Wei Lu; Jing Mao; Chuanxi Wu

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A UNIVERSAL BOUND FOR LOWER NEUMANN EIGENVALUES OF THE LAPLACIAN

Wei Lu, Jing Mao, Chuanxi Wu, Wuhan

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Abstract. Let M be an n-dimensional ($n \geqslant 2$) simply connected Hadamard manifold. If the radial Ricci curvature of M is bounded from below by $(n-1)k(t)$ with respect to some point $p \in M$, where $t = d(\cdot, p)$ is the Riemannian distance on M to p, $k(t)$ is a nonpositive continuous function on $(0, \infty)$, then the first n nonzero Neumann eigenvalues of the Laplacian on the geodesic ball $B(p, l)$, with center p and radius $0 < l < \infty$, satisfy

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} + \ldots + \frac{1}{\mu_n} \geqslant \frac{l^{n+2}}{(n+2)\int_0^l f^{n-1}(t)\,\mathrm{d}t},$$

where $f(t)$ is the solution to

$$\begin{cases} f''(t) + k(t)f(t) = 0 & \text{on } (0, \infty), \\ f(0) = 0, \ f'(0) = 1. \end{cases}$$

Keywords: Hadamard manifold; Neumann eigenvalue; radial Ricci curvature

MSC 2010: 35P15, 53C20

1. Introduction

Let Ω be a bounded domain in an n-dimensional ($n \geqslant 2$) complete Riemannian manifold M. Denote by ∇ the gradient and by Δ the Laplacian on M. The so-called *Neumann eigenvalue problem* of the Laplacian is to find all possible real numbers μ such that

$$(\sharp) \qquad \begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

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has a nontrivial solution u, where $\vec{n}$ denotes the outward unit normal vector of the boundary $\partial\Omega$. It is well-known that for the boundary value problem ($\sharp$), $-\Delta$ has only discrete spectrum. Elements in the spectrum are called eigenvalues and all the eigenvalues μ_i, $i = 0, 1, 2, \ldots$, can be listed increasingly as follows

$$0 = \mu_0 < \mu_1 \leqslant \mu_2 \leqslant \mu_3 \leqslant \ldots \leqslant \mu_n \leqslant \ldots \uparrow +\infty.$$

For a fixed Neumann eigenvalue μ_i, the space of solutions u to ($\sharp$) is called the eigenspace of μ_i. Each eigenspace has finite dimension and we refer to the dimension of each eigenspace as to the multiplicity of the eigenvalue. Each eigenvalue in the above increasing sequence repeats according to its multiplicity.

For the first n nonzero Neumann eigenvalues of ($\sharp$), there exist some interesting estimates. For instance, if $M = \mathbb{R}^2$, the 2-dimensional Euclidean space, and furthermore $\Omega \subset \mathbb{R}^2$ is simply connected, Szegö in [12] proved the estimate

$$\mu_1 \leqslant \frac{\pi F_{1,1}^2}{|\Omega|} \cong \frac{10.65}{|\Omega|} \tag{1.1}$$

with equality if and only if Ω is a disk. His proof relies on the method of conformal transplantation and only works for simply connected domains. Weinberger in [13] extended Szegö's estimate (1.1) to arbitrary bounded domains in $\mathbb{R}^n$ ($n \geqslant 2$). In fact, he proved that

$$\mu_1 \leqslant \left(\frac{C_n}{|\Omega|}\right)^{2/n} F_{n/2,1}^2 \tag{1.2}$$

with equality if and only if Ω is a ball, where $|\Omega|$ is the volume of Ω, C_n is the volume of the unit ball in $\mathbb{R}^n$, $F_{v,k}$ is the kth positive zero of the derivative of $x^{1-v}J_v(x)$ with $J_v(x)$ the Bessel function. Besides, Szegö and Weinberger found that Szegö's proof for the estimate (1.1) can be used to get the bound

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} \geqslant \frac{2|\Omega|}{\pi F_{1,1}^2} \cong \frac{|\Omega|}{5.325} \tag{1.3}$$

for simply connected domains in $\mathbb{R}^2$. Bandle in [3], [4] showed that among all simply connected surfaces of given area A and of Gaussian curvature K_0 with $AK_0 \leqslant 2\pi$, the disk of area A on a complete simply connected surface of constant Gaussian curvature K_0 minimizes $1/\mu_1 + 1/\mu_2$, which improved the estimate (1.3). Ashbaugh and Benguria in [1] proved that for arbitrary domains in $\mathbb{R}^2$, the lower bound estimate

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} \geqslant \frac{|\Omega|}{2\pi} \tag{1.4}$$

holds. They also extended (1.4) to arbitrary domains in $\mathbb{R}^n$ as follows:

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\ldots+\frac{1}{\mu_n}\geqslant\frac{n}{n+2}\Big(\frac{|\Omega|}{C_n}\Big)^{2/n}. \tag{1.5}$$

If $\Omega\subset\mathbb{R}^2$ is a simply connected bounded domain, for the problem ($\sharp$), Girouard, Nadirashvili, and Polterovich in [9] gave a *sharp* upper bound for μ_2 as follows:

$$\mu_2\leqslant\frac{2\pi\mu_1(\mathbb{D})}{|\Omega|}=\frac{2\pi F_{1,1}^2}{|\Omega|}\cong\frac{21.3}{|\Omega|}, \tag{1.6}$$

where $\mu_1(\mathbb{D})$ stands for the first nonzero Neumann eigenvalue of the Laplacian on the unit disk in $\mathbb{R}^2$. Besides, the equality in (1.6) can be obtained for the disjoint union of two identical disks. Polterovich also suggested that the inequality (1.1) can be strengthened in the following sense (see [2], Problem (2), page 405 for details):

$$\mu_1\mu_2\leqslant\frac{\pi^2\mu_1^2(\mathbb{D})}{|\Omega|^2}=\frac{\pi^2F_{1,1}^4}{|\Omega|^2}\cong\frac{113.42}{|\Omega|^2}. \tag{1.7}$$

If furthermore the simply connected domain $\Omega\subset\mathbb{R}^2$ has some symmetry, additional results on $\mu_1(\Omega)$, $\mu_2(\Omega)$ can be expected. For instance, by Ashbaugh-Benguria (see [1]), one has:

- ▷ if $\Omega\subset\mathbb{R}^2$ has k-fold ($k\geqslant 3$) rotational symmetry, then for the problem ($\sharp$), $\mu_1=\mu_2$;
- ▷ if $\Omega\subset\mathbb{R}^2$ has 4-fold rotational symmetry, then for the problem ($\sharp$), $\mu_k\leqslant\mu_k(\Omega^*)$, $k=1,2$, where Ω^* is the disk of the same area as Ω, and $\mu_k(\Omega^*)$ stands for the kth nonzero Neumann eigenvalue of the Laplacian on Ω^*.

Furthermore, by Enache-Philippin (see [7]), if $\Omega\subset\mathbb{R}^2$ has 2-fold rotational symmetry, then for the problem ($\sharp$), one has

- ▷ $r_0^2(\mu_1+\mu_2)\leqslant 2\mu_1(\mathbb{D})=2F_{1,1}^2\cong 6.78$,
- ▷ $r_0^2|\Omega|\mu_1\mu_2\leqslant\pi\mu_1^2(\mathbb{D})=\pi F_{1,1}^4\cong 36.12$,

where r_0 is the conformal radius of Ω at the origin. More explicit but less sharp inequalities are

- ▷ $|\Omega_0|(\mu_1+\mu_2)\leqslant 2\pi\mu_1(\mathbb{D})=2\pi F_{1,1}^2\cong 21.30$,
- ▷ $|\Omega_0|\,|\Omega|\mu_1\mu_2\leqslant\pi^2\mu_1^2(\mathbb{D})=\pi^2F_{1,1}^4\cong 113.42$,

where Ω_0 is the largest disk inscribed in Ω centered at the origin. The second inequality here somehow supports Polterovich's suggestion (1.7) and also shows the hope and the possibility of solving (1.7).

An interesting attempt is trying to generalize the above conclusions for bounded domains in Euclidean spaces to the case of bounded domains on manifolds. This attempt has been carried out already. For instance, Xia in [14] generalized Ashbaugh-Benguria's estimates (1.4) and (1.5) to the following setting: for a bounded domain Ω with smooth boundary in an n-dimensional ($n \geqslant 2$) simply connected Hadamard manifold having Ricci curvature $\mathrm{Ric} \geqslant -(n-1)k^2$, the first nonzero n Neumann eigenvalues satisfy

$$\frac{1}{\mu_1} + \frac{1}{\mu_2} + \ldots + \frac{1}{\mu_n} \geqslant C(n, |\Omega|, |k|), \tag{1.8}$$

where $C(n, |\Omega|, |k|)$ is a computable constant depending only on n, $|\Omega|$ and $|k|$.

The purpose of this paper is to improve Xia's estimate (1.8) under a weaker curvature assumption. Before stating our main result, we need the notion of radial Ricci curvature lower bound – for the precise statement of radial (Ricci or sectional) curvature being bounded and related geometric applications, see e.g. [8], [10].

Definition 1.1. Given a continuous function $k\colon (0, l) \to \mathbb{R}$, we say that the preseribed n-dimensional ($n \geqslant 2$) complete Riemannian manifold M has a radial Ricci curvature lower bound $(n-1)k$ along any unit-speed minimizing geodesic starting from a point $p \in M$ if

$$\mathrm{Ric}(v_x, v_x) \geqslant (n-1)k(t(x)) \quad \forall\, x \in M \setminus (\mathrm{Cut}(p) \cup \{p\}), \tag{1.9}$$

where Ric denotes the Ricci curvature of M, $\mathrm{Cut}(p)$ is the cut-locus of p, $t(x) = d(x, p)$ is the Riemannian distance from p to x, and v_x is the radial unit tangent vector of the geodesic at x.

Remark 1.1. If (1.9) is satisfied, then we say that *M has a radial Ricci curvature lower bound with respect to the point p.* Clearly, if furthermore M is a Hadamard manifold (i.e., the sectional curvature K_M satisfies $K_M \leqslant 0$), then by the Cartan-Hadamard theorem, p has no conjugate point, which implies that $\mathrm{Cut}(p)$ is empty. In this setting, $k(t)$ is a continuous function on $(0, \infty)$.

Our result is the following.

Theorem 1.1. *Let M be an n-dimensional ($n \geqslant 2$) complete simply connected Hadamard manifold having a radial Ricci curvature lower bound $(n-1)k(t)$ w.r.t. some point $p \in M$, where $t = d(\cdot, p)$ is the Riemannian distance on M to p, and $k(t)$ is a nonpositive continuous function on $(0, \infty)$. Let $B(p, l)$ be the geodesic ball with center p and radius l on M. Assume that $f(t)$ is the solution to the initial value problem (IVP for short)*

$$\begin{cases} f''(t) + k(t)f(t) = 0 & \text{on } (0, \infty), \\ f(0) = 0, \ f'(0) = 1. \end{cases} \tag{1.10}$$

Then the first n nonzero Neumann eigenvalues μ_i of the Laplacian on $B(p,l)$ satisfy

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\ldots+\frac{1}{\mu_n}\geqslant\frac{l^{n+2}}{(n+2)\int_0^l f^{n-1}(t)\,\mathrm{d}t}. \tag{1.11}$$

Remark 1.2. If $K_M\equiv 0$, then $f(t)=t$ and the estimate (1.11) becomes

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\ldots+\frac{1}{\mu_n}\geqslant\frac{nl^2}{n+2}.$$

If $\inf\limits_{t\in(0,l)} k(t)=k^-<0$, then

$$f(t)=\frac{\sinh(\sqrt{-k^-}t)}{\sqrt{-k^-}}$$

and the estimate (1.11) becomes

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\ldots+\frac{1}{\mu_n}\geqslant\frac{(\sqrt{-k^-})^{n-1}l^{n+2}}{(n+2)\int_0^l(\sinh(\sqrt{-k^-}t))^{n-1}\,\mathrm{d}t}.$$

2. Proof of Theorem 1.1

Denote by T_pM and $S_pM\subset T_pM$ the tangent space (at p) and its unit sphere (centered at the origin), respectively. Let $\exp_p$ be the exponential map of M at p. Let $\{u_i\}_{i=0}^{\infty}$ be the sequence of orthonormal eigenfunctions corresponding to the Neumann eigenvalues $\{\mu_i\}_{i=0}^{\infty}$ of ($\sharp$) with $\Omega=B(p,l)$. By Rayleigh's theorem and the Max-min theorem (see [6], pages 16–17), we know that the ith Neumann eigenvalue μ_i of ($\sharp$) with $\Omega=B(p,l)$ is given by

$$\mu_i=\min\left\{\frac{\int_{B(p,l)}|\nabla u|^2\,\mathrm{d}v}{\int_{B(p,l)}u^2\,\mathrm{d}v}\colon\ u\in W_0^{1,2}(B(p,l)),\ u\neq 0,\ u\perp\operatorname{span}(u_0,u_1,\ldots,u_{i-1})\right\}, \tag{2.1}$$

where $\mathrm{d}v$ is the volume element on M, $W_0^{1,2}(B(p,l))$ is the completion of $C^\infty(B(p,l))$, the space of all smooth functions defined on $B(p,l)$, under the Sobolev norm

$$|w|_{1,2}:=\int_{B(p,l)}|\nabla w|^2\,\mathrm{d}v+\int_{B(p,l)}w^2\,\mathrm{d}v,$$

and $\operatorname{span}(u_0,u_1,\ldots,u_{i-1})$ is the space spanned by $u_0,u_1,\ldots,u_{i-1}$.

For any $\vec{v} \in S_pM$, let $\gamma(t)$ with $\gamma(0) = p$ and $\gamma'(0) = \vec{v}$ be the unit-speed minimizing geodesic emanating from p. By the Cartan-Hadamard theorem, since $K_M \leqslant 0$, $\gamma(t) = \exp_p(t\vec{v})$ with $\exp_p(\cdot)$ the exponential map gives a diffeomorphism from T_pM to M, and $\gamma(t)$ can extend to infinity. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of T_pM satisfying for some $\vec{v}_0 \in S_pM$ (which is the initial tangent vector of a unit-speed minimizing geodesic $\widetilde{\gamma}(t)$ determined below, i.e., $\widetilde{\gamma}(0) = p$, $\widetilde{\gamma}'(0) = \vec{v}_0$), $\int_{S_pM} \langle \vec{v}_0, e_i \rangle \,\mathrm{d}\sigma = 0$ with $\mathrm{d}\sigma$ the volume element of S_pM. Parallel translate $\{e_1, e_2, \ldots, e_n\}$ along the geodesics $\gamma(t)$ and then a differentiable orthonormal frame field $\{E_1, E_2, \ldots, E_n\}$ on M can be obtained. Define a vector field $Y(q)$ on $B(p, l)$ as

$$Y(q) := \sum_{i=1}^{n} \left(\int_{B(p,l)} \langle \exp_q^{-1}(z), e_i \rangle \,\mathrm{d}v \right) E_i(q),$$

which is continuous. The convexity of $B(p, l)$ implies that on the boundary $\partial B(p, l)$ of $B(p, l)$, Y points into $B(p, l)$. Then using the Brouwer fixed point theorem (see [5]), we know that $Y(q)$ has a zero q_0. Since there exists some $\vec{v}_0 \in S_pM$ such that the unit-speed minimizing geodesic $\widetilde{\gamma}(t)$ with $\widetilde{\gamma}(0) = p$, $\widetilde{\gamma}'(0) = \vec{v}_0$, $\widetilde{\gamma}(t_0) = q_0$ for some $0 < t_0 < l$, joining p and q_0 can be determined uniquely, hence $\beta(t) := \widetilde{\gamma}(t_0 - t)$ must be the unit-speed minimizing geodesic, emanating from q_0, with $\beta(0) = q_0$, $\beta(t_0) = p$, $\beta'(t_0) = -\vec{v}_0$. Therefore, one has

$$\nabla_{\beta'(t)} Y = -\sum_{i=1}^{n} \left(\int_{S_pM} \langle \exp_q^{-1}(z), e_i \rangle \,\mathrm{d}\sigma \right) E_i(q),$$

which implies

$$\nabla_{\vec{v}_0} Y(p) = \sum_{i=1}^{n} \left(\int_{S_pM} \langle \vec{v}_0, e_i \rangle \,\mathrm{d}\sigma \right) e_i = \vec{0},$$

the zero vector. Hence, q_0 coincides with p if one suitably chooses the orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM (which is diffeomorphic to the Euclidean n-space $\mathbb{R}^n$) such that $\int_{S_pM} \langle \vec{v}_0, e_i \rangle \,\mathrm{d}\sigma = 0$. Then $Y(p) = \vec{0}$, which implies

$$\int_{B(p,l)} \langle \exp_p^{-1}(z), e_i \rangle \,\mathrm{d}v = 0$$

for $i = 1, 2, \ldots, n$. Therefore, for the Riemannian normal coordinates $y\colon M \to \mathbb{R}^n$ determined by the orthonormal frame $(p; e_1, e_2, \ldots, e_n)$, we know that their coordinate functions $y^i\colon M \to \mathbb{R}^n$, $i = 1, 2, \ldots, n$, satisfy

$$\int_{B(p,l)} y^i(q) \,\mathrm{d}v = \int_{B(p,l)} \langle \exp_p^{-1}(q), e_i \rangle \,\mathrm{d}v = 0 \tag{2.2}$$

for $i=1,2,\ldots,n$. For any $\vec{v}\in S_pM$, let $\langle \exp_p^{-1},\vec{v}\rangle$ be the function on M defined by $\langle \exp_p^{-1},\vec{v}\rangle(q)=\langle \exp_p^{-1}(q),\vec{v}\rangle$ for any $q\in B(p,l)$. Then (2.2) is equivalent to saying that

$$\int_{B(p,l)}\langle \exp_p^{-1},e_i\rangle\,\mathrm{d}v=0 \tag{2.3}$$

holds for $i=1,2,\ldots,n$. Therefore, for any $\eta\in S_pM$, by (2.3), we have

$$\int_{B(p,l)}\langle \exp_p^{-1},\eta\rangle\,\mathrm{d}v=0. \tag{2.4}$$

As in [1], [14], by the Borsuk-Ulam theorem in [11], page 266, one can find $(n-1)$ unit orthogonal vectors $\eta_2,\ldots,\eta_{n-1}$ in T_pM such that

$$\int_{B(p,l)}\langle \exp_p^{-1},\eta_j\rangle u_i\,\mathrm{d}v=0 \tag{2.5}$$

for $j=2,3,\ldots,n$ and $i=1,2,\ldots,j-1$. In fact, one can define a mapping $f_n\colon S_pM\to\mathbb{R}^{n-1}$ componentwise by

$$f_{n,k}\colon\ \eta\to\int_{\Omega}\langle \exp_p^{-1},\eta\rangle u_k\,\mathrm{d}v\quad\text{for } k=1,2,\ldots,n-1. \tag{2.6}$$

By the Borsuk-Ulam theorem, there exists $\overline{\eta}\in S_pM$ such that $f_n(\overline{\eta})=0$, since S_pM is isometric to the $(n-1)$-dimensional unit Euclidean sphere and f_n is antipode preserving. Taking $\overline{\eta}$ as η_n, then we have found a unit vector $\eta_n\in S_pM$ satisfying

$$\int_{B(p,l)}\langle \exp_p^{-1},\eta_n\rangle u_i\,\mathrm{d}v=0,\quad i=1,2,\ldots,n-1. \tag{2.7}$$

Set $S^{n-2}_{\eta_n}:=\{\eta\in S_pM\colon\ \langle\eta,\eta_n\rangle=0\}$, which is isometric to an $(n-2)$-dimensional unit Euclidean sphere. By an argument similar to the above, one can find a unit vector $\eta_{n-1}\in S^{n-2}_{\eta_n}$ such that

$$\int_{B(p,l)}\langle \exp_p^{-1},\eta_{n-1}\rangle u_i\,\mathrm{d}v=0,\quad i=1,2,\ldots,n-2. \tag{2.8}$$

Repeating the above process, one can finally obtain $(n-1)$ mutually orthogonal unit tangent vectors $\eta_2,\eta_3,\ldots,\eta_n\in S_pM$ satisfying (2.5).

Extending $\{\eta_2,\eta_3,\ldots,\eta_n\}$ to an orthonormal basis $\{\eta_1,\eta_2,\eta_3,\ldots,\eta_n\}$ of T_pM, then for the Riemannian normal coordinates $x\colon M\to\mathbb{R}^n$ on M determined by $(p;\eta_1,\eta_2,\eta_3,\ldots,\eta_n)$, the corresponding coordinate functions $x^i\colon M\to\mathbb{R}$,

$i = 1, 2, \ldots, n$, are given by $x^i(q) = \langle \exp_p^{-1}(q), \eta_i \rangle$. Since the eigenfunction u_0 of the first Neumann eigenvalue μ_0 is the constant function $1/\sqrt{\mathcal{V}(B(p,r))}$ with $\mathcal{V}(\cdot)$ the volume of a given geometric object, by (2.4) and (2.5) one has

$$\int_{B(p,l)} x^j u_i \, \mathrm{d}v = 0 \quad \text{for } j = 1, 2, \ldots, n \quad \text{and} \quad i = 0, 1, 2, \ldots, j-1. \tag{2.9}$$

Choosing trial functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ as $\varphi_j = x^j$, $j = 1, 2, \ldots, n$, then by (2.1) and (2.9) we have

$$\mu_i \leqslant \frac{\int_{B(p,l)} |\nabla x^i|^2 \, \mathrm{d}v}{\int_{B(p,l)} (x^i)^2 \, \mathrm{d}v} \tag{2.10}$$

for $i = 1, 2, \ldots, n$. Denote by $\{\partial/\partial x^k,\ k = 1, 2, \ldots, n\}$ the natural basis of tangent spaces associated to the coordinate chart x. Set $g_{kl} := \langle \partial/\partial x^k, \partial/\partial x^l \rangle$ for $k, l = 1, 2, \ldots, n$. Since $K_M \leqslant 0$, by Rauch's comparison theorem, we know that all the eigenvalues of the matrix (g_{kl}) are greater than or equal to 1. Hence all the eigenvalues of its inverse matrix $(g^{kl}) := (g_{kl})^{-1}$ are less than or equal to 1, which implies that the diagonal elements of (g^{kl}) are all less than or equal to 1, i.e., $g^{kk} \leqslant 1$, $k = 1, 2, \ldots, n$. Therefore, for $i = 1, 2, \ldots, n$, we have

$$|\nabla x^i|^2 = \left\langle \sum_{k=1}^{n} g^{ik} \frac{\partial}{\partial x^k}, \sum_{l=1}^{n} g^{il} \frac{\partial}{\partial x^l} \right\rangle = g^{ii} \leqslant 1. \tag{2.11}$$

Substituting (2.11) into (2.10) yields

$$\mu_i \leqslant \frac{\int_{B(p,l)} \mathrm{d}v}{\int_{B(p,l)} (x^i)^2 \, \mathrm{d}v} = \frac{\mathcal{V}(B(p,l))}{\int_{B(p,l)} (x^i)^2 \, \mathrm{d}v}$$

for $i = 1, 2, \ldots, n$, which implies that

$$\sum_{i=1}^{n} \frac{1}{\mu_i} \geqslant \frac{\int_{B(p,l)} \sum_{i=1}^{n} (x^i)^2 \, \mathrm{d}v}{\mathcal{V}(B(p,l))} = \frac{\int_{B(p,l)} \sum_{i=1}^{n} \langle \exp_p^{-1}, e_i \rangle^2 \, \mathrm{d}v}{\mathcal{V}(B(p,l))} = \frac{\int_{B(p,l)} t^2 \, \mathrm{d}v}{\mathcal{V}(B(p,l))} \tag{2.12}$$

with $t = d(p, \cdot)$ the Riemannian distance to the point p. Since $K_M \leqslant 0$, by Bishop's volume comparison theorem we have that the area of the boundary of the geodesic ball $B(p,t)$ satisfies

$$\mathcal{A}(\partial B(p,r)) \geqslant w_{n-1} r^{n-1},$$

where w_{n-1} denotes the area of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Hence, one has

$$\begin{aligned}(2.13)\qquad \int_{B(p,l)} t^2\,\mathrm{d}v &= \int_0^l \Big(\int_{\partial B(p,r)} t^2\,\mathrm{d}\mathcal{A}_r\Big)\,\mathrm{d}r = \int_0^l r^2\mathcal{A}(\partial B(p,r))\,\mathrm{d}r\\ &\geqslant w_{n-1}\int_0^l r^{n+1}\,\mathrm{d}r = \frac{w_{n-1}}{n+2}l^{n+2},\end{aligned}$$

where $\mathrm{d}\mathcal{A}_t$ is the area element of $\partial B(p,t)$.

On the other hand, since the radial Ricci curvature of M is bounded from below by $(n-1)k(t)$ with respect to p, by the generalized Bishop's volume comparison theorem I (see [8], Theorem 3.3 and Corollary 3.5) we have

$$(2.14)\qquad \mathcal{V}(B(p,l)) \leqslant \mathcal{V}(V_n(p^-,l)),$$

where $V_n(p^-,l)$ is the geodesic ball, with center p^- and radius l, of the spherically symmetric manifold $M^- := [0,\infty)\times_{f(t)}\mathbb{S}^{n-1}$ with the base point p^- and the warping function determined by the IVP (1.10). The equality in (2.14) holds if and only if $B(p,l)$ is isometric to $V_n(p^-,l)$. Since $\mathcal{V}(V_n(p^-,l)) = w_{n-1}\int_0^l f^{n-1}(t)\,\mathrm{d}t$, we have

$$(2.15)\qquad \mathcal{V}(B(p,l)) \leqslant w_{n-1}\int_0^l f^{n-1}(t)\,\mathrm{d}t.$$

By (2.12), (2.13) and (2.15), it is easy to get that

$$\sum_{i=1}^n \frac{1}{\mu_i} \geqslant \frac{l^{n+2}}{(n+2)\int_0^l f^{n-1}(t)\,\mathrm{d}t},$$

which completes the proof of Theorem 1.1. $\square$

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Authors' address: Wei Lu, Jing Mao (corresponding author), Chuanxi Wu, Faculty of Mathematics and Statistics, Key Laboratory of Applied Mathematics of Hubei Province, Hubei University, 368#, You Yi Road, Wuchang District, Wuhan 430062, Hubei Province, P. R.China, e-mail: jiner120@163.com.