ and \mathcal{N}_∞ forms a k -additive ideal on (X, G) , without loss of generality, we may assume that $Y_\alpha \notin \mathcal{N}_\infty$ for every $\alpha < k$. Also \mathcal{S} being admissible, $\mathcal{S} \setminus \mathcal{N}_\alpha$ satisfies the k -chain condition. Hence, for each α there exists a k -sequence $\{g_i^{(\alpha)}\}_{i < k}$ such that $X \setminus \bigcup_{i < k} g_i^{(\alpha)} Y_\alpha \in \mathcal{N}_\infty$. Again as $E \in \tilde{\mathcal{S}} \setminus \mathcal{N}_\infty$, there exists $\alpha_0 < k$ such that $E \notin \mathcal{N}_{\alpha_0}$. But $E = K\Delta P$, where $K \in \mathcal{S}$ and $P \in \mathcal{N}_\infty$. Since \mathcal{N}_α has an \mathcal{S} -base by condition (ii) of Definition 2.3, $P \subseteq Q \in \mathcal{N}_\infty \cap \mathcal{S}$. Hence, by (iv) of Definition 2.2, $E \supseteq K \setminus Q \in \mathcal{S} \setminus \mathcal{N}_{\alpha_0}$. We relabel $K \setminus Q$ as E . Now from the above and by condition (v) of Definition 2.2, it is possible to generate an injective mapping $\lambda: k \mapsto k$ having the property that for each $\alpha < k$ there exists $g \in G$ such that $g^{-1}(Y_\alpha) \cap E \notin \mathcal{N}_{\lambda(\alpha)}$, where $\lambda(\alpha) > \alpha^*$.

We set $\Gamma_\alpha = \{g \in G: g^{-1}(Y_\alpha) \cap E \notin \mathcal{N}_{\lambda(\alpha)}\}$ and claim that for some $\alpha_1 < k$, $\text{card}(\Gamma_{\alpha_1}) = k^+$. For otherwise, $\text{card}(\bigcup \Gamma_\alpha: \alpha < k) \leq k$ and so for any $g \in$

$G \setminus \bigcup_{\alpha < k} \Gamma_\alpha$, $g^{-1}(Y_\alpha) \cap E \in \mathcal{N}_{\lambda(\alpha)}$ which leads to the conclusion that $E = E \cap X = E \cap g^{-1}\left(\bigcup_{\alpha < k} Y_\alpha\right) = \bigcup_{\alpha < k} (g^{-1}(Y_\alpha) \cap E) \in \mathcal{N}_{\alpha_0}$, a contradiction.

From Γ_{α_1} we choose a set $\{g_\alpha: \alpha < k\}$ of cardinality k . By condition (iii) of Definition 2.2, $\bigcup_{\beta > \alpha} (g_\beta^{-1}(Y_{\alpha_1}) \cap E) \notin \mathcal{N}_{(\alpha_1)}$. Since $\{\mathcal{N}_\alpha\}_{\alpha < k}$ is upper semi-continuous relative to \mathcal{S} , the set $E_0 = \bigcap_{\alpha < k} \bigcup_{\beta > \alpha} g_\beta^{-1}(Y_{\alpha_1}) \cap E \in \mathcal{S} \setminus \mathcal{N}_\infty$. We set $W_\alpha = \bigcup_{\beta > \alpha} (g_\beta^{-1}(Y_{\alpha_1}) \cap E)$ so that $E_0 = \bigcap_{\alpha < k} W_\alpha$.

Let H be the subgroup generated by $\{g_\alpha: \alpha < k\}$. Then $\text{card}(H) = k$. From the family of H -orbits, extract out a subfamily members which have nonempty intersection with E_0 and choose a partial selector corresponding to this subfamily such that $V_0 \subseteq E_0$. Let V be an H -selector in X which extends V_0 and we write $F = E \cap V$.

We claim that F cannot belong to any k -additive algebra on (X, G) which contains \mathcal{S} and satisfies the $(*)$ -property. If possible, let Ω be one such k -additive algebra on (X, G) . Then $V_0 = F \cap E_0 \in \Omega$ and therefore $E_0 \subseteq H(V_0)$. Let $V_\alpha = V_0 \cap W_\alpha$. Now as the action of G on X is free, the collection $\{g_\alpha(V_\alpha): \alpha < k\}$ consists of mutually disjoint sets. We claim that for every $\xi < k$ there exists $\alpha < k$ such that $V_\beta \in \mathcal{N}_\xi$ for $\beta > \alpha$. For otherwise, there would exist $\xi_0 < k$ and a cofinal set \mathcal{D} of k such that $V_\alpha \notin \mathcal{N}_{\xi_0}$ for every $\alpha \in \mathcal{D}$ and $\{g_\alpha(V_\alpha): \alpha \in \mathcal{D}\}$ is a family of mutually disjoint subsets of Y_{α_1} . As k is regular, so $\text{card}(\mathcal{D}) = k$ and this contradicts the $(*)$ -property. Hence, $V_0 \in \mathcal{N}_\infty$ and therefore $E_0 \in \Omega \cap \mathcal{N}_\infty$. But earlier we have found that $E_0 \in \mathcal{S} \setminus \mathcal{N}_\infty \subseteq \Omega \setminus \mathcal{N}_\infty$. This is a contradiction.

Hence the theorem. \square

From the deductions in the proof of the above theorem, we find that every set $E_\xi \in \mathcal{S} \setminus \mathcal{N}_\infty$ contains a set X_ξ such that $X_\xi \subseteq \bigcap_{\alpha < k} \bigcup_{\beta^\xi > \alpha} g_{\beta^\xi}^{-1}(Y_\eta)$ for some k -sequence $\{\beta^\xi\}$ and $\eta < \alpha$. Since $\mathcal{S} \setminus \mathcal{N}_\alpha$ satisfies k -chain condition, $X \setminus \bigcup_{\xi < k} X_\xi \in \mathcal{N}_\infty$. Let \widehat{H} be the subgroup generated by $\{g_{\beta^\xi}: \beta^\xi < k, \xi < k\}$.

We show that no \widehat{H} -selector belongs to any k -additive algebra on (X, G) which contains \mathcal{S} and which satisfies the $(*)$ -property. If possible, let Ω be one such k -additive algebra. On account of the relation $X = \bigcup\{g(V): g \in \widehat{H}\}$ we get $V \in \Omega \setminus \mathcal{N}_\infty$. Consequently, $V \cap X_{\xi_0} \notin \mathcal{N}_{\eta_0}$ for some $\xi_0, \eta_0 < k$. Let $W_\alpha^{\xi_0} = \bigcup_{\beta^{\xi_0} > \alpha} g_{\beta^{\xi_0}}^{-1}(Y_{\eta_0}) \cap (V \cap X_{\xi_0})$ so that $V \cap X_{\xi_0} = \bigcap_{\alpha < k} W_\alpha^{\xi_0}$. But this implies by similar reasoning as given in the proof of the above theorem that for every $\xi < k$ there exists $\alpha < k$ such that $W_\beta^{\xi_0} \in \mathcal{N}_\xi$ for $\beta > \alpha$. Hence, $V \cap X_{\xi_0} \in \mathcal{N}_\infty$ and we arrive at a contradiction.

Theorem 2.6. *Let \mathcal{S} be a k -additive algebra on (X, G) such that $\text{card}(G) = k^+ \leq \text{card}(X)$, where k is a regular infinite cardinal. Assume also that*

- (i) G acts freely on X ,
- (ii) \mathcal{S} is admissible with respect to a k -small system $\{\mathcal{N}_\alpha\}_{\alpha < k}$ on (X, G) which is upper semi-continuous relative to \mathcal{S} , and
- (iii) $X \notin \mathcal{N}_\infty$ and $X = \bigcup_{\alpha < k} Y_\alpha$, where $Y_\alpha \in \mathcal{S}$.

Then there exists a subgroup \widehat{H} of G with $\text{card}(\widehat{H}) = k$ such that no \widehat{H} -selector in X belongs to any k -additive algebra on (X, G) which contains \mathcal{S} and satisfies the $(*)$ -property.

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