Applications of Mathematics

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Applications of Mathematics, Vol. 65 (2020), No. 4, 355–377

Persistent URL: http://dml.cz/dmlcz/148338

ON COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF NEGATIVELY SUPERADDITIVE DEPENDENT RANDOM VARIABLES

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Received September 23, 2018. Published online June 15, 2020.

Abstract. In this work, the complete moment convergence and complete convergence for weighted sums of negatively superadditive dependent (NSD) random variables are studied, and some equivalent conditions of these strong convergences are established. These main results generalize and improve the corresponding theorems of Baum and Katz (1965) and Chow (1988) to weighted sums of NSD random variables without the assumption of identical distribution. As an application, a Marcinkiewicz-Zygmund-type strong law of large numbers for weighted sums of NSD random variables is obtained.

Keywords: NSD random variables; complete moment convergence; weighted sum; equivalent conditions

MSC 2020: 60F15

1. Introduction

The concept of complete convergence was first introduced by Hsu and Robbins [13] as follows: a sequence $\{X_n;\ n \geqslant 1\}$ of random variables is said to converge completely to a constant λ if $\sum\limits_{n=1}^{\infty} P(|X_n - \lambda| > \varepsilon) < \infty$ for all $\varepsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $X_n \to \lambda$ almost surely (a.s.). The converse is true if $\{X_n;\ n \geqslant 1\}$ is independent. Hsu and Robbins [13] showed that the sequence

This paper is supported by the Science and Technology Plan Project of Hunan Province (2016TP1020), the Scientific Research Fund of Hunan Provincial Education Department (18C0660), the National Statistical Science Research Project of China (2018LY05), the State Scholarship Fund of China Scholarship Council (No. 201908430242) and the Discovery Grants (RGPIN-2018-06466) from Natural Sciences and Engineering Research Council of Canada.

DOI: 10.21136/AM.2020.0255-18

of arithmetic means of independent and identically distributed random variables converges completely to the expected value of the summands, provided the variance is finite. Erdős [11] proved the converse. The Hsu-Robbins-Erdős theorem is a fundamental result in probability theory, which has been generalized and extended in several ways. One of the most important generalizations was provided by Baum and Katz [4] for the following strong law of large numbers.

Theorem A. *Let* $\frac{1}{2} < \alpha \leqslant 1$ *and* $\alpha p > 1$. *Let* $\{X, X_n;\ n \geqslant 1\}$ *be a sequence of independent and identically distributed random variables with* $EX_n = 0$. *Then the following statements are equivalent:*

$$E|X|^p < \infty, \tag{1.1}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty \quad \forall\, \varepsilon > 0. \tag{1.2}$$

From then on, many researchers investigated and improved the Baum-Katz theorem for independent and dependent random variables. Chow [6] first showed the concept of complete moment convergence by generalizing the result of Baum and Katz [4] as follows: Let $\{X_n;\ n \geqslant 1\}$ be a sequence of random variables, and $a_n > 0$, $b_n > 0$, $q > 0$. If $\sum_{n=1}^{\infty} a_n E(b_n^{-1}|X_n| - \varepsilon)_+^q < \infty$ for all $\varepsilon \geqslant 0$, then $\{X_n;\ n \geqslant 1\}$ is said to have the *property of complete moment convergence*. It is well known that complete moment convergence implies complete convergence. For more details about complete moment convergence, we can refer the readers to Sung [21], Chen and Wang [5], Wu [26], Wang and Wu [24] among others. In addition, Chow [6] obtained the following complete moment convergence result for independent and identically distributed random variables.

Theorem B. *Let* $\alpha > \frac{1}{2}$, $p \geqslant 1$, *and* $\alpha p > 1$. *Let* $\{X, X_n;\ n \geqslant 1\}$ *be a sequence of independent and identically distributed random variables with* $EX_n = 0$. *If* $E(|X|^p + |X|\log(1 + |X|)) < \infty$, *then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} < \infty. \tag{1.3}$$

However, both of Baum and Katz [4] and Chow [6] did not consider the interesting case $\alpha p = 1$. In addition, Chow [6] did not discuss the necessary condition of complete moment convergence.

By weakening the assumptions on validity of limit convergence, we provide an extension for possible applications of probability theory to various fields, especially to statistics research field. In many theoretical statistical frameworks, we assume that variables are independent. But in real studies, this assumption is not plausible. Therefore, many statisticians have revised this assumption in order also to consider dependent cases, such as negatively associated random variables, positively associated random variables, negatively orthant dependent random variables, negatively superadditive dependent (NSD) random variables, and many others.

The main purpose of this paper is to extend and improve the results of Baum and Katz [4] and Chow [6] for independent and identically distributed random variables to weighted sums of NSD random variables without the assumption of identical distribution. Inspired by the above important results, complete moment convergence for weighted sums of NSD random variables is further studied, and some equivalent conditions on the strong convergence for weighted sums of NSD cases are established. In addition, the interesting case $\alpha p = 1$ is also considered in our work. As an application, a Marcinkiewicz-Zygmund-type strong law of large numbers for weighted sums of NSD random variables is obtained.

Before introducing our work, let us recall some definitions of negative dependent structures.

Definition 1.1. A finite family $\{X_i;\ 1 \leqslant i \leqslant n\}$ of random variables is said to be *negatively associated* (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}(f_1(X_i,\ i \in A),\ f_2(X_j,\ j \in B)) \leqslant 0, \tag{1.4}$$

whenever f_1 and f_2 are real coordinatewise nondecreasing functions such that the covariance exists. An infinite family $\{X_n;\ n \geqslant 1\}$ of random variables is NA if every finite subfamily is NA.

The concept of NA random variables was first introduced by Alam and Saxena [1] and studied in detail by Joag-Dev and Proschan [15].

Another negative dependence structure is negatively superadditive dependence (NSD), which is weaker than NA. To introduce the notion of NSD, we shall first recall the class of superadditive functions as follows.

Definition 1.2. A function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ is called *superadditive* if $\phi(x \vee y) + \phi(x \wedge y) \geqslant \phi(x) + \phi(y)$ for all $x, y \in \mathbb{R}^n$, where $\vee$ stands for componentwise maximum and $\wedge$ stands for componentwise minimum.

Based on the class of superadditive functions introduced by Kemperman [16], Hu [14] introduced the following concept of NSD random variables.

Definition 1.3. A random vector $X = (X_1, X_2, \ldots, X_n)$ is said to be NSD if

$$E\phi(X_1, X_2, \ldots, X_n) \leqslant E\phi(X_1^*, X_2^*, \ldots, X_n^*), \tag{1.5}$$

where $X_1^*, X_2^*, \ldots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.5) exist. A sequence $\{X_n;\ n \geqslant 1\}$ of random variables is said to be NSD if for all $n \geqslant 1$, $(X_1, X_2, \ldots, X_n)$ is NSD.

Hu [14] gave an example illustrating that NSD does not imply NA, and posed an problem whether NA implies NSD. Christofides and Vaggelatou [7] solved this problem and indicated that NA implies NSD. Thus, the NSD structure is an extension of the NA structure and sometimes more useful than the latter and can be used to obtain many important probability inequalities. As pointed out and proved by Joag-Dev and Proschan [15], many well-known multivariate distributions have the NA property. Consequently, studying the probability limit convergence properties of NSD random variables is of much interest.

Since the concept of NSD random variables was introduced by Hu [14], many applications for NSD random variables have been established. See, for example, Hu [14] for some basic properties and three structural theorems, Eghbal et al. [9] for two maximal inequalities and a strong law of large numbers of quadratic forms of nonnegative NSD random variables, Eghbal et al. [10] for some Kolmogorov inequalities for quadratic forms and weighted quadratic forms of nonnegative NSD uniformly bounded random variables, Shen et al. [18] for the almost sure convergence and strong stability for weighted sums of NSD random variables, Wang et al. [22] for the complete convergence of arrays of rowwise NSD random variables and the complete consistency for the estimator of nonparametric regression model based on NSD errors, Wang et al. [23] for the complete convergence for weighted sums of NSD random variables and its application in the EV regression model, Shen et al. [20] for some applications of the Rosenthal-type inequality for NSD random variables, Zhang [27] for the strong convergence property of Jamison weighted sums of NSD random variables, Naderi et al. [17], Deng et al. [8] and Zheng et al. [28] for the complete convergence of weighted sums for NSD random variables, Shen et al. [19] for the complete moment convergence for arrays of rowwise NSD random variables, Amini et al. [2] for the complete convergence of moving average processes based on NSD sequences, among others.

Definition 1.4. A sequence $\{X_n;\ n \geqslant 1\}$ of random variables is said to be *stochastically dominated by a random variable* X if there exists a positive constant C such that

$$P(|X_n| > x) \leqslant CP(|X| > x),$$

for all $x \geqslant 0$ and $n \geqslant 1$.

Definition 1.5. A real-valued function $l(x)$, positive and measurable on $(0,\infty)$, is said to be a slowly varying function if

$$\lim_{x\to\infty}\frac{l(\lambda x)}{l(x)}=1 \quad \text{for each } \lambda>0. \tag{1.6}$$

Throughout this paper, the symbols C, $C_1, C_2, \ldots$ represent generic positive constants which may be different in various places, and $a_n = O(b_n)$ denotes $a_n \leqslant Cb_n$ for all $n \geqslant 1$. $I(A)$ is the indicator function on the set A. Set $\log x = \ln\max(x, \mathrm{e})$. The symbol $[x]$ stands for the integer part of x and $a \vee b$ stands for $\max(a, b)$.

2. Preliminary lemmas

Lemma 2.1 (Hu [14])**.** *If* $(X_1, X_2, \ldots, X_n)$ *is NSD and* $f_1, f_2, \ldots, f_n$ *are all nondecreasing (or all nonincreasing) functions, then* $(f_1(X_1), f_2(X_2), \ldots, f_n(X_n))$ *is NSD.*

Lemma 2.2 (Hu [14]; Wang et al. [22])**.** *Let* $\{X_n;\ n \geqslant 1\}$ *be a sequence of NSD random variables with* $E|X_n|^M < \infty$ *for some* $M \geqslant 2$ *and each* $n \geqslant 1$*. Then for all* $n \geqslant 1$,

$$E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}X_i\Big|^M\Big)\leqslant 2\Big(\frac{15M}{\ln M}\Big)^M\Big(\sum_{i=1}^{n}E|X_i|^M+\Big(\sum_{i=1}^{n}EX_i^2\Big)^{M/2}\Big). \tag{2.1}$$

Lemma 2.3 (Bai and Su [3])**.** *Let* $l(x) > 0$ *be a slowly varying function as* $x \to \infty$*. Then the following statements hold:*

(1) $\lim_{x\to\infty} l(x+u)/l(x) = 1$ *for each* $u > 0$;
(2) $\lim_{k\to\infty} \sup 2^k \leqslant x < 2^{k+1} l(x)/l(2^k) = 1$;
(3) $\lim_{x\to\infty} x^{\delta} l(x) = \infty$, $\lim_{x\to\infty} x^{-\delta} l(x) = 0$ *for each* $\delta > 0$;
(4) $C_1 2^{kr} l(2^k\varepsilon) \leqslant \sum_{j=1}^{k} 2^{jr} l(2^j\varepsilon) \leqslant C_2 2^{kr} l(2^k\varepsilon)$ *for all* $r > 0$*, all* $\varepsilon > 0$*, any positive integer* k*, and some positive constants* C_1 *and* C_2;
(5) $C_3 2^{kr} l(2^k\varepsilon) \leqslant \sum_{j=k}^{\infty} 2^{jr} l(2^j\varepsilon) \leqslant C_4 2^{kr} l(2^k\varepsilon)$ *for all* $r < 0$*, all* $\varepsilon > 0$*, any positive integer* k*, and some positive constants* C_3 *and* C_4.

Based on Lemma 2.3, Zhou [29] obtained the following result.

Lemma 2.4. *Let* $l(x) > 0$ *be a slowly varying function as* $x \to \infty$*. Then the following statements hold:*

(1) $\sum\limits_{n=1}^{m} n^{t-1}l(n) \leqslant Cm^t l(m)$ *for* $t > 0$ *and a positive integer* m*;*

(2) $\sum\limits_{n=m}^{\infty} n^{t-1}l(n) \leqslant Cm^t l(m)$ *for* $t < 0$ *and a positive integer* m*.*

Lemma 2.5 (Zheng et al. [28])**.** *Let* $\{X_n;\ n \geqslant 1\}$ *be a sequence of NSD random variables. Then there exists a positive constant* C *such that for any* $x \geqslant 0$ *and all* $n \geqslant 1$*,*

$$\Big(1 - P\Big(\max_{1\leqslant i\leqslant n} |X_i| > x\Big)\Big)^2 \sum_{i=1}^{n} P(|X_i| > x) \leqslant CP\Big(\max_{1\leqslant i\leqslant n} |X_i| > x\Big). \tag{2.2}$$

Lemma 2.6 (Sung [21])**.** *Let* $\{X_n;\ n \geqslant 1\}$ *and* $\{Y_n;\ n \geqslant 1\}$ *be sequences of random variables. Then for all* $\varepsilon > 0$*,* $M > 1$ *and* $a > 0$*,*

$$\begin{aligned} &E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j}(X_i + Y_i)\bigg| - \varepsilon a\bigg)^+ \\ &\qquad \leqslant \Big(\frac{1}{\varepsilon^M} + \frac{1}{M-1}\Big)\frac{1}{a^{M-1}} E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j} X_i\bigg|^M\bigg) + E\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j} Y_i\bigg|\bigg). \end{aligned} \tag{2.3}$$

Lemma 2.7 (Wu [25])**.** *Assume that* $\{X_n;\ n \geqslant 1\}$ *is a sequence of random variables which is stochastically dominated by a random variable* X*. Then for all* $\alpha > 0$*,* $x > 0$*, and* $n \geqslant 1$*, the following two inequalities hold:*

$$E|X_n|^\alpha I(|X_n| \leqslant x) \leqslant C_1(E|X|^\alpha I(|X| \leqslant x) + x^\alpha P(|X| > x)); \tag{2.4}$$

$$E|X_n|^\alpha I(|X_n| > x) \leqslant C_2 E|X|^\alpha I(|X| > x), \tag{2.5}$$

where C_1 *and* C_2 *are positive constants. Consequently,* $E|X_n|^\alpha \leqslant CE|X|^\alpha$*.*

3. Main results

In this section, we state and prove the main results of this paper.

Theorem 3.1. *Let* $\alpha > \frac{1}{2}$, $p > 1$, *and* $\alpha p \geqslant 1$. *Suppose that* $l(x) > 0$ *is a slowly varying function, and* $\{X_n;\ n \geqslant 1\}$ *is a sequence of NSD random variables with* $EX_n = 0$, *which is stochastically dominated by a random variable* X. *Let* $\{a_{ni};\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ *be an array of real numbers such that* $\sum\limits_{i=1}^{n} |a_{ni}|^q = O(n)$ *for some*

$$q > \frac{\alpha p - 1}{\alpha - 1/2} \vee 2.$$

If

$$E|X|^p l(|X|^{1/\alpha}) < \infty, \tag{3.1}$$

then for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} < \infty, \tag{3.2}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty. \tag{3.3}$$

P r o o f of Theorem 3.1. First, we prove that (3.1) implies (3.2). Without loss of generality, assume that $a_{ni} \geqslant 0$ for $1 \leqslant i \leqslant n$ and all $n \geqslant 1$. For all $n \geqslant 1$ and $i \geqslant 1$, define

$$X_{ni} = -n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \leqslant n^{\alpha}) + n^{\alpha} I(X_i > n^{\alpha}); \quad Y_{ni} = X_i - X_{ni}.$$

Noting that $EX_n = 0$ and by Lemma 2.6 (for $M \geqslant 2$), it follows that

$$\begin{aligned} J_0 &= \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha M} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} (a_{ni}X_{ni} - Ea_{ni}X_{ni})\Big|^{M}\Big) \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} (a_{ni}Y_{ni} - Ea_{ni}Y_{ni})\Big|\Big). \end{aligned}$$

To prove (3.2), it suffices to show that

$$J_1 = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha M} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(a_{ni}X_{ni} - Ea_{ni}X_{ni})\Big|^M\Big) < \infty,$$

and

$$J_2 = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(a_{ni}Y_{ni} - Ea_{ni}Y_{ni})\Big|\Big) < \infty.$$

Actually, by the Hölder inequality and $\sum_{i=1}^{n} |a_{ni}|^q = O(n)$, it is easy to check that

$$\sum_{i=1}^{n} |a_{ni}|^s = O(n) \quad \forall\, s \leqslant q,\ s > 0. \tag{3.4}$$

For J_2, note that $|Y_{ni}| \leqslant |X_i| I(|X_i| > n^\alpha)$. By (2.5) of Lemma 2.7, (3.4) (for $s = 1$) and Lemma 2.4, we have that

$$\begin{aligned}
J_2 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \sum_{i=1}^{n} E|a_{ni}Y_{ni}| \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > n^\alpha) \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E|X| I(|X| > n^\alpha) \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{j=n}^{\infty} E|X| I(j^\alpha < |X| \leqslant (j+1)^\alpha) \\
&= C\sum_{j=1}^{\infty} E|X| I(j^\alpha < |X| \leqslant (j+1)^\alpha) \sum_{n=1}^{j} n^{\alpha p-1-\alpha} l(n) \\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha p-\alpha} l(j) E|X| I(j^\alpha < |X| \leqslant (j+1)^\alpha) \\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) < \infty.
\end{aligned} \tag{3.5}$$

For J_1, we shall consider the following three cases.

Case 1: $\alpha > \frac{1}{2}$, $\alpha p > 1$ and $p \geqslant 2$.

Taking $M = q$. Note that

$$q > \max\Big(\frac{\alpha p - 1}{\alpha - 1/2}, 2\Big)$$

implies $q > p$ and $\alpha p - 2 - \alpha q + q/2 < -1$. It follows from Lemma 2.2 that

$$(3.6)\qquad J_1 \leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \sum_{i=1}^{n} E|a_{ni}X_{ni}|^q + C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \Bigl(\sum_{i=1}^{n} E|a_{ni}X_{ni}|^2\Bigr)^{q/2} = J_{11} + J_{12}.$$

For J_{11}, by (2.4) of Lemma 2.7 and Lemma 2.4 analogously to the proof of (3.5),

$$\begin{aligned}(3.7)\quad J_{11} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \sum_{i=1}^{n} |a_{ni}|^q (E|X_i|^q I(|X_i| \leqslant n^\alpha) + n^{\alpha q} P(|X_i| > n^\alpha)) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) E|X|^q I(|X| \leqslant n^\alpha) \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E|X| I(|X| > n^\alpha) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} l(n) \sum_{j=1}^{n} E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) \\ &\quad + CE|X|^p l(|X|^{1/\alpha}) \\ &\leqslant C\sum_{j=1}^{\infty} E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) \sum_{n=j}^{\infty} n^{\alpha p-1-\alpha q} l(n) \\ &\quad + CE|X|^p l(|X|^{1/\alpha}) \\ &\leqslant C\sum_{j=1}^{\infty} j^{\alpha p-\alpha q} l(j) E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) + CE|X|^p l(|X|^{1/\alpha}) \\ &\leqslant 2CE|X|^p l(|X|^{1/\alpha}) < \infty.\end{aligned}$$

For J_{12}, note that $E|X|^2 < \infty$ if $E|X|^p I(|X|^{1/\alpha}) < \infty$ for $p > 2$. It follows that

$$\begin{aligned}(3.8)\qquad J_{12} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \\ &\quad\times \Bigl(\sum_{i=1}^{n} |a_{ni}|^2 (E|X_i|^2 I(|X_i| \leqslant n^\alpha) + n^{2\alpha} P(|X_i| > n^\alpha))\Bigr)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2} l(n) (E|X|^2)^{q/2} < \infty.\end{aligned}$$

On the other hand, $E|X|^{2-\delta} < \infty$ with $0 < \delta < (2\alpha q - q - 2\alpha p + 2)/(\alpha q)$ can be proved if (3.1) holds for $p = 2$. It also follows that

(3.9)

$$\begin{aligned} J_{12} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n)\Bigl(\sum_{i=1}^{n}|a_{ni}|^2(E|X_i|^2 I(|X_i|\leqslant n^{\alpha}) + n^{2\alpha}P(|X_i|>n^{\alpha}))\Bigr)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \\ &\qquad\times\Bigl(\sum_{i=1}^{n}|a_{ni}|^2(E|X_i|^{2-\delta}n^{\alpha\delta} I(|X_i|\leqslant n^{\alpha}) + n^{2\alpha}P(|X|>n^{\alpha}))\Bigr)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) n^{q/2}(E|X|^{2-\delta}n^{\alpha\delta} I(|X|\leqslant n^{\alpha}) + 2n^{2\alpha}P(|X|>n^{\alpha}))^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) n^{q/2}\Bigl(n^{\alpha\delta} + 2n^{2\alpha}\frac{E|X|^{2-\delta}}{n^{\alpha(2-\delta)}}\Bigr)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2+\alpha\delta q/2} l(n) < \infty. \end{aligned}$$

Case 2: $\alpha > \frac{1}{2}$, $\alpha p > 1$ and $1 < p < 2$.

Taking $M = 2$, by Lemma 2.2 and (2.4) of Lemma 2.7, analogously to the proofs of (3.7) and (3.5), we have

$$\begin{aligned} (3.10)\quad J_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) E\Bigl(\max_{1\leqslant j\leqslant n}\Bigl|\sum_{i=1}^{j} a_{ni}(X_{ni}-EX_{ni})\Bigr|^2\Bigr) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) E\Bigl(\max_{1\leqslant j\leqslant n}\Bigl|\sum_{i=1}^{j} a_{ni}X_{ni}\Bigr|^2\Bigr) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n)\sum_{i=1}^{n} E|a_{ni}X_{ni}|^2 \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n)\sum_{i=1}^{n}|a_{ni}|^2(E|X_i|^2 I(|X_i|\leqslant n^{\alpha}) + n^{2\alpha}P(|X_i|>n^{\alpha})) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n)(E|X|^2 I(|X|\leqslant n^{\alpha}) + 2n^{2\alpha}P(|X|>n^{\alpha})) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n) E|X|^2 I(|X|\leqslant n^{\alpha}) \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) E|X| I(|X|>n^{\alpha}) \\ &\leqslant CE|X|^p l(|X|^{1/\alpha}) < \infty. \end{aligned}$$

Case 3: $\alpha > \frac{1}{2}$, $\alpha p = 1$ and $p > 1$.

Take $M = 2$. Note that $\frac{1}{2} < \alpha < 1$ for $\alpha p = 1$ and $p > 1$. Similarly,

$$
\begin{aligned}
(3.11)\quad J_1 &\leqslant C\sum_{n=1}^{\infty} n^{-1-2\alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}(X_{ni}-EX_{ni})\Big|^2\Big) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-2\alpha} l(n)\sum_{i=1}^{n} E|a_{ni}X_{ni}|^2 \\
&= C\sum_{n=1}^{\infty} n^{-1-2\alpha} l(n)\sum_{i=1}^{n} |a_{ni}|^2 (E|X_i|^2 I(|X_i|\leqslant n^{\alpha}) + n^{2\alpha}P(|X_i|>n^{\alpha})) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-2\alpha} l(n) E|X|^2 I(|X|\leqslant n^{\alpha}) + C\sum_{n=1}^{\infty} l(n)P(|X|>n^{\alpha}) \\
&\leqslant C\sum_{n=1}^{\infty} n^{-2\alpha} l(n) E|X|^2 I(|X|\leqslant n^{\alpha}) + C\sum_{n=1}^{\infty} n^{-\alpha} l(n)E|X|I(|X|>n^{\alpha}) \\
&\leqslant CE|X|^{1/\alpha} l(|X|^{1/\alpha}) < \infty.
\end{aligned}
$$

Next, we will prove that (3.2) implies (3.3). Actually, it is easy to check that for any $t \geqslant 0$ and all $\varepsilon > 0$,

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_0^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha} > t\Big)\,\mathrm{d}t \\
&\geqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_0^{\varepsilon n^{\alpha}} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha} > t\Big)\,\mathrm{d}t \\
&\geqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_0^{\varepsilon n^{\alpha}} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > 2\varepsilon n^{\alpha}\Big)\,\mathrm{d}t \\
&\geqslant \varepsilon\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > 2\varepsilon n^{\alpha}\Big).
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, the desired result (3.3) follows from the above statement immediately. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Suppose that the conditions of Theorem* 3.1 *are satisfied. Then equation* (3.2) *is equivalent to equation* (3.3).

P r o o f of Theorem 3.2. By Theorem 3.1, it follows that (3.2) implies (3.3). Hence, it needs only to be proved that (3.3) implies (3.2). In fact, for any $t \geqslant 0$, we have that

$$\begin{aligned}
J_0 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha} > t\Big)\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_0^{n^{\alpha}} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha} > t\Big)\,\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha} > t\Big)\,\mathrm{d}t \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > \varepsilon n^{\alpha}\Big) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > t\Big)\,\mathrm{d}t.
\end{aligned}$$

Therefore, it suffices to show that

$$(3.12)\qquad J_3 = \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > t\Big)\,\mathrm{d}t < \infty.$$

For any $t \geqslant 0$ and all $i \geqslant 1$, define

$$X_{ti} = -tI(X_i < -t) + X_i I(|X_i| \leqslant t) + tI(X_i > t); \quad Y_{ti} = X_i - X_{ti}.$$

Noting that $EX_i = 0$, it follows that

$$\begin{aligned}
J_3 &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}(X_{ti}+Y_{ti})\Big| > t\Big)\,\mathrm{d}t \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}(X_{ti}-EX_{ti})\Big| > \frac{t}{2}\Big)\,\mathrm{d}t \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^{\alpha}}^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}(Y_{ti}-EY_{ti})\Big| > \frac{t}{2}\Big)\,\mathrm{d}t \\
&= J_4 + J_5.
\end{aligned}$$

For J_5, note that $|Y_{ti}| \leqslant |X_i| I(|X_i| > t)$. By the Markov inequality and (2.5) of Lemma 2.7, we obtain that

$$
\begin{aligned}
(3.13) \qquad J_5 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} E\Big(\max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{j} a_{ni}(Y_{ti} - EY_{ti})\Big|\Big)\,\mathrm{d}t \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} \Big(\sum_{i=1}^{n} |a_{ni}| E|Y_{ti}|\Big)\,\mathrm{d}t \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} \Big(\sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > t)\Big)\,\mathrm{d}t \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-1} E|X| I(|X| > t)\,\mathrm{d}t \\
&= C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{k=n}^{\infty} \int_{k^\alpha}^{(k+1)^\alpha} t^{-1} E|X| I(|X| > t)\,\mathrm{d}t \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n) \sum_{k=n}^{\infty} k^{-1} E|X| I(|X| > k^\alpha) \\
&\leqslant C \sum_{k=1}^{\infty} k^{\alpha p-\alpha-1} l(k) E|X| I(|X| > k^\alpha) \\
&= C \sum_{k=1}^{\infty} k^{\alpha p-\alpha-1} l(k) \sum_{m=k}^{\infty} E|X| I(m^\alpha < |X| \leqslant (m+1)^\alpha) \\
&\leqslant C \sum_{m=1}^{\infty} E|X| I(m^\alpha < |X| \leqslant (m+1)^\alpha) m^{\alpha p-\alpha} l(m) \\
&\leqslant C E|X|^p l(|X|^{1/\alpha}) < \infty.
\end{aligned}
$$

For J_4, by the Markov inequality (for $M \geqslant 2$) and Lemma 2.2, we have that

$$
\begin{aligned}
(3.14) \quad J_4 &\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-M} E\Big(\max_{1\leqslant j\leqslant n} \Big|\sum_{i=1}^{j} a_{ni}(X_{ti} - EX_{ti})\Big|^M\Big)\,\mathrm{d}t \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-M} \Big(\sum_{i=1}^{n} |a_{ni}|^M E|X_{ti} - EX_{ti}|^M\Big)\,\mathrm{d}t \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-M} \Big(\sum_{i=1}^{n} |a_{ni}|^2 E|X_{ti} - EX_{ti}|^2\Big)^{M/2}\,\mathrm{d}t
\end{aligned}
$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-M}\Big(\sum_{i=1}^{n}|a_{ni}|^M E|X_{ti}|^M\Big)\,\mathrm{d}t$$
$$+C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-M}\Big(\sum_{i=1}^{n}|a_{ni}|^2 E|X_{ti}|^2\Big)^{M/2}\,\mathrm{d}t$$
$$= CJ_{41}+CJ_{42}.$$

Analogously to the proof of $J_1<\infty$, we will proceed with the following three cases.

Case 1: $\alpha>\frac{1}{2}$, $\alpha p>1$ and $p\geqslant 2$.

Take $M=q$. For J_{41}, note that

$$q>\max\Big(\frac{\alpha p-1}{\alpha-1/2},2\Big)$$

implies $q>p$ and $\alpha p-2-\alpha q+q/2<-1$. By (2.4) of Lemma 2.7 and Lemma 2.4, analogously to the proof of (3.13), we have

$$
\begin{aligned}
(3.15)\quad J_{41} &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-q}\Big(\sum_{i=1}^{n}|a_{ni}|^q E|X_{ti}|^q\Big)\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\\
&\qquad\times\int_{n^\alpha}^{\infty} t^{-q}\sum_{i=1}^{n}(|a_{ni}|^q(E|X_i|^q I(|X_i|\leqslant t)+t^q P(|X_i|>t)))\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-q}E|X|^q I(|X|\leqslant t)\,\mathrm{d}t\\
&\quad+C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\int_{n^\alpha}^{\infty} P(|X|>t)\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\sum_{k=n}^{\infty}\int_{k^\alpha}^{(k+1)^\alpha} t^{-q}E|X|^q I(|X|\leqslant t)\,\mathrm{d}t+CE|X|^p l(|X|^{1/\alpha})\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\sum_{k=n}^{\infty} k^{\alpha-1-\alpha q}E|X|^q I(|X|\leqslant (k+1)^\alpha)\\
&\leqslant C\sum_{k=1}^{\infty} k^{\alpha-1-\alpha q}E|X|^q I(|X|\leqslant (k+1)^\alpha)\sum_{n=1}^{k} n^{\alpha p-1-\alpha} l(n)\\
&\leqslant C\sum_{k=1}^{\infty} k^{\alpha p-\alpha q-1} l(k)E|X|^q I(|X|\leqslant (k+1)^\alpha)
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{k=1}^{\infty} k^{\alpha p-\alpha q-1} l(k) E|X|^q I(k^\alpha < |X| \leqslant (k+1)^\alpha) \\
&\quad + C\sum_{k=1}^{\infty} k^{\alpha p-\alpha q-1} l(k) E|X|^q I(|X| \leqslant k^\alpha) \\
&\leqslant C\sum_{k=1}^{\infty} k^{-1} l(k) E|X|^p I(k^\alpha < |X| \leqslant (k+1)^\alpha) \\
&\quad + C\sum_{k=1}^{\infty} k^{\alpha p-\alpha q-1} l(k) \sum_{j=1}^{k} E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) \\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) + C\sum_{j=1}^{\infty} E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) \sum_{k=j}^{\infty} k^{\alpha p-\alpha q-1} l(k) \\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) + C\sum_{j=1}^{\infty} j^{\alpha p-\alpha q} l(j) E|X|^q I((j-1)^\alpha < |X| \leqslant j^\alpha) \\
&\leqslant 2CE|X|^p l(|X|^{1/\alpha}) < \infty.
\end{aligned}
$$

For J_{42}, note that $E|X|^2 < \infty$ if $E|X|^p I(|X|^{1/\alpha}) < \infty$ for $p > 2$. It follows that

$$
\begin{aligned}
(3.16)\quad J_{42} &= \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \int_{n^\alpha}^{\infty} t^{-q} \Bigl(\sum_{i=1}^{n} |a_{ni}|^2 E|X_{ti}|^2\Bigr)^{q/2} \,\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) \\
&\qquad \times \int_{n^\alpha}^{\infty} t^{-q} \Bigl(\sum_{i=1}^{n} |a_{ni}|^2 (E|X_i|^2 I(|X_i| \leqslant t) + t^2 P(|X_i| > t))\Bigr)^{q/2} \,\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} l(n) \int_{n^\alpha}^{\infty} t^{-q} (E|X|^2)^{q/2} \,\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha+q/2} l(n) \int_{n^\alpha}^{\infty} t^{-q} \,\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2} l(n) < \infty.
\end{aligned}
$$

Note that $E|X|^{2-\delta} < \infty$ if (3.1) holds for $p = 2$, where

$$
0 < \delta < \frac{2\alpha q - q - 2\alpha p + 2}{\alpha q}.
$$

By an argument similar to that in the proof of (3.9), $J_{42} < \infty$ also follows.

Case 2: $\alpha > \frac{1}{2}$, $\alpha p > 1$ and $1 < p < 2$.

Taking $M = 2$, analogously to the proof of (3.10), gives

$$
\begin{aligned}
(3.17)\qquad J_4 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-2} E\biggl(\max_{1\leqslant j\leqslant n}\biggl|\sum_{i=1}^{j} a_{ni}(X_{ti}-EX_{ti})\biggr|^2\biggr)\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-2}\sum_{i=1}^{n}|a_{ni}|^2 E|X_{ti}|^2\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\sum_{k=n}^{\infty} k^{-\alpha-1}E|X|^2 I(|X|\leqslant (k+1)^\alpha)\\
&\leqslant C\sum_{k=1}^{\infty} k^{\alpha p-2\alpha-1} l(k)E|X|^2 I(|X|\leqslant (k+1)^\alpha)\\
&\leqslant C\sum_{k=1}^{\infty} k^{-1} l(k)E|X|^p I(k^\alpha<|X|\leqslant (k+1)^\alpha)\\
&\quad + C\sum_{k=1}^{\infty} k^{\alpha p-2\alpha-1} l(k)\sum_{j=1}^{k} E|X|^2 I((j-1)^\alpha<|X|\leqslant j^\alpha)\\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) + C\sum_{j=1}^{\infty} E|X|^2 I((j-1)^\alpha<|X|\leqslant j^\alpha)\sum_{k=j}^{\infty} k^{\alpha p-2\alpha-1} l(k)\\
&\leqslant C\sum_{j=1}^{\infty} j^{\alpha p-2\alpha} l(j)E|X|^2 I((j-1)^\alpha<|X|\leqslant j^\alpha)\\
&\leqslant CE|X|^p l(|X|^{1/\alpha}) < \infty.
\end{aligned}
$$

Case 3: $\alpha > \frac{1}{2}$, $\alpha p = 1$ and $p > 1$.

Take $M = 2$. Note that $\frac{1}{2} < \alpha < 1$ for $\alpha p = 1$ and $p > 1$. Similarly to the proof of (3.11), we also prove

$$
\begin{aligned}
(3.18)\quad J_4 &\leqslant C\sum_{n=1}^{\infty} n^{-1-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-2} E\biggl(\max_{1\leqslant j\leqslant n}\biggl|\sum_{i=1}^{j} a_{ni}(X_{ti}-EX_{ti})\biggr|^2\biggr)\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-2}\sum_{i=1}^{n} E|a_{ni}X_{ti}|^2\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-2}\sum_{i=1}^{n}|a_{ni}|^2(E|X_i|^2 I(|X_i|\leqslant t)+t^2P(|X_i|>t))\,\mathrm{d}t\\
&\leqslant C\sum_{n=1}^{\infty} n^{-\alpha} l(n)\int_{n^\alpha}^{\infty} t^{-2}(E|X|^2 I(|X|\leqslant t)+2t^2P(|X|>t))\,\mathrm{d}t
\end{aligned}
$$

$$\leqslant C\sum_{n=1}^{\infty} n^{-\alpha}l(n)\int_{n^\alpha}^{\infty} t^{-2}E|X|^2 I(|X|\leqslant t)\,\mathrm{d}t$$

$$+\,C\sum_{n=1}^{\infty} n^{-\alpha}l(n)\int_{n^\alpha}^{\infty} t^{-1}E|X|I(|X|>t)\,\mathrm{d}t\leqslant CE|X|^{1/\alpha}l(|X|^{1/\alpha})<\infty.$$

The proof of Theorem 3.2 is completed. □

Theorem 3.3. *Let* $\alpha>\frac{1}{2}$ *and* $\alpha p\geqslant 1$. *Suppose that* $l(x)>0$ *is a slowly varying function, and* $\{X_n;\ n\geqslant 1\}$ *is a sequence of NSD random variables. Suppose that there exist a random variable* X *and some positive constants* C_1 *and* C_2 *such that* $C_1P(|X|>x)\leqslant \inf_{n\geqslant 1}P(|X_n|>x)\leqslant \sup_{n\geqslant 1}P(|X_n|>x)\leqslant C_2P(|X|>x)$ *for all* $x\geqslant 0$. *Assume further that* $l(x)\geqslant C$ *for some positive constant* C *if* $\alpha p=1$. *If*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}X_i\Big|>\varepsilon n^\alpha\Big)<\infty\quad\forall\,\varepsilon>0, \tag{3.19}$$

then expression (3.1) *holds.*

Proof of Theorem 3.3. For $1\leqslant i\leqslant n$ and all $n\geqslant 1$, it is easy to check that

$$\max_{1\leqslant j\leqslant n}|X_j|\leqslant \max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}X_i\Big|+\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j-1}X_i\Big|.$$

By (3.19), it follows that

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)P\Big(\max_{1\leqslant j\leqslant n}|X_j|>\varepsilon n^\alpha\Big)<\infty. \tag{3.20}$$

For $\alpha p>1$ and $l(x)\geqslant C>0$ if $\alpha p=1$, we have that

$$l(n)P\Big(\max_{1\leqslant j\leqslant n}|X_j|>\varepsilon n^\alpha\Big)\leqslant Cl(n)n^{\alpha p-1}P\Big(\max_{1\leqslant j\leqslant n}|X_j|>\varepsilon n^\alpha\Big)$$
$$\leqslant C\sum_{i=n}^{2n} i^{\alpha p-2}l(i)P\Big(\max_{1\leqslant j\leqslant i}|X_j|>\frac{\varepsilon}{2^\alpha}i^\alpha\Big),$$

which together with (3.20) implies that

$$P\Big(\max_{1\leqslant j\leqslant n}|X_j|>\varepsilon n^\alpha\Big)\to 0\quad\text{as } n\to\infty. \tag{3.21}$$

Hence, for n large enough,

$$P\Big(\max_{1\leqslant j\leqslant n}|X_j|>\varepsilon n^{\alpha}\Big)<\frac{1}{2}. \tag{3.22}$$

By Lemma 2.5, (3.22) and $C_1P(|X|>x)\leqslant \inf_{n\geqslant 1}P(|X_n|>x)$ for all $x\geqslant 0$, we can get that

$$nP(|X|>\varepsilon n^{\alpha})\leqslant C\sum_{j=1}^{n}P(|X_j|>\varepsilon n^{\alpha})\leqslant CP\Big(\max_{1\leqslant j\leqslant n}|X_j|>\varepsilon n^{\alpha}\Big). \tag{3.23}$$

Taking $\varepsilon=1$, by (3.20), (3.23), Lemma 2.3 and some standard computation, we have that

$$\begin{aligned}\infty &> \sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P\Big(\max_{1\leqslant j\leqslant n}|X_j|>n^{\alpha}\Big)\\ &\geqslant C\sum_{n=1}^{\infty}n^{\alpha p-1}l(n)P(|X|>n^{\alpha})\\ &= C\sum_{n=1}^{\infty}n^{\alpha p-1}l(n)\sum_{j=n}^{\infty}P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha})\\ &= C\sum_{j=1}^{\infty}P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha})\sum_{n=1}^{j}n^{\alpha p-1}l(n)\\ &\geqslant C\sum_{j=1}^{\infty}P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha})\sum_{i=1}^{[\log_2 j]}\sum_{n=2^{i-1}}^{2^i-1}n^{\alpha p-1}l(n)\\ &\geqslant C\sum_{j=1}^{\infty}P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha})2^{[\log_2 j]\alpha p}l(2^{[\log_2 j]})\\ &\geqslant C\sum_{j=1}^{\infty}P(j^{\alpha}<|X|\leqslant (j+1)^{\alpha})j^{\alpha p}l(j)\\ &\geqslant CE|X|^{p}l(|X|^{1/\alpha}).\end{aligned} \tag{3.24}$$

The proof of Theorem 3.3 is completed. □

Corollary 3.1. *Under the conditions of Theorem* 3.1,

$$\lim_{n\to\infty}\frac{1}{n^{\alpha}}\sum_{i=1}^{n}a_{ni}X_i=0\quad \text{a.s.}$$

Corollary 3.2. *For the case* $a_{ni}=1$, *under the conditions of Theorem* 3.1, *for all* $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\sup_{j\geqslant n}\Big|j^{-\alpha}\sum_{i=1}^{j} X_i\Big|>\varepsilon\Big)<\infty. \tag{3.25}$$

P r o o f of Corollary 3.2. For writing convenience, denoted by I the expression (3.25). Inspired by the proof of Theorem 12.1 of Gut [12], we get

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\sup_{j\geqslant n}\Big|j^{-\alpha}\sum_{i=1}^{j} X_i\Big|>\varepsilon\Big) \\ &= \sum_{m=1}^{\infty}\sum_{n=2^{m-1}}^{2^m-1} n^{\alpha p-2} l(n) P\Big(\sup_{j\geqslant n}\Big|j^{-\alpha}\sum_{i=1}^{j} X_i\Big|>\varepsilon\Big) \\ &\leqslant C\sum_{m=1}^{\infty} P\Big(\sup_{j\geqslant 2^{m-1}}\Big|j^{-\alpha}\sum_{i=1}^{j} X_i\Big|>\varepsilon\Big)\sum_{n=2^{m-1}}^{2^m-1} 2^{m(\alpha p-2)} l(2^m) \\ &\leqslant C\sum_{m=1}^{\infty} 2^{m(\alpha p-1)} l(2^m) P\Big(\sup_{j\geqslant 2^{m-1}}\Big|j^{-\alpha}\sum_{i=1}^{j} X_i\Big|>\varepsilon\Big) \\ &= C\sum_{m=1}^{\infty} 2^{m(\alpha p-1)} l(2^m) P\Big(\sup_{k\geqslant m}\max_{2^{k-1}\leqslant j<2^k}\Big|j^{-\alpha}\sum_{i=1}^{j} X_i\Big|>\varepsilon\Big) \\ &\leqslant C\sum_{m=1}^{\infty} 2^{m(\alpha p-1)} l(2^m)\sum_{k=m}^{\infty} P\Big(\max_{1\leqslant j<2^k}\Big|\sum_{i=1}^{j} X_i\Big|>\varepsilon 2^{\alpha(k-1)}\Big) \\ &\leqslant C\sum_{k=1}^{\infty} 2^{k(\alpha p-1)} l(2^k) P\Big(\max_{1\leqslant j<2^k}\Big|\sum_{i=1}^{j} X_i\Big|>\varepsilon 2^{\alpha(k-1)}\Big) \\ &\leqslant C\sum_{k=1}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} n^{\alpha p-2} l(n) P\Big(\max_{1\leqslant j<n}\Big|\sum_{i=1}^{j} X_i\Big|>\frac{\varepsilon}{4^{\alpha}} n^{\alpha}\Big) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\Big(\max_{1\leqslant j<n}\Big|\sum_{i=1}^{j} X_i\Big|>\frac{\varepsilon}{4^{\alpha}} n^{\alpha}\Big). \end{aligned} \tag{3.26}$$

Since $\varepsilon>0$ is arbitrary, the desired result (3.25) follows from the above statement and (3.3) with $a_{ni}=1$ immediately. The proof of Corollary 3.2 is completed. □

For $p=1$ and $l(x)=1$, the following specific result can be established.

Theorem 3.4. *Let $\alpha > 0$. Suppose that $\{X_n;\ n \geqslant 1\}$ is a sequence of NSD random variables with $EX_n = 0$, which is stochastically dominated by a random variable X. Let $\{a_{ni};\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of real numbers such that $\sum\limits_{i=1}^{n} |a_{ni}|^2 = O(n)$. If $E|X|(1+\log|X|) < \infty$, then for all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} < \infty, \tag{3.27}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha-2} P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty. \tag{3.28}$$

Proof of Theorem 3.4. Similarly to Theorem 3.1, it needs only to be proved that (3.27) holds. By using the same notation as in the proof of Theorem 3.1, we easily get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} \\
&\qquad \leqslant C\sum_{n=1}^{\infty} n^{-\alpha-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} (a_{ni}X_{ni} - Ea_{ni}X_{ni})\Big|^2\Big) \\
&\qquad\quad + C\sum_{n=1}^{\infty} n^{-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(a_{ni}Y_{ni} - Ea_{ni}Y_{ni})\Big|\Big).
\end{aligned}$$

In the following, we need only to show that

$$J_6 = \sum_{n=1}^{\infty} n^{-\alpha-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} (a_{ni}X_{ni} - Ea_{ni}X_{ni})\Big|^2\Big) < \infty,$$

and

$$J_7 = \sum_{n=1}^{\infty} n^{-2} E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} (a_{ni}Y_{ni} - Ea_{ni}Y_{ni})\Big|\Big) < \infty.$$

For J_7, note that $|Y_{ni}| \leqslant |X_i| I(|X_i| > n^\alpha)$. Thus,

$$
\begin{aligned}
J_7 &\leqslant C \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} E|a_{ni} Y_{ni}| \\
&\leqslant C \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}| E|X_i| I(|X_i| > n^\alpha) \\
&\leqslant C \sum_{n=1}^{\infty} n^{-1} E|X| I(|X| > n^\alpha) \\
&= C \sum_{n=1}^{\infty} n^{-1} \sum_{j=n}^{\infty} E|X| I(j^\alpha < |X| \leqslant (j+1)^\alpha) \\
&\leqslant C \sum_{j=1}^{\infty} \log j E|X| I(j^\alpha < |X| \leqslant (j+1)^\alpha) \\
&\leqslant C E|X|(1 + \log |X|) < \infty.
\end{aligned} \tag{3.29}
$$

For J_6, analogously to the proofs of (3.10) and (3.29), it follows from Lemma 2.2 that

$$
\begin{aligned}
J_6 &\leqslant C \sum_{n=1}^{\infty} n^{-\alpha-2} E\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_{ni} \right|^2 \right) \\
&\leqslant C \sum_{n=1}^{\infty} n^{-\alpha-2} \sum_{i=1}^{n} E|a_{ni} X_{ni}|^2 \\
&= C \sum_{n=1}^{\infty} n^{-\alpha-2} \sum_{i=1}^{n} |a_{ni}|^2 (E|X_i|^2 I(|X_i| \leqslant n^\alpha) + n^{2\alpha} P(|X_i| > n^\alpha)) \\
&\leqslant C \sum_{n=1}^{\infty} n^{-\alpha-1} E|X|^2 I(|X| \leqslant n^\alpha) + C \sum_{n=1}^{\infty} n^{\alpha-1} P(|X| > n^\alpha) \\
&\leqslant C \sum_{n=1}^{\infty} n^{-\alpha-1} E|X|^2 I(|X| \leqslant n^\alpha) + C \sum_{n=1}^{\infty} n^{-1} E|X| I(|X| > n^\alpha) \\
&\leqslant C E|X|(1 + \log |X|) < \infty.
\end{aligned} \tag{3.30}
$$

The proof of Theorem 3.4 is completed. □

Remark 3.1. Compared to the result of Baum and Katz [4], the slowly varying function $l(x)$ is added in Theorems 3.1–3.3 and the assumption of identical distribution of the random variables is not needed in the main results; while taking $l(x) = 1$, Theorems 3.1–3.3 yield the corresponding results of Baum and Katz [4]. In addition, complete moment convergence for weighted sums of NSD random variables is considered for the case $\alpha > \frac{1}{2}$ and $p > 1$ with $\alpha p \geqslant 1$.

R e m a r k 3.2. Compared to the result of Chow [6], the generalizations and improvements are as follows: (1) the slowly varying function $l(x)$ is added and the assumption of identical distribution of the random variables is not needed; (2) the sufficient and necessary conditions of complete moment convergence for weighted sums of NSD random variables are established; (3) the case $\alpha > \frac{1}{2}$, $p \geqslant 1$ with $\alpha p > 1$ is extended to the case $\alpha > \frac{1}{2}$, $p > 1$ with $\alpha p \geqslant 1$ and the case $\alpha > 0$, $p = 1$, respectively; (4) the moment condition of the main results in this work is much weaker than that of Chow [6].

A c k n o w l e d g e m e n t s. The authors are most grateful to the editor and the anonymous referees for carefully reading this manuscript and for offering valuable suggestions that greatly enabled them to improve this paper.

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