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INCOMPRESSIBLE INVISCID LIMIT FOR THE FULL MAGNETOHYDRODYNAMIC FLOWS ON EXPANDING DOMAINS

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Abstract. In this paper we study the incompressible inviscid limit of the full magnetohydrodynamic flows on expanding domains with general initial data. By applying the relative energy method and carrying out detailed analysis on the oscillation part of the velocity, we prove rigorously that the gradient part of the weak solutions of the full magnetohydrodynamic flows converges to the strong solution of the incompressible Euler system in the whole space, as the Mach number, viscosity as well as the heat conductivity go to zero and the domains expand to the whole space. Furthermore, we obtain the convergence rate.

Keywords: full magnetohydrodynamic flows; inviscid limit; expanding domain; incompressible limit

MSC 2020: 35Q30, 35E15

1. Introduction

Magnetohydrodynamic flows arise in science and engineering in a variety of practical applications such as in plasma confinement, liquid-metal cooling of nuclear reactors, and electromagnetic casting. The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. The set of equations that describe MHD are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. These differential equations must be solved simultaneously, either analytically or numerically. The compressible magnetohydrodynamic flows appears in a variety of engineering and physical problems. We consider the motion of viscous compressible fluids in a family of domains $\Omega_M \subset \mathbb{R}^3$, $M \to \infty$, verifying the following properties:

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▷ Ω_M are simply connected, bounded and $\mathcal{C}^2$ domains, uniformly for $M \to \infty$.
▷ there exists $\omega > 0$ such that

$$\{x \in \mathbb{R}^3;\ |x| < \omega M\} \subset \Omega_M, \tag{1.1}$$

▷ there exists $\beta > 0$ such that

$$|\partial\Omega_M|_2 \leqslant \beta M^2, \tag{1.2}$$

where $|\cdot|$ is the surface measure.

A typical family of Ω_M is $M\Omega$ with Ω a fixed simply connected C^2 bounded domain including B_1, the unit ball in $\mathbb{R}^3$. The evolution of the fluids in Ω_M is governed by the scaled complete compressible magnetohydrodynamic flows:

$$\partial_t \varrho + \operatorname{div}(\varrho\mathbf{u}) = 0, \tag{1.3}$$

$$\partial_t(\varrho\mathbf{u} + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u})) + \frac{1}{\varepsilon^2}\nabla P(\varrho,\vartheta) = \varepsilon^a \operatorname{div} \mathbb{S}(\vartheta,\nabla\mathbf{u}) + (\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2, \tag{1.4}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \operatorname{div}(\varrho s(\varrho,\vartheta)\mathbf{u}) + \varepsilon^b \operatorname{div}\Big(\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\Big) = \frac{1}{\vartheta}\Big(\varepsilon^{2+a}\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \varepsilon^b \frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\Big), \tag{1.5}$$

$$\partial_t\mathbf{H} + (\operatorname{div}\mathbf{u})\mathbf{H} + (\mathbf{u}\cdot\nabla)\mathbf{H} - (\mathbf{H}\cdot\nabla)\mathbf{u} = \varepsilon^c\nu\Delta\mathbf{H}. \tag{1.6}$$

Here the density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$, and the absolute temperature $\vartheta = \vartheta(t,x)$ are three typical macroscopic quantities describing the motion of the fluids, while $p = p(\varrho,\vartheta)$ is the pressure, $s = s(\varrho,\vartheta)$ the specific entropy and $\mathbb{S}(\vartheta,\nabla\mathbf{u})$ denotes the viscous stress tensor satisfying *Newton's rheological law*,

$$\mathbb{S}(\vartheta,\nabla\mathbf{u}) = \mu(\vartheta,\mathbf{H})\Big(\nabla\mathbf{u} + \nabla^t\mathbf{u} - \frac{2}{3}\operatorname{div}\mathbf{u}\mathbb{I}\Big). \tag{1.7}$$

According to *Fourier's law*, the heat flux $\mathbf{q} = \mathbf{q}(\vartheta,\nabla\vartheta)$ is of the form

$$\mathbf{q}(\vartheta,\nabla\vartheta) = -\kappa(\vartheta,\mathbf{H})\nabla\vartheta. \tag{1.8}$$

Here the quantities $\mu > 0$, $\eta \geqslant 0$, $\kappa > 0$ are temperature dependent dissipative coefficients and in the following we assume $\eta \equiv 0$ for simplicity. Note that the system contains a small parameter ε related to different *nondimensional numbers* following from the scale analysis: *Mach number* ε, *Reynolds number* ε^{-a} and *Péclet number* ε^{-b} with suitable $a, b > 0$ to be chosen later. We supplement the system with boundary conditions

$$\mathbf{u}|_{\partial\Omega_M} = 0, \quad \mathbf{H}|_{\partial\Omega_M} = 0, \quad \nabla\vartheta\cdot\mathbf{n}|_{\partial\Omega_M} = 0, \tag{1.9}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega_M$, together with the initial data

$$(1.10)\qquad \varrho(0,\cdot)=\varrho_{0,\varepsilon}(x)=\overline{\varrho}+\varepsilon\varrho_{0,\varepsilon}^{(1)}(x),\quad \vartheta(0,x)=\vartheta_{0,\varepsilon}(x)=\overline{\vartheta}+\varepsilon\vartheta_{0,\varepsilon}^{(1)}(x),$$
$$\mathbf{u}(0,x)=\mathbf{u}_{0,\varepsilon}(x),\quad \mathbf{H}(0,x)=\mathbf{H}_{0,\varepsilon}(x),$$

where $\overline{\varrho}>0$, $\overline{\vartheta}>0$ is the anticipated constant density and temperature enforced by the incompressible limit.

The goal of this paper is to rigorously investigate the limit

$$\varrho_\varepsilon\to\overline{\varrho},\qquad \frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\to\mathcal{T},\quad \mathbf{u}_\varepsilon\to\mathbf{u},\quad \mathbf{H}_\varepsilon\to\mathbf{H}$$

in some suitable sense as ε tends to 0 and M goes to infinity such that the given limit $(\mathbf{u},\mathbf{H},\mathcal{T})$ represents the unique strong solution of the incompressible ideal MHD-Boussinesq system in the whole space $\mathbb{R}^3$:

$$(1.11)\qquad \partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\Pi=(\mathbf{H}\cdot\nabla)\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2,\quad \operatorname{div}\mathbf{u}=0,$$

$$(1.12)\qquad \partial_t\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}=0,\quad \operatorname{div}\mathbf{H}=0,$$

$$(1.13)\qquad \partial_t\mathcal{T}+\mathbf{u}\cdot\nabla\mathcal{T}=0,$$

with suitable initial data. Note that the transport equation of $\mathcal{T}$ is decoupled from the Euler equations (1.11).

We first notice that the global-in-time existence solutions for system (1.3)–(1.6), supplemented with physically relevant constitutive relations, has been studied by Hu, Wang [6]. For the incompressible limit problems, there are many recent works by Lions, Masmoudi [10] for isentropic Navier-Stokes equations with constant viscosity and by Feireisl, Novotný [3], [5] for the Navier-Stokes-Fourier systems. There are also works on the inviscid incompressible limit problems by Masmoudi [11] for isentropic compressible Navier-Stokes equations and by Feireisl, Novotný [5] for Navier-Stokes Fourier systems. The models of compressible magnetohydrodynamic flows have been also studied by Jiang, Ju, and Li [8], [9].

In a recent paper [2], Feireisl, Nečasová, and Sun have studied the incompressible inviscid limit problem on expanding domains for isentropic Navier-Stokes equations (1.3)–(1.6). In this paper, we study the inviscid incompressible limit of the compressible magnetohydrodynamic flows (1.3)–(1.6) on the expanding domain when the Mach number is very small and we use the ill-prepared initial data. Our contribution of this paper is physically to derive a rigorous equations (1.11)–(1.13) from compressible magnetohydrodynamic flows based on the relative entropy method and we obtain the convergence rate. This paper is more a developed result than the result mentioned in [2] and the difficulty is to investigate the convergence of magnetic field with the weight even though there is no oscillation of magnetic field.

The rest of this paper is organized as follows. Section 2 is devoted to introducing the preliminaries and the notion of weak solutions of (1.3)–(1.6). In Section 3 we present the main result. In Section 3 we give the rigorous proof of the inviscid incompressible limit of weak solutions of compressible magnetohydrodynamic flows (1.3)–(1.6).

2. Preliminaries

In this section we introduce some preliminary results necessary to perform the singular limit.

2.1. Structural restrictions imposed on constitutive relations. We study the singular limit problem under certain physically motivated restrictions imposed on constitutive equations. Although they might be slightly relaxed if only the convergence towards the target system is studied, we list them in the form presented in [4], Chapter 3, where the interested reader may find more information concerning the physical background.

The pressure $p = p(\varrho, \vartheta)$ is given by the formula

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\Big(\frac{\varrho}{\vartheta^{3/2}}\Big) + \frac{a}{3}\vartheta^4, \quad a > 0; \tag{2.1}$$

the specific internal energy $e = e(\varrho, \vartheta)$ and the specific entropy $s = s(\varrho, \vartheta)$ read

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{3/2}}\Big) + a\vartheta^4, \tag{2.2}$$

$$s(\varrho, \vartheta) = S\Big(\frac{\varrho}{\vartheta^{3/2}}\Big) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.3}$$

where

$$P \in C^1[0,\infty) \cap C^3(0,\infty), \quad P(0) = 0, \quad P'(Z) > 0 \quad \forall\, Z \geqslant 0, \tag{2.4}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0, \tag{2.5}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \quad \forall\, Z > 0, \tag{2.6}$$

and

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2}, \quad \lim_{Z\to\infty} S(Z) = 0. \tag{2.7}$$

Relation (2.6) expresses positivity and uniform boundedness of the specific heat at constant volume.

Integrating (2.7) and employing bound (2.6) we verify that

$$0 < S(Z) \leqslant c(1 + |\log Z|).$$

Consequently, by virtue of (2.3),

$$\varrho|s(\varrho,\vartheta)| \leqslant c(\varrho + \vartheta^3 + \varrho|\log\varrho| + \varrho[\log\vartheta]^+). \tag{2.8}$$

This estimate will be needed later.

The transport coefficients μ and κ are effective functions of the temperature, $\mu,\ \eta \in C^1[0,\infty)$ are globally Lipschitz in $[0,\infty)$, verifying $|\mu'(\vartheta,\mathbf{H})| \leqslant M$,

$$0 < \underline{\mu}(1+\vartheta^\alpha) \leqslant \mu(\vartheta,\mathbf{H}) \leqslant \overline{\mu}(1+\vartheta^\alpha) \quad \forall\,\vartheta \geqslant 0, \tag{2.9}$$

$$\kappa \in C^1[0,\infty), \quad 0 < \underline{\kappa}(1+\vartheta^3) \leqslant \kappa(\varrho,\vartheta,\mathbf{H}) \leqslant \overline{\kappa}(1+\vartheta^3) \quad \forall\,\vartheta \geqslant 0. \tag{2.10}$$

$$0 < \underline{\nu}(1+\vartheta) \leqslant \nu(\varrho,\vartheta,\mathbf{H}) \leqslant \overline{\nu}(1+\vartheta^3) \quad \forall\,\vartheta \geqslant 0, \tag{2.11}$$

where $1 \leqslant \alpha < 3$.

2.2. Energy and dispersive estimates for the acoustic system. To continue we now need to introduce some notations. Let

$$\alpha = \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}, \quad \beta = \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}, \quad \delta = \overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}, \quad \omega = \overline{\varrho}\Big(\alpha + \frac{\beta^2}{\delta}\Big). \tag{2.12}$$

Let $(R_\varepsilon, T_\varepsilon, \Phi_\varepsilon)$ solve the acoustic equation

$$\varepsilon\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon) + \omega\Delta\Phi_\varepsilon = 0, \tag{2.13}$$

$$\varepsilon\partial_t\nabla\Phi_\varepsilon + \nabla(\alpha R_\varepsilon + \beta T_\varepsilon) = 0 \tag{2.14}$$

with the initial data

$$R_\varepsilon(0,\cdot) = \chi_\delta * [\theta_\delta\varrho_0^{(1)}], \quad T_\varepsilon(0,\cdot) = \chi_\delta * [\theta_\delta\vartheta_0^{(1)}], \quad \Phi_{0,\varepsilon} = \chi_\delta * [\theta_\delta\Delta^{-1}\operatorname{div}[\mathbf{u}_0]], \tag{2.15}$$

and

$$\alpha = \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}, \quad \beta = \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}, \quad \omega = \overline{\varrho}\Big(\alpha + \frac{\beta^2}{\delta}\Big),$$

where $\{\chi_\delta\}_{\delta>0}$ is a family of regularizing kernels, and $\theta_\delta \in C_c^\infty(\mathbb{R}^3)$ are the standard cut-off functions $\theta_\delta \nearrow 1$. From now on we assume $\omega = 1$. Indeed, since $\alpha R_{0,\varepsilon} + \beta T_{0,\varepsilon}$

and $\Delta\Phi_{0,\varepsilon}$ have the compact support together with the property of wave equation, we have

$$(2.16)\qquad (\alpha R_\varepsilon + \beta T_\varepsilon)(t,x) = 0, \quad \Delta\Phi_\varepsilon(t,x) = 0 \quad \text{for } |x| \geqslant C + \sqrt{\omega}\frac{t}{\varepsilon}.$$

From now on we remove δ in order to proceed in the most convenient way. Then we have the Strichartz's estimates:

Theorem 2.1 ([1]). *Let $(s_\varepsilon, q_\varepsilon)$ be the solution of system (2.13)–(2.14) with initial data $(s_{\varepsilon,0}, q_{\varepsilon,0})$ given in (2.15). Then, one has*

$$(2.17)\qquad \begin{aligned} &\|\nabla\Phi_\varepsilon(t,\cdot)\|^2_{W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)} + \|(\alpha R_\varepsilon + \beta T_\varepsilon)(t,\cdot)\|^2_{W^{k,2}(\mathbb{R}^3)} \\ &\quad \leqslant \|\nabla\Phi_{0,\varepsilon}(t,\cdot)\|^2_{W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)} + \|(\alpha R_{0,\varepsilon} + \beta T_{0,\varepsilon})(t,\cdot)\|^2_{W^{k,2}(\mathbb{R}^3)} \end{aligned}$$

$$(2.18)\qquad \begin{aligned} &\|\nabla\Phi_\varepsilon\|_{L^l(\mathbb{R};W^{k,p}(\mathbb{R}^3;\mathbb{R}^3))} + \|(\alpha R_\varepsilon + \beta T_\varepsilon)(t,\cdot)\|_{L^l(\mathbb{R};W^{k,p}(\mathbb{R}^3;\mathbb{R}^3)} \\ &\quad \leqslant C\varepsilon^{1/l}(\|\nabla\Phi_{0,\varepsilon}\|_{H^{k+2}(\mathbb{R}^3;\mathbb{R}^3)} + \|(\alpha R_{0,\varepsilon} + \beta T_{0,\varepsilon})(t,\cdot)\|_{H^{k+2}(\mathbb{R}^3;\mathbb{R}^3)}) \end{aligned}$$

with

$$2 < p,\ l \leqslant \infty,\ \frac{1}{p} + \frac{1}{l} = \frac{1}{2},\ k = 0,1,2,\dots$$

Furthermore, we have

$$(2.19)\qquad \begin{aligned} &\|\nabla\Phi_\varepsilon(t,\cdot)\|^2_{W^{k,2}(\Omega_M;\mathbb{R}^3)} + \|(\alpha R_\varepsilon + \beta T_\varepsilon)(t,\cdot)\|^2_{W^{k,2}(\Omega_M)} \\ &\quad \leqslant \|\nabla\Phi_{0,\varepsilon}(t,\cdot)\|^2_{W^{k,2}(\Omega_M;\mathbb{R}^3)} + \|(\alpha R_{0,\varepsilon} + \beta T_{0,\varepsilon})(t,\cdot)\|^2_{W^{k,2}(\Omega_M)}. \end{aligned}$$

Let us consider the transport equation. It is a linearized version of the entropy equation (1.5),

$$(2.20)\qquad \partial_t(\delta T_\varepsilon - \beta R_\varepsilon) + (\mathbf{u} + \nabla\Phi_\varepsilon)\cdot\nabla(\delta T_\varepsilon - \beta R_\varepsilon) + (\delta T_\varepsilon - \beta R_\varepsilon)\Delta\Phi_\varepsilon = 0$$

with

$$\delta = \overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}.$$

Then we have

$$(2.21)\qquad \sup_{t\in[0,T]} \|\delta T_\varepsilon - \beta R_\varepsilon\|_{W^{1,q}(\Omega_M)} \leqslant C(\delta,T)\|\delta T_{0,\varepsilon} - \beta R_{0,\varepsilon}\|_{W^{1,q}(\Omega_M)}, \quad 1 \leqslant q \leqslant \infty.$$

In virtue of (2.16), we get

$$(2.22)\qquad \nabla\Phi_\varepsilon|_{\partial\Omega_M} = \nabla\Phi_{0,\varepsilon}|_{\partial\Omega_M} \quad \forall\, t \in (0,T).$$

We introduce the cut-off functions

$$\mathbf{w}_M^\varepsilon = -\eta_M \mathbf{u} - \eta_M \nabla\Phi_{0,\varepsilon}, \tag{2.23}$$

where

$$\eta_M \in C_C^\infty(\mathbb{R}^3),\ 0 \leqslant \eta_M \leqslant 1,\ \eta_M|_{\partial\Omega_M} = 1,\ \eta_M|_{\partial\Omega_M} = 0 \quad \text{for } \operatorname{dist}[x, \partial\Omega_M] > 1.$$

It also has the boundary

$$\mathbf{w}_M^\varepsilon|_{\partial\Omega_M} = -(\mathbf{u} + \nabla\Phi_\varepsilon)|_{\partial\Omega_M}. \tag{2.24}$$

Following the property of boundary of $\mathbf{w}_M^\varepsilon$ in (4.7), we get

$$\|\partial_t \mathbf{w}_M^\varepsilon(\tau,\cdot)\|_{L^p(\Omega_M;R^3)} + \|\mathbf{w}_M^\varepsilon(\tau,\cdot)\|_{W^{2,p}(\Omega_M;R^3)} \leqslant M^{2(1/p-1)}, \quad 1 \leqslant p \leqslant \infty, \tag{2.25}$$

for any $\tau \in (0,T)$.

2.3. Weak solutions. We say that a family of functions $\{\varrho, \vartheta, \mathbf{u}, \mathbf{H}\}$ represents a *dissipative weak solution* of the Navier-Stokes-Fourier system (1.3)–(1.5) in $(0,T)\times \Omega_M$ if:

▷ $\varrho \geqslant 0$, $\vartheta > 0$ a.e. in $(0,T)\times\Omega_M$,

$$\varrho \in L^\infty(0,T; L^2 + L^{5/3}(\Omega_M)),\ \vartheta \in L^\infty(0,T;L^2 + L^4(\Omega_M)),$$
$$\nabla\vartheta,\ \nabla\log(\vartheta) \in L^2(0,T;L^2(\Omega_M;\mathbb{R}^3)),$$
$$\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega_M;\mathbb{R}^3)),\ \mathbf{u}|_{\partial\Omega_M} = 0;$$

▷ the density $\varrho \in C_{\text{weak}}([0,T];L^1(\Omega_M)$ and the equation of continuity (1.3) holds as a family of integral identities

$$\int_{\Omega_M} [\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_{0,\varepsilon}\varphi(0,\cdot)]\,\mathrm{d}x = \int_0^\tau\int_{\Omega_M} (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.26}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty([0,T]\times\overline{\Omega}_M)$;

▷ the linear momentum $\varrho\mathbf{u} \in C_{\text{weak}}([0,T];L^1(\Omega_M;\mathbb{R}^3))$ and the momentum equation (1.4), together with the initial condition (1.10), are satisfied in the sense of distributions,

$$\begin{aligned}\int_{\Omega_M} \Big[\varrho\mathbf{u}(\tau,\cdot)\cdot\varphi(\tau,\cdot) - \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}\varphi(0,\cdot)\Big]\,\mathrm{d}x \\ = \int_0^\tau\int_{\Omega_M} \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\varphi\frac{1}{\varepsilon^2}p(\varrho,\vartheta)\operatorname{div}\varphi \\ + -\varepsilon^a\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\varphi + \Big[(\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2\Big]\cdot\vec{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.27}$$

for any $\tau \in [0,T]$, and any $\varphi \in C_c^\infty([0,T]\times\overline{\Omega}_M;\mathbb{R}^3)$, $\varphi|_{\partial\Omega_M} = 0$;

▷ the entropy production equation (1.5) is relaxed to the entropy inequality

$$\begin{aligned}\int_{\Omega_M}&[\varrho_{0,\varepsilon}s(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon})\varphi(0,\cdot)-\varrho s(\varrho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot)]\,\mathrm{d}x\\&+\int_0^\tau\int_{\Omega_M}\frac{1}{\vartheta}\Big(\varepsilon^{2+a}\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}-\varepsilon^b\frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\Big)\varphi\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant-\int_0^\tau\int_{\Omega_M}\Big(\varrho s(\varrho,\vartheta)\partial_t\varphi+\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla\varphi+\varepsilon^b\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{2.28}$$

for a.e. $\tau\in[0,T]$ and any test function $\varphi\in C_c^\infty([0,T]\times\overline{\Omega}_M)$, $\varphi\geqslant 0$;

▷ the Maxwell equation verifies

$$\begin{aligned}\int_{\Omega_M}&\mathbf{H}\cdot\vec{\varphi}(\tau,\cdot)\,\mathrm{d}x-\int_\Omega(\mathbf{H})_0\cdot\vec{\varphi}(0,\cdot)\,\mathrm{d}x\\&=\int_0^T\int_{\Omega_M}\mathbf{B}\cdot\partial_t\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_\Omega(-\varepsilon^c\nu\nabla\mathbf{H}:\nabla\vec{\varphi}+(\mathbf{H}\cdot\nabla)\mathbf{u}\cdot\vec{\varphi}\\&\quad-(\mathbf{u}\cdot\nabla)\mathbf{H}\cdot\vec{\varphi}-(\mathbf{H}\cdot\vec{\varphi})\operatorname{div}\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t,\\&\int_0^T\int_{\Omega_M}\mathbf{B}\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t=0,\end{aligned}\tag{2.29}$$

for all $\vec{\varphi}\in[\mathcal{D}([0,T)\times\Omega_M)]^3$, and $\phi\in\mathcal{D}([0,T)\times\Omega_M)$.

3. Main result

In this section we introduce the main result.

Proposition 3.1 ([12]). *Let $\Omega=\mathbb{R}^3$. Assume that the initial datum $\mathbf{u}_0(x),\Phi_0(x)$ satisfies*

$$\mathbf{u}(x,0)=\mathbf{u}_0(x)\in H^s,\quad T(x,0)=\Phi_0(x)\in H^s,\quad\operatorname{div}\mathbf{u}_0=0,\quad s>5/2.\tag{3.1}$$

Then there exist a $T^*\in(0,\infty)$ and a unique solution $\mathbf{u}$ to the incompressible Navier-Stokes equations

$$\begin{aligned}&\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\Pi=(\mathbf{H}\cdot\nabla)\mathbf{H}-\frac{1}{2}\nabla|\mathbf{H}|^2,\quad\operatorname{div}\mathbf{u}=0,\\&\partial_t\mathbf{H}+(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}=0,\\&\partial_tT+\mathbf{u}\cdot\nabla T=0,\end{aligned}\tag{3.2}$$

satisfying the following estimate:

$$\sup_{0<t\leqslant T}(\|\mathbf{u}\|_{H^s}+\|\partial_t\mathbf{u}\|_{H^{s-2}}+\|\partial_t\nabla T\|_{H^s} + \|\nabla T\|_{H^s}+\|\mathbf{H}\|_{H^{s-2}}+\|\partial_t\mathbf{H}\|_{H^{s-2}})\leqslant C(T) \tag{3.3}$$

with $C(T)>0$ *a constant for* $0<T<T^*$.

Theorem 3.1. *Let the thermodynamic functions* p, e, s, *and the transport coefficients* μ, η, κ *satisfy the hypotheses* (2.7), (2.9), (2.10), (2.11) *and* $\{\Omega_M\}_{M>0}$ *be a family of uniformly* C^2 *domains in* $\mathbb{R}^3$ *such that* (4.6) *and* (4.7) *hold for* $M=M(\varepsilon)$,

$$\varepsilon M(\varepsilon)\to\infty \tag{3.4}$$

as ε *tends to* 0. *Let the exponents* a,b *satisfy*

$$0<a<\frac{4}{3},\quad 0<b,\quad 0<c<\frac{5}{4}. \tag{3.5}$$

Let the initial data (1.10) *be chosen in such a way that*

$$\alpha(\varepsilon)=\|\varrho_{0,\varepsilon}^{(1)}-\varrho_0^{(1)}\|_{L^2}^2+\|\vartheta_{0,\varepsilon}^{(1)}-\vartheta_0^{(1)}\|_{L^2}^2+\|\mathbf{u}_{0,\varepsilon}-\widetilde{\mathbf{u}}_0\|_{L^2}^2+\|\mathbf{H}_{0,\varepsilon}-\mathbf{H}_0\|_{L^2}^2, \tag{3.6}$$

where

$$\varrho_0^{(1)},\vartheta_0^{(1)}\in C^\infty(\mathbb{R}^3), \qquad \mathbf{P}[\widetilde{\mathbf{u}}_0]=\mathbf{u}_0\in C^\infty(\mathbb{R}^3;\mathbb{R}^3),\quad \mathbf{H}_0\in C^\infty(\mathbb{R}^3;\mathbb{R}^3) \tag{3.7}$$

for a certain $k\geqslant 3$, *where* $\operatorname{supp}[\mathbf{u}_0]$, $\operatorname{supp}[\varrho_0^{(1)}]$, $\operatorname{supp}[\vartheta_0^{(1)}]$ *are compact subsets in* $\mathbb{R}^3$. *Finally, let* $\{\varrho_\varepsilon,\vartheta_\varepsilon,\mathbf{u}_\varepsilon,\mathbf{H}_\varepsilon\}$ *be a dissipative weak solution of the MHD system* (1.3)–(1.6) *in* $(0,T)\times\mathbb{R}^3$.

Then we have

$$\begin{aligned}\big\|\sqrt{\varrho_\varepsilon}\big(\mathbf{u}_\varepsilon-\nabla\Psi_\varepsilon-\mathbf{u}\big)(\tau,\cdot)\big\|_{L^2(\Omega_M;\mathbb{R}^3)} &+\Big\|\Big(\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big)(\tau,\cdot)-R(\tau,\cdot)\Big\|_{L^2+L^{5/3}(\Omega_M)}\\ &+\|\mathbf{H}_\varepsilon-\mathbf{H}\|_{L^2(\Omega_M;\mathbb{R}^3)}+\Big\|\Big(\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\Big)(\tau,\cdot)-\mathcal{T}(\tau,\cdot)\Big\|_{L^q(\Omega_M)}\\ &\leqslant C\Big[\varepsilon^\theta+\alpha(\varepsilon)+\frac{1}{\varepsilon M(\varepsilon)}\Big]^{1/2},\end{aligned} \tag{3.8}$$

where $\tau \in [0,T]$, $0 < a < 1$, $1 \leqslant q < 2$ *and* $0 < \varepsilon < \varepsilon_0$ *verifying*

$$\theta = \min\Big\{\frac{2-a}{2},\ \frac{2-a-c}{2},\ \frac{a}{2},\ \frac{b}{2},\ \frac{4-3a}{6},\ \frac{1}{4}\Big\}, \tag{3.9}$$

and $(\mathbf{u}, \mathbf{H}, T)$ *verifies the ideal MHD system* (1.11)–(1.13).

Furthermore, the velocity $\mathbf{u}$*, magnetic field* $\mathbf{H}$*, and temperature* T *have the initial data*

$$\mathbf{u}_0 = \mathbf{H}[\widetilde{\mathbf{u}}_0], \quad \mathbf{H}_0, \quad T_0 = \overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\vartheta_0^{(1)} - \frac{1}{\overline{\varrho}}\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\varrho_0^{(1)}. \tag{3.10}$$

To get the target space, we need the convergence of densities, velocity and magnetic field on any compact set of $\mathbb{R}^3$ and so we can easily show the following corollary.

Corollary 3.1. *Assume that* $\alpha(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *under the same assumption as in Theorem* 3.6*, we get, for any* $T < T_*$*, that the weak solution* $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)$ *of system* (1.3)–(1.6) *satisfies*

$$\|\sqrt{\varrho_\varepsilon} - 1\|_{L^\infty(0,T;L^2(K))} \to 0, \tag{3.11}$$

$$\|\vartheta^\varepsilon - 1\|_{L^\infty(0,T;L^p(K))} \to 0, \tag{3.12}$$

$$\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^r(0,T;L^2(K))} \to 0, \tag{3.13}$$

$$\|\mathbf{H}_\varepsilon - \mathbf{H}\|_{L^\infty(0,T;L^2(K))} \to 0 \tag{3.14}$$

with $1 \leqslant p < 2$ *as* ε *tends to* 0 *for all* $2 < r < \infty$ *and any compact* $K \subset \mathbb{R}^3$ *and* $(\mathbf{u}, \mathbf{H}, \mathcal{T})$ *verifying the ideal MHD system* (1.11)–(1.13).

The rest of the paper is devoted to the proof of Theorem 3.1.

4. Proof of Theorem 3.1

In this section, we focus on the proof of Theorem 3.1 based on the relative entropy together with the dispersive estimate (2.18).

4.1. Energy bounds. We derive uniform bounds on the family of dissipative weak solutions $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon, \vartheta_\varepsilon]$ *independent* of the scaling parameter $\varepsilon \to 0$. For $\varrho\colon (0,T)\times\Omega_M \mapsto (0,\infty)$, $r, \Theta, \vartheta\colon (0,T)\times\Omega_M \mapsto (0,\infty)$ and $\mathbf{u}, \mathbf{U}, \mathbf{H}, \mathbf{B}\colon (0,T)\times\Omega_M \mapsto \mathbb{R}^3$, we define the *relative energy functional*

$$\mathcal{E}_\varepsilon(\tau) = \int_{\Omega_M} \Big[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + \frac{1}{\varepsilon^2}E(\varrho,\vartheta \mid r,\Theta) + \frac{1}{2}|\mathbf{H}-\mathbf{B}|^2\Big]\,\mathrm{d}x, \tag{4.1}$$

where

$$(4.2)\qquad E(\varrho,\vartheta \mid r,\Theta) = H_\Theta(\varrho,\vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r,\Theta),$$
$$H_\Theta(\varrho,\vartheta) = \varrho(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta))$$

are the relative energy function and the Helmholtz function, respectively. In virtue of [7], we have the *relative energy inequality*

$$(4.3)\qquad \mathcal{E}_\varepsilon(\tau) + \int_0^\tau\int_{\Omega_M} \Big[\frac{\Theta}{\vartheta}\Big(\varepsilon^a \mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u} - \varepsilon^{b-2}\frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\Big) + \varepsilon^c|\nabla\mathbf{H}|^2\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \mathcal{E}_\varepsilon(0) + \sum_{j=1}^{7} A_\varepsilon^j,$$

where

$$(4.4)\qquad A_\varepsilon^1 = \int_0^\tau\int_{\Omega_M}\Big(\varepsilon^a\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{U} - \varepsilon^{b-2}\frac{\mathbf{q}(\vartheta,\nabla\vartheta)}{\vartheta}\cdot\nabla\Theta\Big)\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_\varepsilon^2 = \int_0^\tau\int_{\Omega_M}(\varrho(\partial_t\mathbf{U} + \mathbf{u}\cdot\nabla\mathbf{U})\cdot(\mathbf{U}-\mathbf{u}))\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_\varepsilon^3 = \frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}\frac{r-\varrho}{r}\mathbf{U}\cdot\nabla p(r,\Theta)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}\Big(\varrho\frac{\nabla p(r,\Theta)}{r}\cdot(\mathbf{U}-\mathbf{u}) + (p(r,\Theta) - p(\varrho,\vartheta))\mathrm{div}\,\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_\varepsilon^4 = -\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}(\varrho(s(\varrho,\vartheta) - s(r,\Theta))\partial_t\Theta + \varrho(s(\varrho,\vartheta) - s(r,\Theta))\mathbf{u}\cdot\nabla\Theta)\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_\varepsilon^5 = \frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}\frac{r-\varrho}{r}\partial_t p(r,\Theta)\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_\varepsilon^6 = -\int_0^\tau\int_{\Omega_M}\Big(\Big((\mathbf{H}\cdot\nabla)\mathbf{H} - \frac{1}{2}\nabla|\mathbf{H}|^2\Big)\cdot\mathbf{U}\Big)\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_\varepsilon^7 = -\int_{\Omega_M}\Big(\mathbf{H}\cdot\mathbf{B} - \mathbf{H}_{0,\varepsilon}\cdot\mathbf{B}_0 - \frac{1}{2}|\mathbf{B}|^2 + \frac{1}{2}|\mathbf{B}_0|^2\Big)\,\mathrm{d}x$$

holds for a.e. $\tau \in (0,T)$ and for any trio of continuously differentiable test functions $(r,\Theta,\mathbf{U})$ defined on $[0,T]\times\overline{\Omega}_M$, such that

$$r > 0,\ \Theta > 0,\quad r,\ \Theta \text{ compactly supported in } \overline{\Omega}_M,$$
$$\mathbf{U},\mathbf{B} \in C([0,T];W^{k,2}(\Omega_M;\mathbb{R}^3)),\ \ \partial_t\mathbf{U} \in C([0,T];W^{k-1,2}(\Omega_M;\mathbb{R}^3)),\quad k \geqslant 3,$$

with $\mathbf{U}|_{\partial\Omega_M} = 0$, $\mathbf{B}|_{\partial\Omega_M} = 0$.

Taking $r = \overline{\varrho}$, $\Theta = \overline{\vartheta}$, $\mathbf{U} = 0$, $\mathbf{B} = 0$ as test functions in the relative entropy inequality (4.3), we obtain

$$\begin{aligned}(4.5)\qquad &\int_{\Omega_M} \Big[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}E(\varrho_\varepsilon,\vartheta_\varepsilon|\overline{\varrho},\overline{\vartheta}) + \frac{1}{2}|\mathbf{H}_\varepsilon|^2\Big]\,\mathrm{d}x \\ &\quad + \overline{\vartheta}\int_0^\tau\int_{\Omega_M} \Big[\frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^a\mathbb{S}(\vartheta_\varepsilon,\nabla\mathbf{u}_\varepsilon) : \\ &\qquad\qquad \nabla\mathbf{u}_\varepsilon - \varepsilon^{b-2}\frac{\mathbf{q}(\vartheta_\varepsilon,\nabla\vartheta_\varepsilon)\cdot\nabla\vartheta_\varepsilon}{\vartheta}\Big) + \varepsilon^c|\nabla\mathbf{H}_\varepsilon|^2\Big]\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant \int_{\Omega_M} \Big[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2}E(\varrho_{0,\varepsilon},\vartheta_{0,\varepsilon}|\overline{\varrho},\overline{\vartheta}) + \frac{1}{2}|\mathbf{H}_{0,\varepsilon}|^2\Big]\,\mathrm{d}x\end{aligned}$$

for a.e. $\tau \in (0,T)$.

In accordance with the structural properties of the thermodynamic functions imposed through (2.7)–(2.11), the relative energy function enjoys the following properties: For any compacts $\mathcal{N} \subset \mathcal{N}' \subset (0,\infty)^2$ there exists a strictly positive constant $c = c(\mathcal{N},\mathcal{N}')$, depending only on $\mathcal{N}$, $\mathcal{N}'$ and the structural properties of P, such that for any $(r,\Theta) \in \mathcal{N}$

$$(4.6)\qquad E(\varrho,\vartheta|r,\Theta) \geqslant c(|\varrho - r|^2 + |\vartheta-\Theta|^2) \quad \text{if } (\varrho,\vartheta)\in\mathcal{N}',$$

$$(4.7)\qquad E(\varrho,\vartheta|r,\Theta) \geqslant c(1+\varrho^\gamma+\vartheta^4) \quad \text{if } (\varrho,\vartheta)\in(0,\infty)^2\setminus\mathcal{N}'.$$

Similarly to [4], Chapter 4.7, we introduce a decomposition of a function h:

$$h = [h]_{\mathrm{ess}} + [h]_{\mathrm{res}} \quad \text{for a measurable function } h,$$

where

$$[h]_{\mathrm{ess}} = h\,1_{\{\overline{\varrho}/2<\varrho_\varepsilon<2\overline{\varrho};\,\overline{\vartheta}/2<\vartheta_\varepsilon<2\overline{\vartheta}\}}, \qquad [h]_{\mathrm{res}} = h - h_{\mathrm{ess}}.$$

Thanks to the hypothesis (3.6), the first integral including initial data on the right-hand side of (4.5) remains bounded uniformly for $\varepsilon \to 0$.

Combining (4.5), (4.6), (4.7) with the hypothesis (2.10)–(2.11), we deduce the following estimates:

$$(4.8)\qquad \operatorname*{ess\,sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon(t,\cdot)\|_{[L^2(\Omega_M)]^3} \leqslant C,$$

$$(4.9)\qquad \operatorname*{ess\,sup}_{t\in(0,T)} \Big\|\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}(t,\cdot)\Big]_{\mathrm{ess}}\Big\|_{L^2(\Omega_M)} + \operatorname*{ess\,sup}_{t\in(0,T)} \Big\|\Big[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}(t,\cdot)\Big]_{\mathrm{ess}}\Big\|_{L^2(\Omega_M)} \leqslant C,$$

$$(4.10)\qquad \operatorname*{ess\,sup}_{t\in(0,T)} \int_{\Omega_M} \big(\big[\varrho_\varepsilon^{5/3}(t,\cdot)\big]_{\mathrm{res}}^{5/3} + [\vartheta_\varepsilon(t,\cdot)]_{\mathrm{res}}^4 + 1_{\mathrm{res}}(t,\cdot)\big)\,\mathrm{d}x \leqslant \varepsilon^2 C,$$

$$\text{(4.11)} \quad \operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{H}_\varepsilon(t)\|_{L^2(\Omega_M)} \leqslant C, \quad \|\nabla \mathbf{H}_\varepsilon\|_{L^2((0,T)\times\Omega_M)} \leqslant C\varepsilon^{-c/2},$$

$$\text{(4.12)} \quad \left\| \varepsilon^{a/2} \sqrt{\frac{\mu(\vartheta,\mathbf{H})}{\vartheta}} \Big(\nabla \mathbf{u}_\varepsilon + \nabla^t \mathbf{u}_\varepsilon - \frac{2}{3} \operatorname{div} \mathbf{u}_\varepsilon \Big) \right\|_{L^2(0,T;[L^2(\Omega_M)]^3)} \leqslant C,$$

$$\text{(4.13)} \quad \|\varepsilon^{a/2} \mathbf{u}_\varepsilon\|_{L^2(0,T;[W^{1,2}(\Omega)]^3)} \leqslant c, \quad \|\varepsilon^{(b-2)/2}(\vartheta_\varepsilon - \overline{\vartheta})\|_{L^2(0,T;W^{1,2}(\Omega_M))} \leqslant C,$$

$$\text{(4.14)} \quad \|\varepsilon^{(b-2)/2}(\log(\vartheta_\varepsilon) - \log(\overline{\vartheta}))\|_{L^2(0,T;W^{1,2}(\Omega_M))} \leqslant C,$$

where the symbol C stands for a generic constant independent of ε.

We remark that (4.11) follows from the generalized Korn's inequality

$$\|\nabla \mathbf{w}\|_{L^2} \leqslant C \Big(\int_{\Omega_M} \varrho_\varepsilon |\mathbf{w}|^2 \, \mathrm{d}x \Big)^{1/2} + C \Big\| \nabla \mathbf{w} + \nabla^t \mathbf{w} - \frac{2}{3} \operatorname{div} \mathbf{w} \mathbb{I} \Big\|_{L^2}$$

for $\mathbf{w} \in W^{1,2}(\Omega_M; \mathbb{R}^3)$, combined with the estimates (4.8), (4.10), (4.12), and the Poincaré type inequality (4.16).

In order to get the second estimate of (4.13) and (4.14), we deduce from (4.5), (1.8), (2.10),

$$\text{(4.15)} \qquad \varepsilon^{(b-2)/2} (\|\nabla \vartheta_\varepsilon\|_{L^2(0,T;[W^{1,2}(\Omega_M)]^3)} + \|\nabla \log \vartheta_\varepsilon\|_{L^2(0,T;[W^{1,2}(\Omega_M)]^3)}) \leqslant c.$$

Now, the bound (4.12) involving $\vartheta_\varepsilon - \overline{\vartheta}$ follows immediately from (4.15) and (4.9), (4.10). To get (4.12) involving $\log \vartheta_\varepsilon - \log \overline{\vartheta}$ we report the following Poincaré type inequality on the layer Ω_M: For any $M > 0$ there exists $C = C(M, \Omega_M) > 0$ such that for all $\omega \in W^{1,2}(\Omega_M)$ and all $V \subset \Omega_M$, $|V| < M$ it holds that

$$\text{(4.16)} \qquad \|\xi\|^2_{L^2(\Omega_M)} \leqslant C \Big(\|\nabla \xi\|^2_{L^2(\Omega_M;\mathbb{R}^3)} + \int_{\Omega_M \setminus V} \xi^2 \, \mathrm{d}x \Big),$$

see e.g. Lemma 3.1 in [7]. To conclude, we take in (4.16), $\xi = \log \vartheta_\varepsilon - \log \overline{\vartheta}$, $V = \{\vartheta_\varepsilon \leqslant \overline{\vartheta}/2\}$ and the desired bound follows by virtue of (4.15), (4.9), (4.10).

4.2. Computation of relative entropy. Our aim is to employ the relative energy inequality (4.3) to prove the convergence of the sequence of weak solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{H}_\varepsilon)$ to the target system. To this end, we take

$$\varrho = \varrho_\varepsilon, \quad \vartheta = \vartheta_\varepsilon, \quad \mathbf{u} = \mathbf{u}_\varepsilon, \quad \mathbf{H} = \mathbf{H}_\varepsilon$$

and choose the test functions $\{r, \Theta, \mathbf{U}\}$ in the following way:

$$\text{(4.17)} \quad r = r_{\varepsilon,\eta} = \overline{\varrho} + \varepsilon R_\varepsilon, \; \Theta = \Theta_\varepsilon = \overline{\vartheta} + \varepsilon T_\varepsilon, \; \mathbf{U} = \mathbf{U}_\varepsilon = \mathbf{u} + \nabla \Phi_\varepsilon + \mathbf{w}^\varepsilon_M, \; \mathbf{B} = \mathbf{H}.$$

This will be carried over in several steps. We omit the subscript η whenever no confusion arises.

Step I. In virtue of the second estimate of (4.13), we first estimate the dissipation term of A^1_ε:

$$
\begin{aligned}
(4.18)\quad & \varepsilon^a \int_0^\tau\!\!\int_{\Omega_M} \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{U}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C\varepsilon^{a/2} \int_0^\tau\!\!\int_{\Omega_M} \frac{\varepsilon^{a/2}\mu(\vartheta_\varepsilon, \mathbf{H}_\varepsilon)}{\vartheta_\varepsilon} \Big| \nabla \mathbf{u}_\varepsilon + \nabla^T \mathbf{u}_\varepsilon - \frac{2}{3}\operatorname{div}\mathbf{u}_\varepsilon \mathbb{I} \Big|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\varepsilon^{a/2} \int_0^\tau\!\!\int_{\Omega_M} [\mu(\vartheta_\varepsilon, \mathbf{H}_\varepsilon)\vartheta_\varepsilon]_{\mathrm{ess}} |\nabla \mathbf{U}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\varepsilon^{a/2} \int_0^\tau\!\!\int_{\Omega_M} [\mu(\vartheta_\varepsilon, \mathbf{H}_\varepsilon)\vartheta_\varepsilon]_{\mathrm{res}} |\nabla \mathbf{U}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C\varepsilon^{a/2} \int_0^\tau\!\!\int_{\Omega_M} ([1]_{\mathrm{res}} + [\vartheta_\varepsilon]^4_{\mathrm{res}}) \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^{a/2} \leqslant C\varepsilon^{a/2} + C\varepsilon^{a/2+2},
\end{aligned}
$$

where we have used (2.9). We also have

$$
\begin{aligned}
(4.19)\quad & -\varepsilon^{b-2} \int_0^\tau\!\!\int_{\Omega_M} \frac{\mathbf{q}(\vartheta_\varepsilon, \nabla\vartheta_\varepsilon)\cdot \nabla \Theta_\varepsilon}{\vartheta_\varepsilon} \,\mathrm{d}x\,\mathrm{d}t \\
&= \varepsilon^{b/2} \int_0^\tau\!\!\int_{\Omega_M} \Big[\varepsilon^{(b-2)/2} \frac{\kappa(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{H}_\varepsilon)}{\vartheta_\varepsilon} \nabla(\vartheta_\varepsilon - \overline{\vartheta}) \Big]_{\mathrm{ess}} \cdot \nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon^{b/2} \int_0^\tau\!\!\int_{\Omega_M} \Big[\varepsilon^{(b-2)/2} \frac{\kappa(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{H}_\varepsilon)}{\vartheta_\varepsilon} \nabla(\vartheta_\varepsilon - \overline{\vartheta}) \Big]_{\mathrm{res}} \cdot \nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leqslant C\varepsilon^{b/2}
\end{aligned}
$$

while (4.15) implies that

$$
\varepsilon^{b/2} \int_0^\tau\!\!\int_{\Omega_M} \Big[\varepsilon^{(b-2)/2} \frac{\kappa(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{H}_\varepsilon)}{\vartheta_\varepsilon} \nabla(\vartheta_\varepsilon - \overline{\vartheta}) \Big]_{\mathrm{ess}} \cdot \nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leqslant C\varepsilon^{b/2}
$$

and

$$
\begin{aligned}
& \varepsilon^{b/2} \int_0^\tau\!\!\int_{\Omega_M} \Big[\varepsilon^{(b-2)/2} \frac{\kappa(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{H}_\varepsilon)}{\vartheta_\varepsilon} \nabla(\vartheta_\varepsilon - \overline{\vartheta}) \Big]_{\mathrm{res}} \cdot \nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \leqslant \varepsilon^{b/2} \int_0^\tau\!\!\int_{\Omega_M} |\varepsilon^{(b-2)/2} \nabla(\ln\vartheta_\varepsilon - \ln\overline{\vartheta})| \|\nabla T_\varepsilon\|_{L^\infty} \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad + \varepsilon^{b/2} \int_0^\tau\!\!\int_{\Omega_M} |\varepsilon^{(b-2)/2} \nabla(\vartheta_\varepsilon - \overline{\vartheta})| \|\nabla T_\varepsilon\|_{L^\infty} \,\mathrm{d}x\,\mathrm{d}t \leqslant C\varepsilon^{b/2},
\end{aligned}
$$

where we have used the estimates in (2.19) and (2.21).

Step II. We next control A_2^ε. For the first term of it, we have

$$
\begin{aligned}
&\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\partial_t \mathbf{U}_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla\mathbf{U}_\varepsilon)(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad = \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\otimes(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon):\nabla\mathbf{U}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \int_0^\tau\int_{\Omega_M} (\varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\partial_t\mathbf{U}_\varepsilon + \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon):\nabla\mathbf{U}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + \sum_{k=1}^5 J_k^\varepsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1^\varepsilon &= \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t,\\
J_2^\varepsilon &= \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\partial_t(\nabla\Phi_\varepsilon + \mathbf{w}_M^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t,\\
J_3^\varepsilon &= \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\otimes(\nabla\Phi_\varepsilon + \mathbf{w}_M^\varepsilon):\nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t,\\
J_4^\varepsilon &= \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\otimes\mathbf{u}:(\nabla^2\Phi_\varepsilon + \nabla\mathbf{w}_M^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t,\\
J_5^\varepsilon &= \frac{1}{2}\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\cdot\nabla|\nabla\Phi_\varepsilon + \mathbf{w}_M^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

For J_1^ε we have

$$
\begin{aligned}
|J_1^\varepsilon| &\leqslant \left|\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)\cdot(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\leqslant \left|\int_0^\tau\int_{\Omega_M} (\varrho_\varepsilon - \overline{\varrho})\mathbf{U}_\varepsilon\cdot\nabla\Pi\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_0^\tau\int_{\Omega} (\nabla\Phi_\varepsilon + \mathbf{w}_M^\varepsilon)\cdot\nabla\Pi\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad + \left|\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla\Pi\,\mathrm{d}x\,\mathrm{d}t\right|.
\end{aligned}
\tag{4.20}
$$

For the first term of the right-hand side of (4.20), we use estimates (4.9) and (4.10) to obtain that

$$
\begin{aligned}
&\left|\int_0^\tau\int_{\Omega_M} (\varrho_\varepsilon - \overline{\varrho})\mathbf{U}_\varepsilon\cdot(\nabla\Pi - (\nabla\times\mathbf{H})\times\mathbf{H})\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\quad \leqslant C\|(\varrho_\varepsilon - 1)1_{|\varrho_\varepsilon - \overline{\varrho}|\leqslant 1/2}\|_{L^2L^2}\|\nabla\Pi - (\nabla\times\mathbf{H})\times\mathbf{H}\|_{L^2L^2}\\
&\qquad + C\int_0^\tau\int_{\Omega_M} ([1]_{\mathrm{res}} + [\varrho_\varepsilon]_{\mathrm{res}}^{5/3})\|\mathbf{U}_\varepsilon - (\nabla\times\mathbf{H})\times\mathbf{H}\|_{L^\infty L^\infty}\|\nabla\Pi\|_{L^\infty L^\infty}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad \leqslant C(\varepsilon + \varepsilon^2).
\end{aligned}
$$

The second term of the right-hand side of (4.20), using equation (2.13) and the dispersive regularity (2.1), can be estimated as follows:

$$
\begin{aligned}
\left|\int_0^\tau\int_{\Omega_M}(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon)\cdot\nabla\Pi\,\mathrm{d}x\,\mathrm{d}t\right| &\leqslant \varepsilon\left[\int_{\Omega_M}|s_\varepsilon||\Pi|\,\mathrm{d}x\right]_0^\tau+\varepsilon\int_0^\tau\int_\Omega|s_\varepsilon||\partial_t\Pi|\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \varepsilon\big(\|s_\varepsilon\|_{L^\infty L^2}\|\Pi\|_{L^\infty L^2}+\|s_{0,\varepsilon}\|_{L^\infty L^2}\|\Pi_0\|_{L^\infty L^2}\\
&\quad+\|s_\varepsilon\|_{L^2L^2}\|\partial_t\Pi\|_{L^2L^2}\big)+C\|\mathbf{w}_M^\varepsilon\|_{W^{1,\infty}}\leqslant C\Big(\varepsilon+\frac{1}{M^2}\Big),
\end{aligned}
$$

where we have used $\operatorname{div}\mathbf{u}=0$ and $\mathbf{U}_\varepsilon=0$ on $\partial\Omega_M$. Similarly, we obtain

$$
\left|\int_0^\tau\int_{\Omega_M}(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon)\cdot(\nabla\times\mathbf{H})\times\mathbf{H}\,\mathrm{d}x\,\mathrm{d}t\right|\leqslant C\Big(\varepsilon+\frac{1}{M^2}\Big).
$$

Furthermore, we here denote s_ε by $s_\varepsilon:=\alpha R_\varepsilon+\beta T_\varepsilon$.

Third term of the right-hand side of (4.20), using the continuity equation (1.3), (4.9) and (4.10), can be bounded as follows:

$$
\begin{aligned}
\left|\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla\Pi\,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_0^\tau\int_{\Omega_M}(\varrho_\varepsilon-\overline{\varrho})\cdot\partial_t\Pi\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\leqslant C\varepsilon\left\|\Big[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\Big]_{\mathrm{ess}}\right\|_{L^2L^2}\|\partial_t\Pi\|_{L^2L^2}\\
&\quad+C\int_0^\tau\int_{\Omega_M}([1]_{\mathrm{res}}+[\varrho_\varepsilon]_{\mathrm{res}}^{5/3})\|\mathbf{U}_\varepsilon\|_{L^\infty L^\infty}\|\partial_t\Pi\|_{L^\infty L^\infty}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C(\varepsilon+\varepsilon^2).
\end{aligned}
$$

Thus, we get

$$
|J_1^\varepsilon|\leqslant C\Big(\varepsilon+\frac{1}{M^2}\Big)
$$

for small number $0<\varepsilon<1$.

Next, we have

$$
\begin{aligned}
J_2^\varepsilon = &-\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}\cdot\partial_t(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&-\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot[(\nabla\times\mathbf{H})\times\mathbf{H}]\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\partial_t|\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.21}
$$

Using estimate (2.25) together with (4.8), (4.9) and (4.10), it is easy to show

$$
\begin{aligned}
&-\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\mathbf{w}_M^\varepsilon\,\mathrm{d}x\,\mathrm{d}t+\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{v}\cdot\partial_t\mathbf{w}_M^\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\nabla\Phi_\varepsilon\cdot\partial_t\mathbf{w}_M^\varepsilon\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\partial_t|\mathbf{w}_M^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant C\Big(\frac{1}{M}+\frac{1}{M^2}\Big).
\end{aligned}
$$

In virtue of (2.1), (4.9), and (4.10), we get

$$\begin{aligned}
&\left|\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \mathbf{u}\cdot\partial_t\nabla\Phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leqslant \left|\int_0^\tau\int_{\Omega_M} \frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\mathbf{u}\cdot\nabla s_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_0^\tau\int_{\partial\Omega_M} \mathbf{u}\cdot\mathbf{n}\,\partial_t\nabla\Phi_{0,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leqslant \left\|\left[\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\right]_{\mathrm{ess}}\right\|_{L^\infty(0,T;L^2(\Omega_M))}\|\mathbf{u}\|_{L^{4/3}(0,T;L^4(\Omega_M))}\|\nabla s_\varepsilon\|_{L^4(0,T;L^4(\Omega_M))} \\
&\qquad+\frac{C}{\varepsilon}\|[1]_{\mathrm{res}}+[\varrho_\varepsilon]_{\mathrm{res}}^{5/3}\|_{L^\infty(0,T;L^1(\Omega_M))}\leqslant C(\varepsilon^{1/4}+\varepsilon),
\end{aligned}$$

where we have used (2.22) for large $M \gg 1/\varepsilon$. Similarly, we have

$$\frac{1}{2}\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\partial_t|\nabla\Phi_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\frac{\overline{\varrho}}{2}\left[\int_{\Omega_M}|\nabla\Phi_\varepsilon|^2\,\mathrm{d}x\right]_0^\tau+C(\varepsilon^{1/4}+\varepsilon).$$

Thus, the term J_2^ε is also bounded by:

$$\begin{aligned}
J_2^\varepsilon = &\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon(\mathbf{U}-\mathbf{u}_\varepsilon)\cdot\partial_t\nabla\Phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\leqslant\frac{\overline{\varrho}}{2}\left[\int_{\Omega_M}|\nabla\Phi_\varepsilon|^2\,\mathrm{d}x\right]_0^\tau \\
&-\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla\Phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t+C\Big(\varepsilon^{1/4}+\varepsilon+\frac{1}{M}+\frac{1}{M^2}\Big).
\end{aligned}$$

Using the regularity (3.3), the dispersive regularity (2.18), (4.8), (4.9), and (2.25), the term J_3^ε can be estimated as:

$$\begin{aligned}
J_3^\varepsilon = &\int_0^\tau\int_{\Omega_M}(\varrho_\varepsilon-\overline{\varrho})\mathbf{U}\otimes(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon):\nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_{\Omega_M}\mathbf{U}\otimes(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon):\nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_{\Omega_M}(\sqrt{\varrho_\varepsilon}-\sqrt{\overline{\varrho}})\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\otimes(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon):\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \\
&+\int_0^\tau\int_{\Omega_M}\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\otimes(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon):\nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t \\
\leqslant\, & C\Big(\varepsilon^{1/4}+\varepsilon+\frac{1}{M^2}\Big).
\end{aligned}$$

Similarly, we get

$$J_4^\varepsilon+J_5^\varepsilon\leqslant C\Big(\varepsilon^{1/4}+\varepsilon+\frac{1}{M^2}\Big).$$

So, the term A_1^ε can be estimated as follows:

$$
\begin{aligned}
A_2^\varepsilon \leqslant \frac{\overline{\varrho}}{2}\Big[\int_{\Omega_M} |\nabla\Phi_\varepsilon|^2 \,\mathrm{d}x\Big]_0^\tau - \int_0^\tau\!\!\int_{\Omega_M} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \nabla\Phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
- \int_0^\tau\!\!\int_{\Omega_M} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot [(\nabla\times\mathbf{H})\times\mathbf{H}] \,\mathrm{d}x\,\mathrm{d}t \\
+ C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^{1/4} + \varepsilon + \frac{1}{M} + \frac{1}{M^2}\Big).
\end{aligned}
$$

Step III. In this part, we handle the pressure A_ε^3.

Notice that

$$
\begin{aligned}
A_\varepsilon^3 = & -\int_0^\tau\!\!\int_{\Omega_M} (p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta}))\Delta\Phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau\!\!\int_{\Omega_M} p(\varrho_\varepsilon,\vartheta_\varepsilon)\operatorname{div}\mathbf{w}_M^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
& - \int_0^\tau\!\!\int_{\Omega_M} \frac{\varrho_\varepsilon}{r_\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla p(r_\varepsilon,\Theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t := \sum_{j=1}^3 A_{3,j}^\varepsilon
\end{aligned}
$$

for big $M \gg 1/\varepsilon$.

We now estimate $A_{3,2}^\varepsilon$. Using the estimate of (4.9), it follows that

$$
\begin{aligned}
A_{2,2}^\varepsilon = & -\int_0^\tau\!\!\int_{\Omega_M} [p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})]_{\mathrm{ess}}\operatorname{div}\mathbf{w}_M^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
& -\int_0^\tau\!\!\int_{\Omega_M} [p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})]_{\mathrm{res}}\operatorname{div}\mathbf{w}_M^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
& -\int_0^\tau\!\!\int_{\Omega_M} p(\overline{\varrho},\overline{\vartheta})\operatorname{div}\mathbf{w}_M^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
\leqslant & \frac{C}{M} + C\|\mathbf{w}_M^\varepsilon\|_{W^{1,\infty}(\Omega_M)}\int_0^\tau\!\!\int_{\Omega_M} ([\varrho_\varepsilon]_{\mathrm{res}}^{5/3} + [\vartheta_\varepsilon]_{\mathrm{res}}^4 + [1]_{\mathrm{res}})\,\mathrm{d}x\,\mathrm{d}t \\
\leqslant & C\Big(\frac{1}{M} + \frac{1}{M^2}\Big)
\end{aligned}
\tag{4.22}
$$

for small number $\varepsilon < 1$, while the divergence Theorem implies that

$$
\begin{aligned}
\int_0^\tau\!\!\int_{\Omega_M} p(\overline{\varrho},\overline{\vartheta})\operatorname{div}\mathbf{w}_M^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\!\!\int_{\partial\Omega_M} p(\overline{\varrho},\overline{\vartheta})\nabla\Phi_\varepsilon\cdot\mathbf{n}\,\mathrm{d}S_x \\
= \int_0^\tau\!\!\int_{\partial\Omega_M} p(\overline{\varrho},\overline{\vartheta})\nabla\Phi_{0,\varepsilon}\cdot\mathbf{n}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\!\!\int_{\Omega_M} p(\overline{\varrho},\overline{\vartheta})\operatorname{div}\mathbf{u}_{0,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t \\
= \int_0^\tau\!\!\int_{\partial\Omega_M} p(\overline{\varrho},\overline{\vartheta})\mathbf{u}_0\cdot\mathbf{n}\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}
$$

where we have used (3.6) and $\mathbf{u}_0 \in C_c^\infty(\mathbb{R}^3)$.

Finally, it remains to estimate $A^\varepsilon_{3,3}$. Note that

$$\begin{aligned}A^\varepsilon_{3,3} &= -\frac{1}{\varepsilon^2}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\frac{1}{r_\varepsilon}\nabla p(r_\varepsilon,\Theta_\varepsilon) = -\frac{1}{\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\Big(\frac{\overline{\varrho}}{r_\varepsilon}-1\Big)\nabla s_\varepsilon - \frac{1}{\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla s_\varepsilon\\ &\quad -\frac{1}{\varepsilon}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\frac{1}{r_\varepsilon}\Big[\Big(\frac{\partial p(r_\varepsilon,\Theta_\varepsilon)}{\partial\varrho}-\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\Big)\nabla R_\varepsilon + \Big(\frac{\partial p(r_\varepsilon,\Theta_\varepsilon)}{\partial\vartheta}-\frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\Big)\nabla T_\varepsilon\Big]\\ &:= \sum_{j=1}^{3} B^\varepsilon_j.\end{aligned}$$

For B^ε_1, we can compute the estimate

$$\begin{aligned}&\int_0^\tau\!\!\int_{\Omega_M} B^\varepsilon_1\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^\tau\!\!\int_{\Omega_M}(\varrho_\varepsilon-\overline{\varrho})\mathbf{u}_\varepsilon\frac{R_\varepsilon}{\overline{\varrho}+\varepsilon R_\varepsilon}\nabla s_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\!\!\int_{\Omega_M}\overline{\varrho}\mathbf{u}_\varepsilon\frac{R_\varepsilon}{\overline{\varrho}+\varepsilon R_\varepsilon}\nabla s_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\varepsilon^{-a/2}\|[\varrho_\varepsilon-\overline{\varrho}]_{\mathrm{ess}}\|_{L^\infty L^2}\|\varepsilon^{a/2}\mathbf{u}_\varepsilon\|_{L^2L^6}\|\nabla s_\varepsilon\|_{L^2L^3}\\ &\quad + C\varepsilon^{-a/2}(\|[\varrho_\varepsilon]_{\mathrm{res}}\|_{L^\infty L^{5/3}} + \|[1]_{\mathrm{res}}\|_{L^\infty L^{5/3}})\|\varepsilon^{a/2}\mathbf{u}_\varepsilon\|_{L^2L^6}\|\nabla s_\varepsilon\|_{L^2L^{30/7}}\\ &\quad + C\varepsilon^{-a/2}\|\varepsilon^{a/2}\mathbf{u}_\varepsilon\|_{L^2L^6}\|R_\varepsilon\|_{L^\infty L^2}\|\nabla s_\varepsilon\|_{L^2L^3}\\ &\leqslant C\varepsilon^{(1-a)/2}\end{aligned}$$

for small number $\varepsilon<1$, where we have used estimates (4.8), (4.9), (4.10), and (2.18). Following (2.14), the term of B^ε_2 is the same as

$$B^\varepsilon_2 = \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\nabla\Phi_\varepsilon,$$

which adding the second term of A^2_ε and B^ε_2 vanishes. Using the Taylor expansion, we get

$$\begin{aligned}\int_0^\tau\!\!\int_{\Omega_M} B^\varepsilon_3\,\mathrm{d}x\,\mathrm{d}t &\leqslant \int_0^\tau\!\!\int_{\Omega_M}\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot(\omega_1\nabla R_\varepsilon^2+\omega_2\nabla(R_\varepsilon T_\varepsilon)+\omega_3\nabla T_\varepsilon^2)\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon\\ &= -\int_0^\tau\!\!\int_{\Omega_M}(\varrho_\varepsilon-\overline{\varrho})\cdot\partial_t\big(\omega_1\nabla R_\varepsilon^2+\omega_2\nabla(R_\varepsilon T_\varepsilon)+\omega_3\nabla T_\varepsilon^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon\\ &\leqslant C(\varepsilon+\varepsilon^2)\end{aligned}$$

with

$$\omega_1 = \frac{1}{2}\frac{\partial^2 p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho^2},\quad \omega_2 = \frac{\partial^2 p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho\partial\vartheta},\quad \omega_3 = \frac{1}{2}\frac{\partial^2 p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta^2},$$

where we have used estimates (4.9), (4.10), (2.17), and transport equations (2.20), (2.21). Consequently, we get

$$A^\varepsilon_3 \leqslant A^\varepsilon_{3,1} + C\Big(\varepsilon+\frac{1}{M}\Big) \tag{4.23}$$

for small $0<\varepsilon<1$ and $M>1$.

Step IV. In this part, we estimate A_4^ε. Our intention in this section is to "replace" $\mathbf{u}_\varepsilon$ by $\mathbf{U}_\varepsilon$ in the remaining (last) convective term in A_4^ε. To this end, we write

$$\begin{aligned}&\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\mathbf{u}_\varepsilon\cdot\nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\mathbf{U}_\varepsilon\cdot\nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon)\cdot\nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

while the estimates of (4.6) and (4.9) together with Taylor expansion provide

$$\begin{aligned}&\left|\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \Big[\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\Big]_{\mathrm{ess}}(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon)\cdot\nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right|\\ &\leqslant C\int_0^\tau \|\nabla T_\varepsilon\|_{L^\infty(\Omega)}\int_{\Omega_M}\left(\varrho_\varepsilon|\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon|^2+\left|\Big[\frac{\varrho_\varepsilon-r_\varepsilon}{\varepsilon}\Big]_{\mathrm{ess}}\right|^2+\left|\Big[\frac{\vartheta_\varepsilon-\Theta_\varepsilon}{\varepsilon}\Big]_{\mathrm{ess}}\right|^2\right)\mathrm{d}t\,\mathrm{d}x\\ &\leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t\end{aligned}$$

and by using (2.8), (4.9), and (4.11), it follows that

$$\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \Big[\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon)-s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\Big]_{\mathrm{res}}(\mathbf{u}_\varepsilon-\mathbf{U}_\varepsilon)\cdot\nabla T_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\leqslant C(\varepsilon^{2/3-a/2}+\varepsilon),$$

where

$$\begin{aligned}&\frac{1}{\varepsilon}\|[\varrho_\varepsilon|s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}\|_{L^1(0,\tau;L^1(\Omega_M)}-\frac{1}{\varepsilon}\|[\varrho_\varepsilon|s(r_\varepsilon,\Theta_\varepsilon)]_{\mathrm{res}}\|_{L^1(0,\tau;L^1(\Omega_M)}\\ &\qquad\leqslant\frac{C}{\varepsilon}\int_0^\tau\int_{\Omega_M}([\vartheta_\varepsilon^4]_{\mathrm{res}}+[\varrho_\varepsilon^{5/3}]_{\mathrm{res}}+[1]_{\mathrm{res}})\,\mathrm{d}x\,\mathrm{d}t\leqslant C\varepsilon\end{aligned}$$

and

$$\begin{aligned}(4.24)\qquad&\frac{1}{\varepsilon}\|[\varrho_\varepsilon|s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\mathrm{res}}\mathbf{u}_\varepsilon\|_{L^1(0,\tau;L^1(\Omega_M)}-\frac{1}{\varepsilon}\|[\varrho_\varepsilon|s(r_\varepsilon,\Theta_\varepsilon)|]_{\mathrm{res}}\mathbf{u}_\varepsilon\|_{L^1(0,\tau;L^1(\Omega_M)}\\ &\leqslant C\varepsilon^{(-2-a)/2}(\|[\vartheta_\varepsilon^3]_{\mathrm{res}}\|_{L^{6/5}(\Omega_M)}+\|[\varrho_\varepsilon\ln\varrho_\varepsilon]_{\mathrm{res}}\|_{L^{6/5}(\Omega_M)}\\ &\quad+\|[\varrho_\varepsilon\ln\vartheta_\varepsilon]_{\mathrm{res}}\|_{L^{6/5}(\Omega_M)}+\|[1]_{\mathrm{res}}\|_{L^{6/5}(\Omega_M)})\|\varepsilon^{a/2}\mathbf{u}_\varepsilon\|_{W^{1,2}(\Omega_M;\mathbb{R}^3)}\\ &\leqslant C\varepsilon^{2/3-a/2}.\end{aligned}$$

Indeed, for (4.24), we can choose $\delta>0$ such that

$$\varrho_\varepsilon|\ln\varrho_\varepsilon|\leqslant C\varrho_\varepsilon^{1+\delta}\leqslant C\varrho_\varepsilon^{25/18}$$

for $\varrho_\varepsilon > 2\overline{\varrho}$ and so we get

$$\|[\varrho_\varepsilon \ln \varrho_\varepsilon]_{\rm res}\|_{L^{6/5}(\Omega_M)} \leqslant C\|[\varrho_\varepsilon^{5/3}]_{\rm res}\|_{L^1(\Omega_M)}^{5/6} \leqslant C\varepsilon^{5/3}.$$

Furthermore, we can estimate the term of $[\varrho_\varepsilon \ln \vartheta_\varepsilon]_{\rm res}$ as follows:

$$\|[\varrho_\varepsilon \ln \varrho_\varepsilon]_{\rm res}\|_{L^{6/5}(\Omega_M)} \leqslant C\|[\varrho_\varepsilon^{5/3}]_{\rm res}\|_{L^1(\Omega_M)}^{5/6} + C\|[\vartheta_\varepsilon^4]_{\rm res}\|_{L^1(\Omega_M)}^{5/6} \leqslant C\varepsilon^{5/3}$$

for $\varrho_\varepsilon > 2\overline{\varrho}, \vartheta_\varepsilon > 2\overline{\vartheta}$, where we have used Young's inequality. Thus, we have

$$\begin{aligned} A_4^\varepsilon \leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,{\rm d}t &+ \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r_\varepsilon,\Theta_\varepsilon)}{\varepsilon}\mathbf{U}_\varepsilon\cdot\nabla T_\varepsilon\,{\rm d}x\,{\rm d}t \\ &- \frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_M} \varrho_\varepsilon(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r_\varepsilon,\Theta_\varepsilon))\partial_t T_\varepsilon\,{\rm d}x\,{\rm d}t + C(\varepsilon^{(2/3-a/2)} + \varepsilon). \end{aligned} \tag{4.25}$$

Step V. We first handle the residual part of the remaining integrals. To this end, we employ estimates (3.3), (2.18) together with the system of equations (2.13), (2.14), and (2.20), to deduce

$$\sup_{t\in(0,T)} \varepsilon\|\partial_t R_\varepsilon(t,\cdot)\|_{L^\infty(\Omega_M)}, \quad \sup_{t\in(0,T)} \varepsilon\|\partial_t T_\varepsilon(t,\cdot)\|_{L^\infty(\Omega_M)} \leqslant C, \tag{4.26}$$

$$\varepsilon\|\partial_t R_\varepsilon\|_{L^\infty(\Omega_M)} \to 0, \quad \varepsilon\|\partial_t T_\varepsilon\|_{L^\infty(\Omega_M)} \to 0 \quad \text{in } L^p(0,T),\ 1\leqslant p<\infty. \tag{4.27}$$

Using the same method as in Step III and Step IV, we finally get the desired result, namely

$$\begin{aligned} &-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_M} [\varrho_\varepsilon(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r_\varepsilon,\Theta_\varepsilon))\partial_t T_\varepsilon]_{\rm res}\,{\rm d}x\,{\rm d}t \\ &-\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_M} [\varrho_\varepsilon(s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(r_\varepsilon,\Theta_\varepsilon))\mathbf{U}_\varepsilon\cdot\nabla T_\varepsilon]_{\rm res}\,{\rm d}x\,{\rm d}t \leqslant C\varepsilon \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} &-\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M} \Big[\frac{\varrho_\varepsilon - r_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\Big]_{\rm res}\,{\rm d}x\,{\rm d}t \\ &-\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M} [(p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})){\rm div}\mathbf{U}_\varepsilon]_{\rm res}\,{\rm d}x\,{\rm d}t \leqslant C\varepsilon. \end{aligned} \tag{4.29}$$

In view of the preceding section, we have to handle solely the essential part of the integral of the pressure and the entropy part whose integrands can be, roughly speaking, replaced by their linearization. We now handle the remaining terms of $A_2^\varepsilon, \ldots, A_\varepsilon^5$. We start with the following observations that can be obtained by using

Taylor formula and Definition (4.17) as many times as needed—to express R_ε and T_ε as a linear combination of $\alpha R_\varepsilon + \beta T_\varepsilon$ and $Z_\varepsilon = \delta T_\varepsilon - \beta R_\varepsilon$, together with estimates (3.3), (4.9), (4.10), (2.18), (2.21), (2.25), together with using equations (2.20), (2.13), and (2.14).

▷ First,

$$
\begin{aligned}
(4.30)\qquad & \frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M} \frac{r_\varepsilon - \varrho_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_0^\tau\int_{\Omega_M}\Big(R_\varepsilon - \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon \\
&= \frac{\delta}{\beta^2+\alpha\delta}\int_0^\tau\int_{\Omega_M}(\alpha R_\varepsilon + \beta T_\varepsilon)\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau\int_{\Omega_M}(\delta T_\varepsilon - \beta R_\varepsilon)\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau\int_{\Omega_M}\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon.
\end{aligned}
$$

▷ Second,

$$
\begin{aligned}
(4.31)\qquad & \frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}(p(r_\varepsilon,\Theta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta}))\Delta\Phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \frac{\delta}{\beta^2+\alpha\delta}\int_0^\tau\int_{\Omega_M}\Big(\alpha\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} + \beta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon.
\end{aligned}
$$

▷ Third,

$$
\begin{aligned}
(4.32)\quad & \frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon(s(r_\varepsilon,\Theta_\varepsilon) - s(\varrho_\varepsilon,\vartheta_\varepsilon))(\partial_t\Theta_\varepsilon + \mathbf{U}_\varepsilon\cdot\nabla\Theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_0^\tau\int_{\Omega_M}\Big(\delta T_\varepsilon - \beta R_\varepsilon + \beta\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} - \delta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)(\partial_t T_\varepsilon + \mathbf{U}_\varepsilon\cdot\nabla T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon \\
&= \frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau\int_{\Omega_M}\Big(\delta T_\varepsilon - \beta R_\varepsilon + \beta\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} - \delta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)(\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon) \\
&\quad + \mathbf{U}_\varepsilon\cdot\nabla(\alpha R_\varepsilon + \beta T_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \frac{\beta}{\beta^2+\alpha\delta}\int_0^\tau\int_{\Omega_M}\Big(\delta T_\varepsilon - \beta R_\varepsilon + \beta\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} - \delta\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\Big)\partial_t(\alpha R_\varepsilon + \beta T_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\Big(\varepsilon^{1/4} + \frac{1}{M^2}\Big).
\end{aligned}
$$

Summing together formulas (4.30)–(4.32), we get

$$\begin{aligned}
(4.33)\quad & \frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}\Big(\frac{r_\varepsilon-\varrho_\varepsilon}{r_\varepsilon}\partial_t p(r_\varepsilon,\Theta_\varepsilon)+(p(r_\varepsilon,\Theta_\varepsilon)-p(\varrho_\varepsilon,\vartheta_\varepsilon))\mathrm{div}\mathbf{U}_\varepsilon\,\mathrm{d}x\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\varepsilon^2}\int_0^\tau\int_{\Omega_M}\varrho_\varepsilon(s(r_\varepsilon,\Theta_\varepsilon)-s(\varrho_\varepsilon,\vartheta_\varepsilon))(\partial_t\Theta_\varepsilon+\mathbf{U}_\varepsilon\cdot\nabla\Theta_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\Big(\varepsilon^{1/4}+\varepsilon+\frac{1}{\varepsilon M}+\frac{1}{M}+\frac{1}{M^2}\Big)+\frac{\delta}{2(\beta^2+\alpha\delta)}\Big[\int_{\Omega_M}|\alpha R_\varepsilon+\beta T_\varepsilon|^2\,\mathrm{d}x\Big]_0^\tau.
\end{aligned}$$

Step VI. In this section we will handle the magnetic field terms $A_6^\varepsilon, A_\varepsilon^7$. We first compute the first part of A_6^ε. Applying integration by parts yields that

$$\begin{aligned}
(4.34)\quad & \int_0^\tau\int_{\Omega_M}\Big[-(\mathbf{H}_\varepsilon\cdot\nabla)\mathbf{H}_\varepsilon+\frac{1}{2}\nabla|\mathbf{H}_\varepsilon|^2\Big]\cdot\mathbf{U}\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^\tau\int_{\Omega_M}\Big[-(\mathbf{H}_\varepsilon\cdot\nabla)\mathbf{H}_\varepsilon\cdot\mathbf{u}-\frac{1}{2}|\mathbf{H}_\varepsilon|^2\,\mathrm{div}\,\mathbf{u}\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\Omega_M}\Big[(\mathbf{H}_\varepsilon\cdot\nabla)\nabla\Phi_\varepsilon\cdot\mathbf{H}_\varepsilon-\frac{1}{2}|\mathbf{H}_\varepsilon|^2\Delta\Phi_\varepsilon\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\Omega_M}\Big[(\mathbf{H}_\varepsilon\cdot\nabla)\mathbf{w}_M^\varepsilon\cdot\mathbf{H}_\varepsilon-\frac{1}{2}|\mathbf{H}_\varepsilon|^2\,\mathrm{div}\,\mathbf{w}_M^\varepsilon\Big]\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant-\int_0^\tau\int_{\Omega_M}(\mathbf{H}_\varepsilon\cdot\nabla)\mathbf{H}_\varepsilon\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t+C\Big(\varepsilon^{1/4}+\frac{1}{M^2}\Big),
\end{aligned}$$

where we have used the dispersive regularity (2.18) and (2.25).

Adapting $\mathbf{H}$ as a test function to equation (1.5), we obtain that

$$\begin{aligned}
(4.35)\quad & -\int_0^\tau\int_{\Omega_M}(\mathbf{H}_\varepsilon\cdot\mathbf{H}-\mathbf{H}_{0,\varepsilon}\cdot\mathbf{H}_0)\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^\tau\int_{\Omega_M}(\mathbf{H}_\varepsilon\cdot[(\mathbf{u}\cdot\nabla)\mathbf{H}-(\mathbf{H}\cdot\nabla)\mathbf{u}]+\varepsilon^c\nu\nabla\mathbf{H}_\varepsilon\cdot\nabla\mathbf{H})\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\Omega_M}\mathbf{H}\cdot[\mathrm{div}\,\mathbf{u}_\varepsilon\,\mathbf{H}_\varepsilon+(\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{H}_\varepsilon-(\mathbf{H}_\varepsilon\cdot\nabla)\mathbf{u}_\varepsilon]\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

while

$$(4.36)\quad \int_0^\tau\int_{\Omega_M}\varepsilon^c\nu\nabla\mathbf{H}_\varepsilon\cdot\nabla\mathbf{H}\,\mathrm{d}x\,\mathrm{d}t\leqslant\frac{\varepsilon^c\nu}{2}\int_0^\tau\int_{\Omega_M}|\nabla\mathbf{H}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+C\varepsilon^c.$$

Applying integration by parts and using $\operatorname{div}\mathbf{H}_\varepsilon = 0$, $\operatorname{div}\mathbf{H} = 0$, we get

$$
\begin{aligned}
(4.37)\quad & A_\varepsilon^6 + A_\varepsilon^7 - \int_0^\tau\int_{\Omega_M} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot [(\nabla\times\mathbf{H})\times\mathbf{H}]\,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant \int_0^\tau\int_{\Omega_M} (\overline{\varrho}-\varrho_\varepsilon)\mathbf{u}_\varepsilon\cdot(\mathbf{H}\cdot\nabla)\mathbf{H}\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^\tau\int_{\Omega_M} (\varrho_\varepsilon-\overline{\varrho})\mathbf{u}_\varepsilon\cdot\nabla|\mathbf{H}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau\int_{\Omega_M} \overline{\varrho}(\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{u}\cdot(\mathbf{H}_\varepsilon-\mathbf{H})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau\int_{\Omega_M} \overline{\varrho}(\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{H}\cdot(\mathbf{u}_\varepsilon-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\tau\int_{\Omega_M} \overline{\varrho}(\mathbf{u}_\varepsilon-\mathbf{u})\cdot\nabla\mathbf{H}\cdot(\mathbf{H}_\varepsilon-\mathbf{H})\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_{\Omega_M} \overline{\varrho}(\mathbf{u}\cdot\nabla)\mathbf{H}\cdot\mathbf{H}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^\theta + \frac{1}{\varepsilon M}\Big) \\
&:= \sum_{j=1}^{6} D_j + C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^\theta + \frac{1}{\varepsilon M}\Big),
\end{aligned}
$$

where we have used

$$(\nabla\times\mathbf{H})\times\mathbf{H} = (\mathbf{H}\cdot\nabla)\mathbf{H} - \tfrac{1}{2}\nabla|\mathbf{H}|^2.$$

For D_3, we can easily show

$$D_3 \leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t.$$

For the term D_4, from (2.17), (2.18), (3.3), (4.9), (4.11), and (4.13) together with the Sobolev embedding and Hölder's inequality, it follows that

$$
\begin{aligned}
(4.38)\quad & \int_0^\tau\int_{\Omega_M} (\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{H}\cdot(\mathbf{u}_\varepsilon-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau\int_{\Omega_M} (\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{H}\cdot\sqrt{\varrho_\varepsilon}(\mathbf{u}_\varepsilon-\mathbf{U})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau\int_{\Omega_M} \big(\sqrt{\overline{\varrho}}-\sqrt{\varrho_\varepsilon}\big)(\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{H}\cdot(\mathbf{u}_\varepsilon-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau\int_{\Omega_M} (\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{H}\cdot\big(\sqrt{\varrho_\varepsilon}-\sqrt{\overline{\varrho}}\big)(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau\int_{\Omega_M} \sqrt{\overline{\varrho}}(\mathbf{H}_\varepsilon-\mathbf{H})\cdot\nabla\mathbf{H}\cdot(\nabla\Phi_\varepsilon+\mathbf{w}_M^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t := \sum_{j=1}^{4} K_\varepsilon^j.
\end{aligned}
$$

It is easy to show

$$K_\varepsilon^1 + K_\varepsilon^4 \leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^{1/2} + \frac{1}{M^2}\Big).$$

We use the Sobolev imbedding theorem and interpolation of the Lebesgue integral to obtain the estimate of K_ε^2:

$$
\begin{aligned}
K_\varepsilon^2 &= \int_0^\tau\!\!\int_{\Omega_M} (\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon})(\mathbf{H}_\varepsilon - \mathbf{H}) \cdot \nabla \mathbf{H} \cdot (\mathbf{u}_\varepsilon - \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}t \\
&\leqslant \int_0^\tau \|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^2} \|\mathbf{H}_\varepsilon\|_{L^6} \|\nabla \mathbf{H}\|_{L^6} \|\mathbf{u}_\varepsilon\|_{L^6} \,\mathrm{d}t \\
&\quad + C\int_0^\tau \|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^2} \|\mathbf{H}_\varepsilon\|_{L^2} \,\mathrm{d}t + C\int_0^\tau \|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^2} \|\mathbf{H}\|_{L^2} \,\mathrm{d}t \\
&\quad + C\int_0^\tau \|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^2} \|\mathbf{H}\|_{L^6} \|\nabla \mathbf{H}\|_{L^6} \|\mathbf{u}_\varepsilon\|_{L^6} \,\mathrm{d}t \\
&\leqslant C\|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^\infty L^2} \|\nabla \mathbf{H}_\varepsilon\|_{L^2 L^2} \|\nabla \mathbf{u}_\varepsilon\|_{L^2 L^2} \\
&\quad + C\|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^\infty L^2} \|\nabla \mathbf{u}_\varepsilon\|_{L^2 L^2} + C\|\sqrt{\overline{\varrho}} - \sqrt{\varrho_\varepsilon}\|_{L^\infty L^2} \\
&\leqslant C(\varepsilon^{1-(a+c)/2} + \varepsilon^{1-a/2} + \varepsilon).
\end{aligned}
$$

For the term of K_ε^3, we can also use interpolation of the Lebesgue integral and the dispersive estimate (2.18):

$$K_\varepsilon^3 \leqslant C\Big(\varepsilon^{1/2} + \frac{1}{M^2}\Big).$$

Thus, we get

$$D_4 \leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^{1-(a+c)/2} + \varepsilon^{1-a/2} + \varepsilon^{1/2} + \frac{1}{M^2}\Big).$$

Similarly, the term D_5 can also be controlled by

$$D_5 \leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^{1-(a+c)/2} + \varepsilon^{1-a/2} + \varepsilon^{1/2} + \frac{1}{M^2}\Big). \tag{4.39}$$

Finally, we estimate D_1 and can also bound D_2 with the same method. From estimates (4.8), (4.9), (4.10) and (4.13), we get

$$D_1 \leqslant C\|\sqrt{\varrho_\varepsilon} - \sqrt{\overline{\varrho}}\|_{L^\infty(0,T;L^2)} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^2)} \leqslant C\varepsilon^{1-a/2}. \tag{4.40}$$

So, we prove that

$$
\begin{aligned}
A_\varepsilon^6 + A_\varepsilon^7 &- \int_0^\tau\!\!\int_{\Omega_M} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot [(\nabla \times \mathbf{H}) \times \mathbf{H}] \,\mathrm{d}x\,\mathrm{d}t \\
&\leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^{1-(a+c)/2} + \varepsilon^{1-a/2} + \varepsilon^{1/2} + \frac{1}{M^2}\Big) \\
&\leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^\theta + \frac{1}{\varepsilon M}\Big).
\end{aligned} \tag{4.41}
$$

Consequently, we get the following relative entropy:

(4.42)

$$\mathcal{E}_\varepsilon(\tau) + \int_0^\tau\int_{\Omega_M} \Big[\frac{\Theta}{\vartheta}\Big(\varepsilon^a \mathbb{S}(\vartheta, \nabla\mathbf{u}) : \nabla\mathbf{u} - \varepsilon^{b-2}\frac{\mathbf{q}(\vartheta,\nabla\vartheta)\cdot\nabla\vartheta}{\vartheta}\Big) + \varepsilon^c|\nabla\mathbf{H}|^2\Big]\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\int_0^\tau \mathcal{E}_\varepsilon(t)\,\mathrm{d}t + C\Big(\varepsilon^\theta + \frac{1}{\varepsilon M}\Big).$$

Step VII. From (3.6), the initial data part of A_1^ε can be handled as follows:

$$\begin{aligned}
&\big\|\sqrt{\varrho_{0,\varepsilon}}\big(\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 - \nabla\Phi_{0,\varepsilon} - \mathbf{w}_M^{0,\varepsilon}\big)\big\|^2_{L^2(\Omega_M}\\
&\quad\leqslant C\big\|\big(\sqrt{\varrho_{0,\varepsilon}} - \sqrt{\overline{\varrho}}\big)\big(\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 - \nabla\Phi_{0,\varepsilon} - \mathbf{w}_M^{0,\varepsilon}\big)\big\|^2_{L^2(\Omega_M)}\\
&\qquad + C\|\mathbf{u}_{0,\varepsilon} - \widetilde{\mathbf{u}}_0\|^2_{L^2(\Omega_M)} + C\|\nabla\Phi_{0,\varepsilon} - \chi_\delta * \nabla\Phi_{0,\varepsilon}\|_{L^2(\Omega_M)} + C\|\mathbf{w}_M^{0,\varepsilon}\|^2_{L^2(\Omega_M)}\\
&\quad\leqslant C\Big(\varepsilon^2 + \alpha(\varepsilon) + \frac{1}{M^2}\Big) + \chi(\delta)
\end{aligned}$$

with

$$\begin{aligned}
&\lim_{\delta\to 0}\chi(\delta) = 0,\\
&\int_{\Omega_M}\Big[\frac{1}{\varepsilon^2}E(\varrho,\vartheta \mid r,\Theta)\Big]\,\mathrm{d}x\\
&\qquad\leqslant C\|\varrho_{0,\varepsilon}^{(1)} - R_{0,\varepsilon}\|^2_{L^2(\Omega_M)} + C\|\vartheta_{0,\varepsilon}^{(1)} - T_{0,\varepsilon}\|^2_{L^2(\Omega_M)}\\
&\qquad\leqslant C\alpha(\varepsilon) + \chi(\delta),
\end{aligned} \tag{4.43}$$

and

$$\|\mathbf{H}_{0,\varepsilon} - \mathbf{H}_0\|^2_{L^2(\Omega_M)} \leqslant \alpha(\varepsilon).$$

Let us apply Gronwall's inequality to (4.36) in order to obtain

$$\mathcal{E}_\varepsilon(\tau) \leqslant C\Big(\varepsilon^\theta + \alpha(\varepsilon) + \frac{1}{\varepsilon M(\varepsilon)}\Big) \tag{4.44}$$

for any $\tau \in (0, T_*]$, where the number θ is defined in (3.9). The constant depends on

$$\|\nabla\Phi_{0,\varepsilon}\|_{H^{k+2}(\Omega;\mathbb{R}^3)} + \|s_{0,\varepsilon}\|_{H^{k+2}(\Omega;\mathbb{R}^3)}$$

and it is uniformly bounded by a constant number when $\delta\to 0$. This completes the proof of Theorem 3.1. □

References

[1] *B. Desjardins, E. Grenier*: Low Mach number limit of viscous compressible flows in the whole space. Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. *455* (1999), 2271–2279. zbl MR doi

[2] *E. Feireisl, Š. Nečasová, Y. Sun*: Inviscid incompressible limits on expanding domains. Nonlinearity *27* (2014), 2465–2477. zbl MR doi

[3] *E. Feireisl, A. Novotný*: The low Mach number limit for the full Navier-Stokes-Fourier system. Arch. Ration. Mech. Anal. *186* (2007), 77–107. zbl MR doi

[4] *E. Feireisl, A. Novotný*: Singular Limits in Thermodynamics of Viscous Fluids. Advances in Mathematical Fluid Mechanics. Birkhäuser, Basel, 2009. zbl MR doi

[5] *E. Feireisl, A. Novotný*: Inviscid incompressible limits of the full Navier-Stokes-Fourier system. Commun. Math. Phys. *321* (2013), 605–628. zbl MR doi

[6] *X. Hu, D. Wang*: Global solutions to the three-dimensional full compressible magnetohydrodynamic flows. Commun. Math. Phys. *283* (2008), 255–284. zbl MR doi

[7] *D. Jesslé, B. J. Jin, A. Novotný*: Navier-Stokes-Fourier system on unbounded domains: weak solutions, relative entropies, weak-strong uniqueness. SIAM J. Math. Anal. *45* (2013), 1907–1951. zbl MR doi

[8] *S. Jiang, Q. Ju, F. Li*: Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. Commun. Math. Phys. *297* (2010), 371–400. zbl MR doi

[9] *S. Jiang, Q. Ju, F. Li*: Incompressible limit of the compressible magnetohydrodynamic equations with vanishing viscosity coefficients. SIAM J. Math. Anal. *42* (2010), 2539–2553. zbl MR doi

[10] *P.-L. Lions, N. Masmoudi*: Incompressible limit for a viscous compressible fluid. J. Math. Pures Appl. (9) *77* (1998), 585–627. zbl MR doi

[11] *N. Masmoudi*: Incompressible, inviscid limit of the compressible Navier-Stokes system. Ann. Inst. Henri Poincaré, Anal. Non Linéaire *18* (2001), 199–224. zbl MR doi

[12] *F. J. McGrath*: Nonstationary plane flow of viscous and ideal fluids. Arch. Ration. Mech. Anal. *27* (1967), 329–348. zbl MR doi

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