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PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL SYSTEMS BY THE METHOD OF AVERAGING

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Abstract. In many engineering problems, when studying the existence of periodic solutions to a nonlinear system with a small parameter via the local averaging theorem, it is necessary to verify some properties of the fundamental solution matrix to the corresponding linearized system along the periodic solution of the unperturbed system. But sometimes, it is difficult or it requires a lot of calculations. In this paper, a few simple and effective methods are introduced to investigate the existence of periodic solutions for a kind of small parametric systems. In order to prove the existence of periodic solutions by these ideas, we also introduce a forced autoparametric vibrating system and a generalized model of the tuned mass absorber with pendulum discussed by Brzeski, Perlikowski, and Kapitaniak. Then, we also propose an averaging method to study the existence of periodic solutions.

Keywords: periodic solution; local averaging theorem; forced autoparametric vibrating system; tuned mass absorber

MSC 2020: 34C29, 34C25

1. Introduction

In recent years, averaging theory plays an indispensable role in practical applications. Both classical averaging theorem [24] and local averaging theorem [4] proposed by Buică, Françoise, and Llibre have been applied widely for various fields, see e.g. [16], [18], [23], [6], [5], [17]. There are a lot of papers on the local averaging theorem, see [20], [13], [21]. Recently, Llibre and Zhang [22] have proved that the Michelson system has a zero-Hopf bifurcation periodic solution by the local averaging theorem. In [7], Euzebio and Llibre analyse five cases of the Vallis system by the local averaging theorem, the existence and properties of periodic orbits in each case

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are discussed in detail. These works show the important role of the local averaging theorem in practical applications.

The nonlinear system with a small parameter used in the local averaging theorem is

$$\frac{\mathrm{d}X}{\mathrm{d}t} = G_0(t, X) + G_1(t, X)\varepsilon + G_2(t, X, \varepsilon)\varepsilon^2, \tag{1.1}$$

where G_0, $G_1\colon \mathbb{R}\times\Omega \to \mathbb{R}^n$, $G_2\colon \mathbb{R}\times\Omega\times(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are all C^2 smooth functions and T-periodic with respect to t, Ω is an open subset in $\mathbb{R}^n$, $\varepsilon_0 > 0$, and ε is a small parameter. In [15], some smoothness conditions in the local averaging theorem are reduced by the topological degree theory.

It is necessary to determine some properties of the fundamental solution matrix of the corresponding linearized system along the periodic solution of the unperturbed system when we use the local averaging theorem to study the existence of periodic solutions of a nonlinear system with a small parameter. In the papers mentioned above, the fundamental solution matrix of the corresponding linearized system along the periodic solution of the unperturbed system is known. However, in many cases there is no explicit expression for the fundamental solution matrix or it is not easily obtained.

To overcome the difficulties mentioned above, we only consider a special form of system (1.1) as follows:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = F_0(t, x) + F_1(t, x, y)\varepsilon + F_2(t, x, y, \varepsilon)\varepsilon^2, \\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}t} = G_0(t, x, y) + G_1(t, x, y)\varepsilon + G_2(t, x, y, \varepsilon)\varepsilon^2, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ and $F_0\colon \mathbb{R}\times\mathbb{R}^k \to \mathbb{R}^k$, $F_1\colon \mathbb{R}\times\Omega \to \mathbb{R}^k$, $G_0, G_1\colon \mathbb{R}\times\Omega \to \mathbb{R}^{n-k}$, $F_2\colon \mathbb{R}\times\Omega\times(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^k$, $G_2\colon \mathbb{R}\times\Omega\times(-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^{n-k}$, which are all T-periodic C^2 smooth functions with respect to the first variable t, $\varepsilon_0 > 0$, and ε is a parameter. Let $X(t, z, \varepsilon)$ be a solution of system (1.2) with the initial value z.

Note that the existence of periodic solutions of system (1.2) has been studied in [4], Corollary 4, but it is still difficult to verify the conditions when the fundamental solution matrix of the corresponding linearized system along the periodic solution of the unperturbed system does not have an explicit expression. In this paper, we will change the conditions to make the verification possible.

Assume that system (1.2) satisfies the following two conditions:

(F_1) There is a bounded open set $V \subset \mathbb{R}^k$ such that for each $\alpha \in \overline{V}$, the solution $x(t; \alpha)$ of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F_0(t, x) \tag{1.3}$$

with the initial value α is a T-periodic solution.

(F$_2$) There exists a C^2 mapping $\beta_0\colon \overline{V} \to \mathbb{R}^{n-k}$ satisfying $Z = \{z_\alpha = (\alpha, \beta_0(\alpha));\ \alpha \in \overline{V}\} \subset \Omega$. For each $z_\alpha \in Z$, the solution $X(t, z_\alpha, 0)$ of the unperturbed system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = F_0(t,x), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = G_0(t,x,y), \end{cases} \tag{1.4}$$

with the initial value z_α is T-periodic.

The corresponding linearized system along the periodic solution $X(t, z_\alpha, 0)$ of the unperturbed system of system (1.2) is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = D_x F_0(t, x(t;\alpha))u, \tag{1.5}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = D_x G_0(t, X(t, z_\alpha, 0))u + D_y G_0(t, X(t, z_\alpha, 0))v. \tag{1.6}$$

We also consider the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = D_y G_0(t, X(t, z_\alpha, 0))v. \tag{1.7}$$

Let $M_\alpha(t)$ be a fundamental solution matrix of (1.5) ($M_\alpha(0) = E_k$) and let $\Phi_\alpha(t)$ ($\Phi_\alpha(0) = E_{n-k}$) be a fundamental solution matrix of (1.7), where E_i is the ith order identity matrix. Define the function $F\colon \overline{V} \to \mathbb{R}^k$ by

$$F(\alpha) = \int_0^T M_\alpha^{-1}(t) F_1(t, X(t, z_\alpha, 0))\,\mathrm{d}t. \tag{1.8}$$

The following Picard approximation principle [27] is used when we consider approximate solutions of (1.7).

Consider the differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + f(t) \quad (a \leqslant t \leqslant b), \tag{1.9}$$

where $A(t)$ is a continuous and differentiable $n \times n$ matrix function, $f(t)$ is a continuous and differentiable n-dimensional column vector function. By the following recursive operation

$$\begin{cases} \varphi_0(t) = \eta, & a \leqslant t \leqslant b, \\ \varphi_n(t) = \eta + \displaystyle\int_{t_0}^t [A(s)\varphi_{n-1}(s) + f(s)]\,\mathrm{d}s, & a \leqslant t \leqslant b, \end{cases} \tag{1.10}$$

one obtains the nth order approximate solution $\varphi_n(t)$ of the differential system (1.9) with the initial value η, which satisfies the estimate

$$\|\varphi_n(t) - \phi(t)\| \leqslant \frac{ML^nT^{n+1}}{(n+1)!} \quad (n = 1, 2, \ldots), \tag{1.11}$$

where $\phi(t)$ is a solution of the differential system (1.9) with the initial value η,

$$\max_{a\leqslant t\leqslant b} \|A(t)\| \leqslant L, \quad \max_{a\leqslant t\leqslant b} \|f(t)\| \leqslant K, \quad M = L\|\eta\| + K, \quad T = b - a.$$

For the convenience of narration, we just consider the case of $k = n - 2$; the general case can be discussed it similarly. Let us write the fundamental solution matrix $\Phi_\alpha(t)$ of (1.7) in the form of block matrices as follows:

$$\Phi_\alpha(t) = \begin{pmatrix} d_1(t;\alpha) & h_1(t;\alpha) \\ d_2(t;\alpha) & h_2(t;\alpha) \end{pmatrix},$$

where we use $\begin{pmatrix} d_{1m}(t;\alpha) \\ d_{2m}(t;\alpha) \end{pmatrix}$, $\begin{pmatrix} h_{1m}(t;\alpha) \\ h_{2m}(t;\alpha) \end{pmatrix}$ to represent the mth order approximate solution of $\begin{pmatrix} d_1(t;\alpha) \\ d_2(t;\alpha) \end{pmatrix}$, $\begin{pmatrix} h_1(t;\alpha) \\ h_2(t;\alpha) \end{pmatrix}$ in the interval $[0, T]$, respectively, and

$$M = L = \max_{t\in[0,T],\alpha\in\overline{V}} \{D_yG_0(t, X(t, z_\alpha, 0))\}.$$

Now we state one of our results.

Theorem 1.1. *Let $k = n - 2$ and let F_i, G_i $(i = 0, 1, 2)$ be C^2 smooth functions and T-periodic with respect to t. Assume that conditions* (F_1) *and* (F_2) *hold and* $\mathrm{tr}(D_yG_0(t, X(t, z_\alpha, 0))) \equiv 0$. *If there exists a positive integer m such that*

$$d_{1m}(T;\alpha) + h_{2m}(T;\alpha) + \frac{2ML^mT^{m+1}}{(m+1)!} < 2 \tag{1.12}$$

holds for any $\alpha \in \overline{V}$ or

$$d_{1m}(T;\alpha) + h_{2m}(T;\alpha) - \frac{2ML^mT^{m+1}}{(m+1)!} > 2 \tag{1.13}$$

holds for any $\alpha \in \overline{V}$, and there exists $\alpha_0 \in V$ such that

$$F(\alpha_0) = 0, \quad \det(D_\alpha F(\alpha_0)) \neq 0,$$

then, for every sufficiently small $\varepsilon > 0$, system (1.2) *has at least a T-periodic solution $(x(t;\varepsilon), y(t;\varepsilon))$, which satisfies*

$$\lim_{\varepsilon\to 0}(x(0;\varepsilon), y(0;\varepsilon)) = (\alpha_0, \beta_0(\alpha_0)).$$

R e m a r k 1.1. When m is large enough, more general conditions for Theorem 1.1 can be obtained. At the same time, we conclude that the approximation method must be able to solve this kind of difficult problem if there are some conditions on the existence of periodic solutions of the nonlinear system. Especially, if the period T is small enough, generally, we can obtain some results by the second-order approximations.

In fact, when the fundamental solution matrix of the unperturbed system (1.4) is difficult to be obtained, it is not easy to find β_0 for (F_2). For example, for the unperturbed system of a nonlinear system with a small parameter as follows:

$$\begin{cases} x_1' = -x_2, \\ x_2' = x_1, \\ y_1' = (x_1 - x_2)\cos^2 t + \frac{1}{2}(x_1 + x_2)\sin(2t) - \frac{1}{2}y_1 \sin(2t) + y_2 \sin^2 t, \\ y_2' = \frac{1}{2}(x_2 - x_1)\sin(2t) - (x_1 + x_2)\sin^2 t - y_1 \cos^2 t + \frac{1}{2} y_2 \sin(2t), \end{cases} \tag{1.14}$$

it is hard to construct the mapping $\beta_0\colon \mathbb{R}^2 \to \mathbb{R}^2$. Here we have a corollary of Theorem 1.1, but we need to propose a stronger condition (F_3) instead of (F_2).

(F_3) There exists a mapping $\varphi\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$ such that $\varphi(x(t;\alpha))$ is a T-periodic solution of $\mathrm{d}y/\mathrm{d}t = G_0(t, x(t;\alpha), y)$ for any $\alpha \in \overline{V}$, and φ is a C^2 smooth function in $\overline{V}$.

The associated linearized system along the periodic solution $(x(t;\alpha), \varphi(x(t;\alpha)))$ of the unperturbed system (1.4) is

$$\frac{\mathrm{d}u}{\mathrm{d}t} = D_x F_0(t, x(t;\alpha))u, \tag{1.15}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = D_x G_0(t, x(t;\alpha), \varphi(x(t;\alpha)))u + D_y G_0(t, x(t;\alpha), \varphi(x(t;\alpha)))v. \tag{1.16}$$

Similarly, we also consider the equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = D_y G_0(t, x(t;\alpha), \varphi(x(t;\alpha)))v. \tag{1.17}$$

Let $M_\alpha(t)$ be a fundamental solution matrix of (1.15) ($M_\alpha(0) = E_k$) and let $\Phi_\alpha(t)$ be a fundamental solution matrix of (1.17) ($\Phi_\alpha(0) = E_{n-k}$), where E_i denotes the ith order identity matrix. Define the function $F\colon \overline{V} \to \mathbb{R}^k$ by

$$F(\alpha) = \int_0^T M_\alpha^{-1}(t) F_1(t, x(t;\alpha), \varphi(x(t;\alpha)))\,\mathrm{d}t. \tag{1.18}$$

Notice that we first need to find the function φ. Here we use a polynomial-undetermined-coefficient method to construct φ, that is,

$$\varphi'_x(x(t;\alpha))F_0(t;x(t;\alpha)) = G_0(t,x(t;\alpha),\varphi(x(t;\alpha))) \tag{1.19}$$

according to (F_3), $\varphi(x_1,x_2,\ldots,x_k)$ is some undetermined-coefficient polynomial with $x_1,x_2,\ldots,x_k$ $(x_i\in\mathbb{R},\ i=1,\ldots,k)$, then, we look for a group of appropriate coefficients. For example, we find φ: $(x_1,x_2)\mapsto(x_1+x_2,x_1-x_2)$ by this method for the equation (1.14) mentioned above, where

$$(x_1,x_2)=(x_{10}\cos t - x_{20}\sin t, x_{10}\sin t - x_{20}\cos t),$$

x_{10} and x_{20} are constants.

Corollary 1.1. *Let $k=n-2$ and let F_i, G_i $(i=0,1,2)$ be C^2 smooth functions and T-periodic with respect to t. Assume that (F_1) and (F_3) hold and $\mathrm{tr}(D_yG_0(t,x(t;\alpha),\varphi(x(t;\alpha))))\equiv 0$. If there exists a positive integer m such that*

$$d_{1m}(T;\alpha)+h_{2m}(T;\alpha)+\frac{2ML^mT^{m+1}}{(m+1)!}<2 \tag{1.20}$$

or

$$d_{1m}(T;\alpha)+h_{2m}(T;\alpha)-\frac{2ML^mT^{m+1}}{(m+1)!}>2 \tag{1.21}$$

for any $\alpha\in\overline{V}$, and there exists $\alpha_0\in V$ such that

$$F(\alpha_0)=0,\quad \det(D_\alpha F(\alpha_0))\neq 0,$$

then for every sufficiently small $\varepsilon>0$ system (1.2) has at least a T-periodic solution $(x(t;\varepsilon),y(t;\varepsilon))$ which satisfies

$$\lim_{\varepsilon\to 0}(x(0;\varepsilon),y(0;\varepsilon))=(\alpha_0,\varphi(\alpha_0)).$$

Corollary 1.2. *Let F_i,G_i $(i=0,1,2)$ be C^2 smooth functions and T-periodic with respect to t. Assume that (F_1) and (F_2) hold, and the linearized system (1.5)–(1.6) has no nontrivial T-periodic solutions for any $\alpha\in\overline{V}$. If there exists a $\alpha_0\in V$ such that*

$$F(\alpha_0)=0,\quad \det(D_\alpha F(\alpha_0))\neq 0,$$

then for every sufficiently small $\varepsilon>0$, system (1.2) has at least a T-periodic solution $(x(t;\varepsilon),y(t;\varepsilon))$ such that

$$\lim_{\varepsilon\to 0}(x(0;\varepsilon),y(0;\varepsilon))=(\alpha_0,\beta_0(\alpha_0)).$$

In order to reduce calculations, the following result is obtained.

Theorem 1.2. *Assume that F_i, G_i $(i = 0, 1, 2)$ are C^2 smooth functions and T-periodic with respect to t. Let $X(t, z, \varepsilon)$ be the solution of system (1.2) with the initial value z and let $\beta_0\colon \overline{V} \to \mathbb{R}^{n-k}$ be a C^2 mapping with a bounded open subset V in $\mathbb{R}^k$. We assume the following conditions hold.*

(1) *Let $Z = \{z_\alpha = (\alpha, \beta_0(\alpha));\ \alpha \in \overline{V}\} \subset \Omega$. For each $z_\alpha \in Z$, the solution $X(t, z_\alpha, 0)$ of the unperturbed system (1.4) is T-periodic.*

(2) *Assume that*

$$D_y G_0(t, X(t, z_\alpha, 0)) = D^{-1}(t, \alpha) J(t, \alpha) D(t, \alpha) + C(t, \alpha),$$
$$D(t, \alpha) C(t, \alpha) + \frac{\partial D}{\partial t}(t, \alpha) = G(t, \alpha) D(t, \alpha),$$

where $J(t, \alpha)$ and $G(t, \alpha)$ are all diagonal matrices (or all irrelevant to t) and T-periodic with respect to t, $D(t, \alpha)$ and $C(t, \alpha)$ are also T-periodic with respect to t. For any $j \in \{1, 2, \ldots, n-k\}$ and $\alpha \in \overline{V}$, $\lambda_j(\alpha) \neq 2l\pi\mathrm{i}$ $(l \in \mathbb{Z})$, where $\lambda_j(\alpha)$ are eigenvalues of $\int_0^T [J(s, \alpha) + G(s, \alpha)]\,\mathrm{d}s$.

Consider the function $F\colon \overline{V} \to \mathbb{R}^k$ defined by

$$F(\alpha) = \int_0^T M_\alpha^{-1}(t) F_1(t, X(t, z_\alpha, 0))\,\mathrm{d}t,$$

where $M_\alpha(t)$ is the fundamental solution matrix of

$$\frac{\mathrm{d}u}{\mathrm{d}t} = D_x F_0(t, X(t, z_\alpha, 0))u \quad (\alpha \in \overline{V}).$$

If there exists $\alpha_0 \in V$ such that

$$F(\alpha_0) = 0, \quad \det(D_\alpha F(\alpha_0)) \neq 0,$$

then, for every sufficiently small ε, system (1.2) has at least a T-periodic solution $X(t, \varepsilon)$ such that

$$\lim_{\varepsilon \to 0} X(0, \varepsilon) = z_{\alpha_0}.$$

Remark 1.2. When $D_y G_0(t, X(t, z_\alpha, 0))$ is a diagonal matrix or matrix without variable t, the calculations in the proof of Theorem 1.2 can be reduced.

Let $X(t,\alpha,\varepsilon)$ be the solution of system (1.1) with the initial value α and V be a bounded open subset in $\mathbb{R}^n$. As a supplement for the local averaging theorem [4], we have a trivial result as follows:

Theorem 1.3. *Assume that system* (1.1) *satisfies the following conditions:*

(1) *For each $\alpha \in \overline{V}$, $X(t,\alpha,0)$ is a T-periodic solution of the unperturbed system*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = G_0(t,X) \tag{1.22}$$

of system (1.1).

(2) *Define the function $\chi\colon \overline{V} \to \mathbb{R}^n$ by*

$$\chi(\alpha) = \int_0^T M_\alpha^{-1}(t) G_1(t, X(t,\alpha,0))\,\mathrm{d}t, \quad \alpha \in \overline{V}, \tag{1.23}$$

and the corresponding linearized system along the periodic solution $X(t,\alpha,0)$ of the unperturbed system (1.22) *is given by*

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = D_X G_0(t, X(t,\alpha,0))Y. \tag{1.24}$$

We denote by $M_\alpha(t)$ the fundamental solution matrix of (1.24). *We assume that there exists an $a \in V$, which satisfies $\chi(a) = 0$ and*

$$\det\Big(\frac{\mathrm{d}}{\mathrm{d}\alpha}\chi(\alpha)\Big|_{\alpha=a}\Big) \neq 0.$$

Then, for every sufficiently small ε, system (1.1) *has at least one T-periodic solution $X(t,\varepsilon)$, which satisfies $\lim\limits_{\varepsilon\to 0} X(0,\varepsilon) = a$.*

Remark 1.3. We remark that Theorem 1.3 is not contained in Theorem 1.3.6 of [19], which is due to the fact that

$$\int_0^T G_1(t, X(t,\alpha,0))\,\mathrm{d}t$$

and

$$\int_0^T M_\alpha^{-1}(t) G_1(t, X(t,\alpha,0))\,\mathrm{d}t$$

are not equivalent. When

$$\det\Big(\frac{\mathrm{d}}{\mathrm{d}\alpha}\int_0^T G_1(t, X(t,\alpha,0))\,\mathrm{d}t\Big) \equiv 0,$$

for the first-order average, Theorem 1.3.6 in [19] will be no longer suitable for application. Theorem 1.3 is a degenerate form of the local averaging theorem [4], and it cannot be proved by the method used to prove the local averaging theorem. In this paper, we will give a concise proof of Theorem 1.3 by the local averaging theorem and study comprehensively the existence of a periodic solution of a TMA with n pendulums by Theorem 1.3. Especially, when $G_0(t,X) \equiv 0$, we can choose $M_\alpha(t) \equiv E_n$, and obtain that the classical averaging theorem [24] is a special form of Theorem 1.3.

2. The proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3

First, we prove that the linearized system (1.5)–(1.6) has a fundamental solution matrix

$$M_{z_\alpha}(t) = \begin{pmatrix} M_\alpha(t) & 0 \\ C(t) & \Phi_\alpha(t) \end{pmatrix}.$$

Denote the general solution of system (1.5)–(1.6) with the initial value $(x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ by $(x(t,x_0,y_0), y(t,x_0,y_0))$. Let $(x_0,y_0) = I_j$ $(j = 1,2,\ldots,k)$, where I_j represents the n-dimensional vector and only the first j coordinate are 1 and the remaining coordinates are all zero. Therefore, the k solutions of system (1.5)–(1.6) are obtained sequentially. It is obvious that the $n \times (n-k)$ matrix composed of these column vectors can be expressed by $\begin{pmatrix} M_\alpha(t) \\ C(t) \end{pmatrix}$. We also prove that every column vector of $\begin{pmatrix} 0 \\ \Phi_\alpha(t) \end{pmatrix}$ is a solution of system (1.5)–(1.6). Then, a fundamental solution matrix $\begin{pmatrix} M_\alpha(t) & 0 \\ C(t) & \Phi_\alpha(t) \end{pmatrix}$ of system (1.5)–(1.6) is obtained, and also it is easy to obtain its inverse matrix

$$H_\alpha(t) = \begin{pmatrix} M_\alpha^{-1}(t) & 0 \\ -\Phi_\alpha^{-1}(t)C(t)M_\alpha^{-1}(t) & \Phi_\alpha^{-1}(t) \end{pmatrix}.$$

Obviously, the right upper corner $k \times k$ matrix of $H_\alpha(T) - H_\alpha(0)$ is a zero matrix, thus, we only need to verify that the determinant of the right lower $(n-k)\times(n-k)$ matrix $\Phi_\alpha^{-1}(T) - \Phi_\alpha^{-1}(0) = \Phi_\alpha^{-1}(T) - E_{n-k}$ of $H_\alpha(T) - H_\alpha(0)$ is not zero. As we know from matrix theory, the proof of $\det(\Phi_\alpha^{-1}(T) - E_{n-k}) \neq 0$ is equivalent to proving that 1 is not an eigenvalue of $\Phi_\alpha^{-1}(T)$ for any $\alpha \in \overline{V}$, that is, proving that 1 is not an eigenvalue of $\Phi_\alpha(T)$ for any $\alpha \in \overline{V}$.

P r o o f of Theorem 1.1. Assume that 1 is an eigenvalue of $\Phi_\alpha(T)$, and we can obtain

$$1 - (d_1(T;\alpha) + h_2(T;\alpha)) + \det(\Phi_\alpha(T)) = 0.$$

Then by Liouville's formula, we obtain

$$\det(\Phi_\alpha(T)) = \exp\biggl(\int_0^T \operatorname{tr}(D_y G_0(s, X(s, z_\alpha, 0)))\,\mathrm{d}s\biggr) = 1.$$

Thus, we have

$$d_1(T;\alpha) + h_2(T;\alpha) = 2.$$

According to the Picard approximation principle, we obtain

$$\begin{aligned} d_{1m}(T;\alpha) + h_{2m}(T;\alpha) - \frac{2ML^m T^{m+1}}{(m+1)!} &\leqslant d_1(T;\alpha) + h_2(T;\alpha) \\ &\leqslant d_{1m}(T;\alpha) + h_{2m}(T;\alpha) + \frac{2ML^m T^{m+1}}{(m+1)!}. \end{aligned}$$

By condition (1.12) (or (1.13)), we know that $d_1(t;\alpha) + h_2(t;\alpha) \neq 2$. Therefore, 1 is not an eigenvalue of $\Phi_\alpha(T)$ for any $\alpha \in \overline{V}$.

We denote the function $\chi\colon \overline{V} \to \mathbb{R}^k$ by

$$\chi(\alpha) = P\biggl(\int_0^T M_{z_\alpha}^{-1}(t) \begin{pmatrix} F_1(t, X(t, z_\alpha, 0)) \\ G_1(t, X(t, z_\alpha, 0)) \end{pmatrix} \mathrm{d}t\biggr) \quad (\alpha \in \overline{V}).$$

Obviously,

$$\chi(\alpha) = F(\alpha).$$

Theorem 1.1 can be proved by the local averaging theorem under the condition that there is $\alpha_0 \in V$ such that $F(\alpha_0) = 0$ and $\det(D_\alpha F(\alpha_0)) \neq 0$. Then we complete the proof. $\square$

R e m a r k 2.1. To prove Corollary 1.1, we construct a C^2 smooth mapping $\beta_0\colon \overline{V} \to \mathbb{R}^{n-k}$, that is, $\beta_0(\alpha) = \varphi(\alpha), \alpha \in \overline{V}$. Let $Z = \{z_\alpha = (\alpha, \beta_0(\alpha));\ \alpha \in \overline{V}\}$, then $Z \subset \Omega$ according to the condition $\{(\alpha, \varphi(\alpha));\ \alpha \in \overline{V}\} \subset \Omega$. For all $z_\alpha \in Z$, by the conditions (F_1) and (F_3) we obtain that the solution of the unperturbed system (1.4) with the initial value z_α is T-periodic. Finally, we prove Corollary 1.1 using the method in the proof of Theorem 1.1.

R e m a r k 2.2. To prove Corollary 1.2, obviously, $w(t;\eta) = \Phi_\alpha(t)\eta$ is the solution of the linearized system (1.5)–(1.6) with the initial value η, and $w(T;\eta) - w(0;\eta) = (\Phi_\alpha(T) - \Phi_\alpha(0))\eta$. We know that the equation $(\Phi_\alpha(T) - E_{n-k})\eta = 0$ has no nontrivial solutions according to the condition of Corollary 1.2. Therefore, we have $\det(\Phi_\alpha(T) - \Phi_\alpha(0)) \neq 0$.

P r o o f of Theorem 1.2. Similar to the proof of Theorem 1.1, the key of proving Theorem 1.2 is that 1 is not an eigenvalue of $\Phi_\alpha(T)$ for any $\alpha \in \overline{V}$, where $\Phi_\alpha(t)$ ($\Phi_\alpha(0) = E_{n-k}$) is some fundamental solution matrix of the system $\mathrm{d}v/\mathrm{d}t = D_y G_0(t, X(t, z_\alpha, 0))v$. Making the transformation $z = D(t,\alpha)v$, and by the equalities

$$D_y G_0(t, X(t, z_\alpha, 0)) = D^{-1}(t,\alpha)J(t,\alpha)D(t,\alpha) + C(t,\alpha),$$
$$D(t,\alpha)C(t,\alpha) + \frac{\partial D}{\partial t}(t,\alpha) = G(t,\alpha)D(t,\alpha),$$

we know that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = D_y G_0(t, X(t, z_\alpha, 0))v$$

is changed into

$$\frac{\mathrm{d}z}{\mathrm{d}t} = [J(t,\alpha) + G(t,\alpha)]z.$$

Because $J(t,\alpha)$ and $G(t,\alpha)$ are all diagonal matrices or irrelevant to t, we obtain that the fundamental solution matrix of the system $\mathrm{d}z/\mathrm{d}t = [J(t,\alpha) + G(t,\alpha)]z$ is $\exp(\int_0^t [J(s,\alpha) + G(s,\alpha)]\,\mathrm{d}s)$. Then, the fundamental solution matrix is given by

$$\Phi_\alpha(t) = D^{-1}(t,\alpha)\exp\biggl(\int_0^t [J(s,\alpha) + G(s,\alpha)]\,\mathrm{d}s\biggr)D(0,\alpha).$$

Notice that $D^{-1}(t,\alpha)$ is T-periodic. Then we have

$$\Phi_\alpha(T) = D^{-1}(0,\alpha)\exp\biggl(\int_0^T [J(s,\alpha) + G(s,\alpha)]\,\mathrm{d}s\biggr)D(0,\alpha).$$

Thus, the proof that 1 is not an eigenvalue of $\Phi_\alpha(T)$ for any $\alpha \in \overline{V}$ is changed into the proof that 1 is not an eigenvalue of $\exp(\int_0^t [J(s,\alpha) + G(s,\alpha)]\,\mathrm{d}s)$ for any $\alpha \in \overline{V}$. Obviously, we complete the proof using condition (2). $\square$

P r o o f of Theorem 1.3. Consider the following system

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = G_0(t,X) + \varepsilon G_1(t,X) + \varepsilon^2 G_2(t,X,\varepsilon), \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = w, \end{cases} \tag{2.1}$$

where $w\colon \mathbb{R} \to \mathbb{R}$. It is easy to see that system (2.1) is still a T-periodic differential system with respect to t, and the equation $\mathrm{d}w/\mathrm{d}t = w$ has only a T-periodic solution $w = 0$.

The unperturbed system of system (2.1) is

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}t} = G_0(t, X), \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = w. \end{cases} \tag{2.2}$$

We construct a C^2 smooth function $\beta_0 \colon \overline{V} \to \{0\} \subset \mathbb{R}$, and

$$Z = \{z_\alpha = (\alpha, \beta_0(\alpha));\ \alpha \in \overline{V}\} \subset \Omega \times \mathbb{R}.$$

According to condition (1) of Theorem 1.3, we know that the solution $(X(t, \alpha, 0), 0)$ of the unperturbed system (2.2) with the initial value $z_\alpha \in Z$ is T-periodic. It is easy to obtain that some fundamental solution matrix of the corresponding linearized system

$$\begin{cases} \dfrac{\mathrm{d}Y}{\mathrm{d}t} = D_X G_0(t, X(t, \alpha, 0))Y, \\ \dfrac{\mathrm{d}w}{\mathrm{d}t} = w, \end{cases} \tag{2.3}$$

along the periodic solution $(X(t, \alpha, 0), 0)$ of the unperturbed system (2.2) is

$$\Phi_{z_\alpha}(t) = \begin{pmatrix} M_\alpha(t) & 0 \\ 0 & \mathrm{e}^t \end{pmatrix}.$$

Obviously, the right upper corner of the matrix $\Phi_{z_\alpha}(T) - \Phi_{z_\alpha}(0)$ is the $n \times 1$ zero matrix and the right lower $\mathrm{e}^{-T} - 1$ satisfies $\mathrm{e}^{-T} - 1 \neq 0$.

It's easy to obtain that

$$P \int_0^T \Phi_{z_\alpha}^{-1}(t) \begin{pmatrix} G_1(t, (X(t, \alpha, 0), 0)) \\ 0 \end{pmatrix} \mathrm{d}t = \int_0^T M_\alpha^{-1}(t) G_1(t, X(t, \alpha, 0))\, \mathrm{d}t = \chi(\alpha).$$

Since $a \in V$, $\chi(a) = 0$ and $\det(\mathrm{d}/\mathrm{d}\alpha)\chi(\alpha)|_{\alpha=a} \neq 0$, by the local averaging theorem, system (2.1) has at least a T-periodic solution $(X(t, \varepsilon), 0)$ for each sufficiently small $\varepsilon > 0$, which satisfies that $(X(0, \varepsilon), 0) \to z_a = (a, 0)$ as $\varepsilon \to 0$. It is easy to see that $X(t, \varepsilon)$ is a T-periodic solution of system (1.1), and $\lim\limits_{\varepsilon \to 0} X(0, \varepsilon) = a$. This completes the proof of Theorem 1.3. $\square$

In the following, we give two practical examples to illustrate our main results.

3.1. Periodic motions of a forced autoparametric vibrating system. There are lots of papers related to forced autoparametric vibrating systems, see [8], [2], [1]. The following system is based on the forced autoparametric vibrating system in [14] and the appropriate frequency modulation is carried out:

$$
(3.1)\qquad \begin{cases} \eta'' + \dfrac{2\zeta_1}{p(1+R)}\eta' + \dfrac{1}{p^2(1+R)}\eta - \dfrac{R}{1+R}(\theta'' \sin\theta + \theta'^2 \cos\theta) \\ \qquad = \dfrac{F}{p^2(1+R)} \cos\Big(\dfrac{1}{p\sqrt{1+R}}\tau\Big), \\ \theta'' + \dfrac{2\zeta_2 q}{p\sqrt{1+R}}\theta' + \Big(\dfrac{q^2}{p^2(1+R)} - \eta''\Big) \sin\theta \\ \qquad = \dfrac{F_0}{p^2\sqrt{(1+R)R}} \cos\Big(\dfrac{1}{p\sqrt{1+R}}\tau\Big), \end{cases}
$$

where

$$
\tau = \omega t, \quad \eta = \frac{x}{l}, \quad R = \frac{m}{M}, \quad F = \frac{R_0}{k_1 l}, \quad p = \frac{w}{\Omega_1}, \quad \Omega_1 = \sqrt{\frac{k_1}{M}},
$$
$$
q = \frac{w_2}{w_1}, \quad w_1 = \sqrt{\frac{k_1}{M+m}}, \quad \omega_2 = \sqrt{\frac{g}{l}}, \quad \zeta_1 = \frac{c_1}{2M\Omega_1}, \quad \zeta_2 = \frac{c_2}{2ml^2 w_2},
$$

M, m are the masses, l is the pendulum's length, R is the mass ratio of the two bodies, Ω_1 is the natural frequency of the primary mass system, ω_1 is the frequency of the so-called "locked pendulum" system, ω_2 is the natural frequency of the pendulum alone, q is the ratio of the two linear natural frequencies of the combined system, p is the frequency ratio specifying the excitation frequency, and a prime denotes the derivative with respect to the nondimensional time τ.

In order to study small motions around the static equilibrium position of the unforced system, we introduce a small parameter ε $(0 < \varepsilon \ll 1)$ such that

$$
\eta = \widehat{\eta}, \quad \theta = \overline{\theta}\sqrt{\varepsilon}, \quad \zeta_1 = \overline{\zeta}_1 \varepsilon, \quad \zeta_2 = \overline{\zeta}_2 \varepsilon, \quad F = \widehat{F}\varepsilon, \quad F_0 = \widehat{F}_0 \varepsilon^2,
$$
$$
\widehat{\theta} = \frac{1}{2}\sqrt{\frac{R}{2(1+R)}}\overline{\theta}, \quad \widehat{\zeta}_1 = p\overline{\zeta}_1, \quad \widehat{\zeta}_2 = \frac{1}{2}\overline{\zeta}_2.
$$

Then, system (3.1) is transformed into the following system:

$$
(3.2)\quad \begin{cases}
\widehat{\eta}'' = & -\dfrac{1}{p^2(1+R)}\widehat{\eta} + \Big(\dfrac{\widehat{F}}{p^2(1+R)}\cos\Big(\dfrac{1}{p\sqrt{1+R}}\tau\Big) - \dfrac{8q^2}{p^2(1+R)}\widehat{\theta}^2 \\
& -\dfrac{8}{p^2(1+R)}\widehat{\theta}^2\widehat{\eta} + 8(\widehat{\theta}')^2 - \dfrac{2\widehat{\zeta}_1}{p^2(1+R)}\widehat{\eta}'\Big)\varepsilon + \mathcal{O}(\varepsilon^2), \\
\widehat{\theta}'' = & -\widehat{\theta}\Big(\dfrac{q^2}{p^2(1+R)} + \dfrac{1}{p^2(1+R)}\widehat{\eta}\Big) + \Big(\dfrac{\widehat{F}}{p^2(1+R)}\widehat{\theta}\cos\Big(\dfrac{1}{p\sqrt{1+R}}\tau\Big) \\
& -\dfrac{4q\widehat{\zeta}_2}{p\sqrt{1+R}}\widehat{\theta}' - \dfrac{20q^2}{3p^2(1+R)}\widehat{\theta}^3 + \dfrac{4q^2}{3p^2R(1+R)}\widehat{\theta}^3 - \dfrac{20}{3p^2(1+R)}\widehat{\theta}^3\widehat{\eta} \\
& + \dfrac{4}{3p^2R(1+R)}\widehat{\theta}^3\widehat{\eta} + 8\widehat{\theta}(\widehat{\theta}')^2 - \dfrac{2\widehat{\zeta}_1}{p^2(1+R)}\widehat{\theta}\widehat{\eta}'\Big)\varepsilon + \mathcal{O}(\varepsilon^2).
\end{cases}
$$

Theorem 3.1. *Assume that the following inequality holds:*

$$
(3.3)\qquad \sqrt{p^2(1+R) + \frac{q^4}{p^2(1+R)}} < \sqrt[3]{\frac{3q^2}{2\pi}}
$$

and $\widehat{F}/\widehat{\zeta}_1$ is small enough (see Remark 3.1), then for each sufficiently small $\varepsilon > 0$, system (3.2) has at least a $T = 2\pi p\sqrt{1+R}$-periodic solution $(\widehat{\eta}(\tau,\varepsilon), \widehat{\theta}(\tau,\varepsilon))$ such that

$$
\lim_{\varepsilon\to 0}(\widehat{\eta}(0,\varepsilon), \widehat{\eta}'(0,\varepsilon), \widehat{\theta}(0,\varepsilon), \widehat{\theta}'(0,\varepsilon)) = \Big(0, \frac{\widehat{F}}{2\widehat{\zeta}_1}, 0, 0\Big).
$$

Remark 3.1. In [14], we described a system which is the same as (3.1) (up to a scaling factor $(p\sqrt{1+R})^{-1}$), and the existence of a periodic solution for it is obtained by a direct application of the local average theorem in [4] because the form of the fundamental matrix for the linearized system is known. However, in the present case, the second equation (3.8) of the linearized system for the average system (3.5) is a non-autonomous Hill equation and the fundamental matrix is unknown. To overcome this difficulty, we will apply Theorem 1.1 to prove Theorem 3.1.

Moreover, we have the estimate

$$
\frac{\widehat{F}}{\widehat{\zeta}_1} < \frac{2p\sqrt{(3/(2\pi))^{2/3}q^{4/3}(1+R) - p^2(1+R)^2} - 2q^2}{\sqrt{1+p^2(1+R)}}.
$$

Furthermore, the interval estimate of the period is $T \in (0, 2/\sqrt{3})$ under condition (3.3). For the process of choosing p, q, R by condition (3.3), see Fig. 1, where

$$\delta_1(q) = \frac{1}{2}\Big(\frac{3q^2}{2\pi}\Big)^{2/3} - \frac{1}{2}\sqrt{\Big(\frac{3q^2}{2\pi}\Big)^{4/3} - 4q^4} \quad \Big(0 < q < \frac{3\sqrt{2}}{8\pi}\Big),$$
$$\delta_2(q) = \frac{1}{2}\Big(\frac{3q^2}{2\pi}\Big)^{2/3} + \frac{1}{2}\sqrt{\Big(\frac{3q^2}{2\pi}\Big)^{4/3} - 4q^4} \quad \Big(0 < q < \frac{3\sqrt{2}}{8\pi}\Big).$$

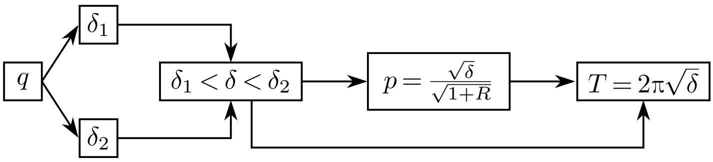

Figure 1. The process of choosing p, q, R by condition (3.3).

Similar to the transformation in [1], we also introduce a small parameter ε ($0 < \varepsilon \ll 1$) such that

$$\eta = \widehat{\eta}\varepsilon, \quad \theta = \overline{\theta}\varepsilon, \quad \zeta_1 = \overline{\zeta}_1\varepsilon, \quad \zeta_2 = \overline{\zeta}_2\varepsilon, \quad F = \widehat{F}\varepsilon^2, \quad F_0 = \widehat{F}_0\varepsilon^2,$$
$$\widehat{\theta} = \frac{1}{2}\sqrt{\frac{R}{2(1+R)}}\overline{\theta}, \quad \widehat{\zeta}_1 = p\overline{\zeta}_1, \quad \widehat{\zeta}_2 = \frac{1}{2}\overline{\zeta}_2.$$

Then system (3.1) can be rewritten as

$$\text{(3.4)} \qquad \begin{cases} \widehat{\eta}'' = -\dfrac{1}{p^2(1+R)}\widehat{\eta} + \Big(\dfrac{\widehat{F}}{p^2(1+R)}\cos\Big(\dfrac{1}{p\sqrt{1+R}}\tau\Big) - \dfrac{2\widehat{\zeta}_1}{p^2(1+R)}\widehat{\eta}' \\ \qquad\quad - \dfrac{8q^2}{p^2(1+R)}\widehat{\theta}^2 + 8\widehat{\theta}'^2\Big)\varepsilon + \mathcal{O}(\varepsilon^2), \\ \widehat{\theta}'' = -\dfrac{q^2}{p^2(1+R)}\widehat{\theta} + \Big(-\dfrac{1}{p^2(1+R)}\widehat{\theta}\widehat{\eta} - \dfrac{4q\widehat{\zeta}_2}{p\sqrt{1+R}}\widehat{\theta}'\Big)\varepsilon + \mathcal{O}(\varepsilon^2). \end{cases}$$

We also get the following result:

Theorem 3.2. *Assume that $q \notin \mathbb{N}_+$, then, for each sufficiently small $\varepsilon > 0$, the system (3.4) has at least a $T = 2\pi p\sqrt{1+R}$-periodic solution $(\widehat{\eta}(\tau,\varepsilon), \widehat{\theta}(\tau,\varepsilon))$ such that*

$$\lim_{\varepsilon\to 0}(\widehat{\eta}(0,\varepsilon), \widehat{\eta}'(0,\varepsilon), \widehat{\theta}(0,\varepsilon), \widehat{\theta}'(0,\varepsilon)) = \Big(0, \frac{\widehat{F}}{2\widehat{\zeta}_1}, 0, 0\Big).$$

P r o o f of Theorem 3.1. Let

$$\widehat{\eta}_1 = \widehat{\eta}, \quad \widehat{\eta}_2 = \widehat{\eta}', \quad \widehat{\theta}_1 = \widehat{\theta}, \quad \widehat{\theta}_2 = \widehat{\theta}', \quad x = (\widehat{\eta}_1, \widehat{\eta}_2), \quad y = (\widehat{\theta}_1, \widehat{\theta}_2),$$
$$D_1 = \frac{1}{p\sqrt{1+R}} \quad \text{and} \quad D_2 = \frac{q}{p\sqrt{1+R}}.$$

Rewrite system (3.2) in the form of a first-order system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = F_0(\tau, x) + F_1(\tau, x, y)\varepsilon + \mathcal{O}(\varepsilon^2), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = G_0(\tau, x, y) + G_1(\tau, x, y)\varepsilon + \mathcal{O}(\varepsilon^2), \end{cases} \tag{3.5}$$

where

$$F_0(\tau, x) = \begin{pmatrix} \widehat{\eta}_2 \\ -D_1^2\widehat{\eta}_1 \end{pmatrix}, \quad G_0(\tau, x, y) = \begin{pmatrix} \widehat{\theta}_2 \\ -\left(D_2^2 + D_1^2\widehat{\eta}_1\right)\widehat{\theta}_1 \end{pmatrix},$$
$$F_1(\tau, x, y) = \begin{pmatrix} 0 \\ \widehat{F}D_1^2\cos(D_1\tau) - 8D_2^2\widehat{\theta}_1^2 - 8D_1^2\widehat{\theta}_1^2\widehat{\eta}_1 + 8\widehat{\theta}_2^2 - 2\widehat{\zeta}_1 D_1^2\widehat{\eta}_2 \end{pmatrix},$$
$$G_1(\tau, x, y) = \begin{pmatrix} 0 \\ \widehat{F}D_1^2\cos(D_1\tau)\widehat{\theta}_1 - 4\widehat{\zeta}_2 D_2\widehat{\theta}_2 - \dfrac{20}{3}D_2^2\widehat{\theta}_1^3 + \dfrac{4D_2^2\widehat{\theta}_1^3}{3R} \\ -\dfrac{20}{3}D_1^2\widehat{\theta}_1^3\widehat{\eta}_1 + \dfrac{4D_1^2\widehat{\theta}_1^3\widehat{\eta}_1}{3R} + 8\widehat{\theta}_1\widehat{\theta}_2^2 - 2\widehat{\zeta}_1 D_1^2\widehat{\theta}_1\widehat{\eta}_2 \end{pmatrix}.$$

Due to the fact that the parts with τ of system (3.1) are all trigonometric functions with the period $T = 2\pi p\sqrt{1+R}$, so $F_0(\tau, x)$, $G_0(\tau, x, y)$, $F_1(\tau, x, y)$, $G_1(\tau, x, y)$, $\mathcal{O}(\varepsilon^2)$ are all $T = 2\pi p\sqrt{1+R}$-periodic functions, which are also C^2 smooth functions.

Obviously, system (3.5) is exactly the model discussed by Theorem 1.1. The proof of Theorem 3.1 is given as follows.

Firstly, let

$$V = \left\{(\widehat{\eta}_{10}, \widehat{\eta}_{20});\ \sqrt{\widehat{\eta}_{10}^2 + \widehat{\eta}_{20}^2} < r, \right.$$
$$\left. \frac{\widehat{F}}{2\widehat{\zeta}_1} < r \leqslant \frac{p\sqrt{(3/(2\pi))^{2/3}q^{4/3}(1+R) - p^2(1+R)^2 - q^2}}{\sqrt{1+p^2(1+R)}}\right\} \subset \mathbb{R}^2.$$

Secondly, it is easy to find the general solution of

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = F_0(\tau, x), \tag{3.6}$$

that is,

$$x(\tau, \alpha) = \begin{pmatrix} \widehat{\eta}_{10}\cos(D_1\tau) + \widehat{\eta}_{20}\frac{1}{D_1}\sin(D_1\tau) \\ -\widehat{\eta}_{10}D_1\sin(D_1\tau) + \widehat{\eta}_{20}\cos(D_1\tau) \end{pmatrix},$$

where $\alpha = (\widehat{\eta}_{10}, \widehat{\eta}_{20})$ is an initial value. Obviously, it is a solution with the period $T = 2\pi p\sqrt{1+R}$, and $(0,0) \in \mathbb{R}^2$ is a rest point of

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = G_0(\tau, x, y). \tag{3.7}$$

We construct a mapping $\varphi\colon \mathbb{R}^2 \to (0,0) \in \mathbb{R}^2$, for each $\alpha \in \overline{V}$, $\varphi(x(\tau;\alpha))$ is a $T = 2\pi p\sqrt{1+R}$-periodic solution of $\mathrm{d}y/\mathrm{d}\tau = G_0(\tau, x(\tau;\alpha), y)$ and φ is a C^2 smooth function in $\overline{V}$.

Thirdly, we prove that the fundamental solution matrix $\Phi_\alpha(t)$ of

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = D_y G_0(\tau, x(\tau;\alpha), \varphi(x(\tau;\alpha)))y \quad (\alpha \in \overline{V}) \tag{3.8}$$

satisfies (1.12). Let

$$\Phi_\alpha(\tau) = \begin{pmatrix} d_1(\tau,1) & h_1(\tau,0) \\ d_2(\tau,0) & h_2(\tau,1) \end{pmatrix},$$

where $d_1(\tau,1)$, $d_2(\tau,0)$, $h_1(\tau,0)$, $h_2(\tau,1)$ fulfil $d_1(0,1)=1$, $d_2(0,0)=0$, $h_1(0,0)=0$, $h_2(0,1)=1$, respectively.

Here, we use the Picard approximation principle [27] to obtain the second-order approximate solution $\begin{pmatrix} d_{12}(\tau) \\ d_{22}(\tau) \end{pmatrix}$ with the initial value $(1,0)$ and the second-order approximate solution $\begin{pmatrix} h_{12}(\tau) \\ h_{22}(\tau) \end{pmatrix}$ of system (3.8) with the initial value $(0,1)$, respectively, which satisfy the following estimates:

$$\left\| \begin{pmatrix} d_{12}(\tau) - d_1(\tau,1) \\ d_{22}(\tau) - d_2(\tau,0) \end{pmatrix} \right\| \leqslant \frac{L^3T^3}{3!}, \tag{3.9}$$

$$\left\| \begin{pmatrix} h_{12}(\tau) - h_1(\tau,0) \\ h_{22}(\tau) - h_2(\tau,1) \end{pmatrix} \right\| \leqslant \frac{L^3T^3}{3!}, \tag{3.10}$$

where $L = \sqrt{1 + \bigl(D_2^2 + D_1\sqrt{\widehat{\eta}_{10}^2 D_1^2 + \widehat{\eta}_{20}^2}\bigr)^2}$.

According to the estimates (3.9) and (3.10), we obtain that

$$|d_{12}(T) - d_1(T,1)| \leqslant \frac{L^3T^3}{3!}, \quad |h_{22}(T) - h_2(T,1)| \leqslant \frac{L^3T^3}{3!}.$$

Therefore, we have that

$$d_{12}(T) + h_{22}(T) - \frac{L^3T^3}{3} \leqslant d_1(T,1) + h_2(T,1) \leqslant d_{12}(T) + h_{22}(T) + \frac{L^3T^3}{3}.$$

By condition (3.3) of Theorem 3.1, we also obtain

$$d_{12}(T) + h_{22}(T) + \frac{L^3T^3}{3} = -D_2^2T^2 + \frac{L^3T^3}{3} + 2 < 2.$$

Finally, we prove that there exists an $\alpha_0 \in \overline{V}$ such that

$$F(\alpha_0) = 0, \quad \det(D_\alpha F(\alpha_0)) \neq 0,$$

where

$$F(\alpha) = \int_0^T M_\alpha^{-1}(\tau) F_1(\tau, x(\tau;\alpha), \varphi(x(\tau;\alpha)))\,\mathrm{d}\tau, \tag{3.11}$$

and $M_\alpha(\tau) = \begin{pmatrix} \cos(D_1\tau) & D_1^{-1}\sin(D_1\tau) \\ -D_1\sin(D_1\tau) & \cos(D_1\tau) \end{pmatrix}$.

By (3.11), we infer that

$$\begin{aligned} F(\alpha) &= \int_0^T \begin{pmatrix} -\frac{1}{2}\widehat{F}D_1\sin(2D_1\tau) - 2\widehat{\zeta}_1 D_1^2 \sin^2(D_1\tau)\widehat{\eta}_{10} + \widehat{\zeta}_1 D_1 \sin(2D_1\tau)\widehat{\eta}_{20} \\ \widehat{F}D_1^2\cos^2(D_1\tau) + \widehat{\zeta}_1 D_1^3 \sin(2D_1\tau)\widehat{\eta}_{10} - 2\widehat{\zeta}_1 D_1^2 \cos^2(D_1\tau)\widehat{\eta}_{20} \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} -\widehat{\zeta}_1 D_1^2 T\widehat{\eta}_{10} \\ \dfrac{\widehat{F}D_1^2 T}{2} - \widehat{\zeta}_1 D_1^2 T\widehat{\eta}_{20} \end{pmatrix}. \end{aligned}$$

Solving the equation $F(\alpha) = 0$, we obtain $\alpha_0 = (0, \widehat{F}/2\widehat{\zeta}_1)$. Obviously,

$$\det \frac{\partial F(\alpha)}{\partial \alpha}\Big|_{\alpha=\alpha_0} \neq 0.$$

Now, all conditions of Theorem 1.1 hold, system (3.5) has at least a $T = 2\pi p\sqrt{1+R}$-periodic solution $(x(\tau,\varepsilon), y(\tau,\varepsilon))$, which satisfies

$$\lim_{\varepsilon\to 0}(x(0,\varepsilon), y(0,\varepsilon)) = \Big(0, \frac{\widehat{F}}{2\widehat{\zeta}_1}, 0, 0\Big),$$

that is, system (3.2) has a solution $(\widehat{\eta}(\tau,\varepsilon), \widehat{\theta}(\tau,\varepsilon))$ with the period $T = 2\pi p\sqrt{1+R}$ such that

$$\lim_{\varepsilon\to 0}(\widehat{\eta}(0,\varepsilon), \widehat{\eta}'(0,\varepsilon), \widehat{\theta}(0,\varepsilon), \widehat{\theta}'(0,\varepsilon)) = \Big(0, \frac{\widehat{F}}{2\widehat{\zeta}_1}, 0, 0\Big).$$

This completes the proof of Theorem 3.1. □

We give the main proof of $T \in (0, 2/\sqrt{3})$ in Remark 3.1. According to (3.3), we obtain that

$$\begin{aligned} \frac{1}{2}\Big(\frac{3q^2}{2\pi}\Big)^{2/3} - \frac{1}{2}\sqrt{\Big(\frac{3q^2}{2\pi}\Big)^{4/3} - 4q^4} < \delta < \frac{1}{2}\Big(\frac{3q^2}{2\pi}\Big)^{2/3} \\ + \frac{1}{2}\sqrt{\Big(\frac{3q^2}{2\pi}\Big)^{4/3} - 4q^4},\ 0 < q < \frac{3\sqrt{2}}{8\pi}, \end{aligned}$$

where $\delta = p^2(1+R)$. Denoting that

$$\delta_2(q) = \frac{1}{2}\Big(\frac{3q^2}{2\pi}\Big)^{2/3} + \frac{1}{2}\sqrt{\Big(\frac{3q^2}{2\pi}\Big)^{4/3} - 4q^4}, \quad \Big(0 < q < \frac{3\sqrt{2}}{8\pi}\Big),$$

$\delta_2(q)$ has a maximum value $1/(3\pi^2)$ in $q = 1/(\sqrt[4]{18}\pi)$. Therefore, we have $T = 2\pi\sqrt{\delta} < 2/\sqrt{3}$.

P r o o f of Theorem 3.2. Let

$$\widehat{\eta}_1 = \widehat{\eta}, \quad \widehat{\eta}_2 = \widehat{\eta}', \quad \widehat{\theta}_1 = \widehat{\theta}, \quad \widehat{\theta}_2 = \widehat{\theta}', \quad x = (\widehat{\eta}_1, \widehat{\eta}_2), \quad y = (\widehat{\theta}_1, \widehat{\theta}_2),$$
$$D_1 = \frac{1}{p\sqrt{1+R}}, \quad D_2 = \frac{q}{p\sqrt{1+R}},$$

and rewrite system (3.4) in the form of the first-order system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tau} = F_0(\tau, x) + F_1(\tau, x, y)\varepsilon + \mathcal{O}(\varepsilon^2), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = G_0(\tau, x, y) + G_1(\tau, x, y)\varepsilon + \mathcal{O}(\varepsilon^2), \end{cases} \tag{3.12}$$

where

$$F_0(\tau, x) = \begin{pmatrix} \widehat{\eta}_2 \\ -D_1^2\widehat{\eta}_1 \end{pmatrix}, \quad G_0(\tau, x, y) = \begin{pmatrix} \widehat{\theta}_2 \\ -D_2^2\widehat{\theta}_1 \end{pmatrix},$$
$$F_1(\tau, x, y) = \begin{pmatrix} 0 \\ \widehat{F}D_1^2\cos(D_1\tau) - 2\widehat{\zeta}_1 D_1^2\widehat{\eta}_2 - 8D_2^2\widehat{\theta}_1^2 + 8\widehat{\theta}_2^2 \end{pmatrix},$$
$$G_1(\tau, x, y) = \begin{pmatrix} 0 \\ -D_1^2\widehat{\theta}_1^2\widehat{\eta}_1 - 4\widehat{\zeta}_2 D_2\widehat{\theta}_2 \end{pmatrix}.$$

It is easy to prove Theorem 3.2 by the local averaging Theorem [4]. □

Notice that there is a periodic solution $(\eta(\tau), \theta(\tau))$ such that $\eta'(0, \varepsilon) \neq 0$ for the corresponding system (3.1) under the transformation

$$\eta = \widehat{\eta}, \quad \theta = \overline{\theta}\sqrt{\varepsilon}, \quad \zeta_1 = \overline{\zeta}_1\varepsilon, \quad \zeta_2 = \overline{\zeta}_2\varepsilon, \quad F = \widehat{F}\varepsilon, \quad F_0 = \widehat{F}_0\varepsilon^2,$$
$$\widehat{\theta} = \frac{1}{2}\sqrt{\frac{R}{2(1+R)}}\bar{\theta}, \quad \widehat{\zeta}_1 = p\overline{\zeta}_1, \quad \widehat{\zeta}_2 = \frac{1}{2}\overline{\zeta}_2$$

by Theorem 3.1, but we cannot ensure that when Theorem 3.2 holds.

3.2. Nonlinear oscillator of a tuned mass absorber. The tuned mass damper (TMD) was patented by Frahm in 1909. Scientists and engineers have made a lot of effort to improve the properties of TMD by adding control, e.g., the linear (nonlinear) oscillator of the tuned mass absorber (TMA) proposed in [12], [9], [10], [11] is replaced by a pendulum, which is called the classical TMA. At the same time, there are also a lot of publications on the multiple TMA, e.g., the works of Vyas and Bajaj [25], [26], where the authors increase the efficiency of TMA by differentiating pendulums' lengths. In [3], Brzeski, Perlikowski and Kapitaniak discuss the dynamics of TMA replaced by dual pendulums, see Fig. 2, the purpose of their analysis is to study and compare energy absorption properties of the system and show that by a careful choice of parameters one can achieve large decrease of the Duffing system amplitude. Here our task is to investigate the existence of a periodic solution of the generalized model (see Fig. 2) of the tuned mass absorber system discussed in [3].

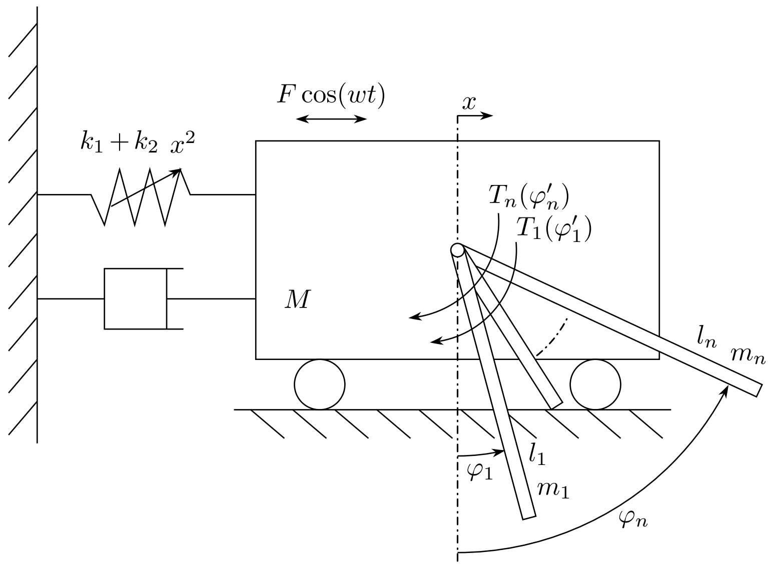

Figure 2. TMA with n pendulums (k_1, k_2 are linear and nonlinear parts of spring stiffness, respectively).

The corresponding n-dimensional equations of the model are as follows:

$$
(3.13)\quad \begin{cases}
\left(1+\sum_{i=1}^{n} m_{iD}\right)x'' + \sum_{i=1}^{n} \frac{1}{2} m_{iD} l_{iD}(\varphi_i'' \cos\varphi_i - \varphi_i'^2 \sin\varphi_i) \\
\qquad\qquad + x + k_{2D}x^3 + c_D x' = F_D \cos(w\tau), \\
\frac{1}{2} m_{1D} l_{1D} x'' \cos\varphi_1 + \frac{1}{3} m_{1D} l_{1D}^2 \varphi_1'' + \frac{1}{2} m_{1D} l_{1D} g_D \sin\varphi_1 + c_{P1D}\varphi_1' = 0, \\
\quad \vdots \\
\frac{1}{2} m_{nD} l_{nD} x'' \cos\varphi_n + \frac{1}{3} m_{nD} l_{nD}^2 \varphi_n'' + \frac{1}{2} m_{nD} l_{nD} g_D \sin\varphi_n + c_{PnD}\varphi_n' = 0,
\end{cases}
$$

where m_{iD}, l_{iD}, c_{PiD} is the mass, length and damping coefficient of the first ith pendulum, respectively, φ_i is the angle of the first i pendulum, the vertical position of the Duffing oscillator is described by the coordinate x.

Making the transformation

$$c_D = \varepsilon\overline{c}_D,\ F_D = \varepsilon^2\overline{F}_D,\ x = \varepsilon\overline{x},\ \varphi_i = \varepsilon\overline{\varphi}_i,\ m_{iD} = \varepsilon\overline{m}_{iD},\\ c_{PiD} = \varepsilon^2\overline{c}_{PiD},\quad i = 1,\dots,n,$$

we obtain

$$\text{(3.14)}\qquad \begin{cases} \overline{x}'' = -\overline{x} + \left(\overline{F}_D\cos(w\tau) - \overline{c}_D\overline{x}' + \dfrac{1}{4}\displaystyle\sum_{i=1}^{n}\overline{m}_{iD}\overline{x} + \dfrac{3g_D}{4}\sum_{i=1}^{n}\overline{m}_{iD}\overline{\varphi}_i\right)\varepsilon \\ \qquad + \mathcal{O}(\varepsilon^2), \\ \overline{\varphi}_1'' = \dfrac{3}{2l_{1D}}\overline{x} - \dfrac{3g_D}{2l_{1D}}\overline{\varphi}_1 - \left(\dfrac{3\overline{F}_D}{2l_{1D}}cos(w\tau) - \dfrac{3\overline{c}_D}{2l_{1D}}\overline{x}' + \dfrac{3}{8l_{1D}}\displaystyle\sum_{i=1}^{n}\overline{m}_{iD}\overline{x}\right. \\ \qquad \left. + \dfrac{9g_D}{8l_{1D}}\displaystyle\sum_{i=1}^{n}\overline{m}_{iD}\overline{\varphi}_i + \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}\overline{\varphi}_1'\right)\varepsilon + \mathcal{O}(\varepsilon^2), \\ \quad\vdots \\ \overline{\varphi}_n'' = \dfrac{3}{2l_{nD}}\overline{x} - \dfrac{3g_D}{2l_{nD}}\overline{\varphi}_n - \left(\dfrac{3\overline{F}_D}{2l_{nD}}cos(w\tau) - \dfrac{3\overline{c}_D}{2l_{nD}}\overline{x}' + \dfrac{3}{8l_{2D}}\displaystyle\sum_{i=1}^{n}\overline{m}_{iD}\overline{x}\right. \\ \qquad \left. + \dfrac{9g_D}{8l_{nD}}\displaystyle\sum_{i=1}^{n}\overline{m}_{iD}\overline{\varphi}_i + \dfrac{3\overline{c}_{PnD}}{\overline{m}_{nD}l_{nD}^2}\overline{\varphi}_n'\right)\varepsilon + \mathcal{O}(\varepsilon^2). \end{cases}$$

We state some results as follows:

Theorem 3.3. *Assume that $w = q/p$ ($p, q \in \mathbb{N}_+$ and coprime), $p\sqrt{\frac{3}{2}g_D/l_{iD}} \notin \mathbb{N}_+$ ($i = 1, 2, \dots, n$). If $w \neq 1$, then, for every sufficiently small $\varepsilon > 0$, system (3.14) has at least a $T = 2p\pi$-periodic solution*

$$(\overline{x}(\tau,\varepsilon), \overline{\varphi}_1(\tau,\varepsilon), \dots, \overline{\varphi}_n(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon), \overline{x}'(0,\varepsilon), \overline{\varphi}_1(0,\varepsilon), \overline{\varphi}_1'(0,\varepsilon), \dots, \overline{\varphi}_n(0,\varepsilon), \overline{\varphi}_n'(0,\varepsilon)) = (0,0,0,0,\dots,0,0);$$

if $w = 1$, then, for every sufficiently small $\varepsilon > 0$, system (3.14) has at least a $T = 2\pi$-periodic solution

$$(\overline{x}(\tau,\varepsilon), \overline{\varphi}_1(\tau,\varepsilon), \dots, \overline{\varphi}_n(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\dots,\overline{\varphi}_n(0,\varepsilon),\overline{\varphi}_n'(0,\varepsilon))$$
$$=\Big(-K_2\overline{F}_D,K_1\overline{F}_D,\frac{3K_2\overline{F}_D}{2l_{1D}-3g_D},-\frac{3K_1\overline{F}_D}{2l_{1D}-3g_D},\dots,\frac{3K_2\overline{F}_D}{2l_{nD}-3g_D},-\frac{3K_1\overline{F}_D}{2l_{nD}-3g_D}\Big),$$

where

$$K_1=\frac{\overline{c}_D}{\overline{c}_D^2+b^2},\quad K_2=\frac{b}{\overline{c}_D^2+b^2},$$
$$b=\frac{1}{2}\sum_{i=1}^{n}\overline{m}_{iD}\frac{l_{iD}-6g_D}{2l_{iD}-3g_D}.$$

R e m a r k 3.2. Obviously, Theorem 3.3 holds for $\sqrt{\frac{3}{2}g_D/l_{iD}}\notin\mathbb{Q}_+$ $(i=1,\dots,n)$.

Theorem 3.4. *Assume that* $w=q/p$ *(*$p,q\in\mathbb{N}_+$ *and coprime),* $l_{iD}\neq l_{jD}$ $(i\neq j)$, $i,j\in\{1,\dots,n\}$, $\sqrt{\frac{3}{2}g_D/l_{iD}}=q_i/p_i\neq 1$ *(*$p_i,q_i\in\mathbb{N}_+$ *and coprime,* $i=1,2,\dots,n$*). If* $w=1$*, then, for every sufficiently small* $\varepsilon>0$*, system* (3.14) *has at least a* $T=2p_1\dots p_n\pi$*-periodic solution*

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\dots,\overline{\varphi}_n(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\dots,\overline{\varphi}_n(0,\varepsilon),\overline{\varphi}_n'(0,\varepsilon))$$
$$=(-K_2\overline{F}_D,K_1\overline{F}_D,0,0,\dots,0,0);$$

if $w\neq 1$, $\sqrt{\frac{3}{2}g_D/l_{iD}}\neq w$ $(i=1,2,\dots,n)$*, then, for every sufficiently small* $\varepsilon>0$*, system* (3.14) *has at least a* $T=2pp_1\dots p_n\pi$*-periodic solution*

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\dots,\overline{\varphi}_n(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\dots,\overline{\varphi}_n(0,\varepsilon),\overline{\varphi}_n'(0,\varepsilon))=(0,0,0,0,\dots,0,0);$$

if $w\neq 1$, $\sqrt{\frac{3}{2}g_D/l_{1D}}=w$*, then, for every sufficiently small* $\varepsilon>0$*, system* (3.14) *has at least a* $T=2pp_1\dots p_n\pi$*-periodic solution*

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\dots,\overline{\varphi}_n(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\ldots,\overline{\varphi}_n(0,\varepsilon),\overline{\varphi}_n'(0,\varepsilon))$$
$$= (0,0,a_1,a_2,0,0,\ldots,0,0),$$

where

$$K_1 = \frac{\overline{c}_D}{\overline{c}_D^2+b^2},\quad K_2 = \frac{b}{\overline{c}_D^2+b^2},$$
$$b = \frac{1}{2}\sum_{i=1}^{n}\overline{m}_{iD}\frac{l_{iD}-6g_D}{2l_{iD}-3g_D}$$

$$a_1 = \frac{\det\begin{pmatrix} 0 & \dfrac{9g_D\overline{m}_{1D}}{4(2l_{1D}-3g_D)} \\ \dfrac{9g_D\overline{F}_D}{2l_{1D}(2l_{1D}-3g_D)} & \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}\end{pmatrix}}{\det\begin{pmatrix} \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2} & \dfrac{9g_D\overline{m}_{1D}}{4(2l_{1D}-3g_D)} \\ -\dfrac{27g_D^2\overline{m}_{1D}}{8l_{1D}(2l_{1D}-3g_D)} & \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}\end{pmatrix}}$$

$$a_2 = \frac{\det\begin{pmatrix} \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2} & 0 \\ -\dfrac{27g_D^2\overline{m}_{1D}}{8l_{1D}(2l_{1D}-3g_D)} & \dfrac{9g_D\overline{F}_D}{2l_{1D}(2l_{1D}-3g_D)}\end{pmatrix}}{\det\begin{pmatrix} \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2} & \dfrac{9g_D\overline{m}_{1D}}{4(2l_{1D}-3g_D)} \\ -\dfrac{27g_D^2\overline{m}_{1D}}{8l_{1D}(2l_{1D}-3g_D)} & \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}\end{pmatrix}}$$

Remark 3.3. The periodicity $T = 2p\pi$ in the conclusions of Theorem 3.3 and Theorem 3.4 above in the case of different pendulum lengths with $p/p_i \in \mathbb{N}_+$ $(i = 1,2,\ldots,n)$, and the conditions of Theorems 3.3 and 3.4 are mutually exclusive, which implies that Theorem 1.3 and the local averaging theorem [4] are two theorems that do not contain each other.

Here, we only prove the case with $n = 2$, which is just the case discussed by Brzeski, Perlikowski and Kapitaniak [3], the proofs of other cases are similar.

P r o o f of Theorem 3.3. Let $\overline{x} = x_1$, $\overline{x}' = x_2$, $\overline{\varphi}_1 = y_{11}$, $\overline{\varphi}_1' = y_{12}$, $\overline{\varphi}_2 = y_{21}$, $\overline{\varphi}_2' = y_{22}$. Then system (3.14) is normalized for

$$
\tag{3.15}
\begin{cases}
x_1' = x_2, \\
x_2' = -x_1 + \Big(\overline{F}_D \cos(w\tau) + \dfrac{1}{4}\displaystyle\sum_{i=1}^{2} \overline{m}_{iD} x_1 - \overline{c}_D x_2 + \dfrac{3g_D}{4}\sum_{i=1}^{2}\overline{m}_{iD} y_{i1}\Big)\varepsilon \\
\qquad +\mathcal{O}(\varepsilon^2), \\
y_{11}' = y_{12}, \\
y_{12}' = \dfrac{3}{2l_{1D}} x_1 - \dfrac{3g_D}{2l_{1D}} y_{11} - \Big(\dfrac{3\overline{F}_D}{2l_{1D}}\cos(w\tau) + \dfrac{3}{8l_{1D}}\displaystyle\sum_{i=1}^{2}\overline{m}_{iD}x_1 - \dfrac{3\overline{c}_D}{2l_{1D}}x_2 \\
\qquad + \dfrac{9g_D}{8l_{1D}}\displaystyle\sum_{i=1}^{2}\overline{m}_{iD}y_{i1} + \dfrac{3\overline{c}_{P1D}}{\overline{m}_{1D} l_{1D}^2} y_{12}\Big)\varepsilon + \mathcal{O}(\varepsilon^2), \\
y_{21}' = y_{22}, \\
y_{22}' = \dfrac{3}{2l_{2D}} x_1 - \dfrac{3g_D}{2l_{2D}} y_{21} - \Big(\dfrac{3\overline{F}_D}{2l_{2D}}\cos(w\tau) + \dfrac{3}{8l_{2D}}\displaystyle\sum_{i=1}^{2}\overline{m}_{iD}x_1 - \dfrac{3\overline{c}_D}{2l_{2D}}x_2 \\
\qquad + \dfrac{9g_D}{8l_{2D}}\displaystyle\sum_{i=1}^{2}\overline{m}_{iD}y_{i1} + \dfrac{3\overline{c}_{P2D}}{\overline{m}_{2D} l_{2D}^2} y_{22}\Big)\varepsilon + \mathcal{O}(\varepsilon^2).
\end{cases}
$$

The unperturbed system of system (3.15) is

$$
\tag{3.16}
\begin{cases}
x_1' = x_2, \\
x_2' = -x_1, \\
y_{11}' = y_{12}, \\
y_{12}' = \dfrac{3}{2l_{1D}} x_1 - \dfrac{3g_D}{2l_{1D}} y_{11}, \\
y_{21}' = y_{22}, \\
y_{22}' = \dfrac{3}{2l_{2D}} x_1 - \dfrac{3g_D}{2l_{2D}} y_{21}.
\end{cases}
$$

We construct a C^2 smooth mapping $\beta_0\colon \overline{V}_0 \to \mathbb{R}^4$, that is,

$$
\beta_0(x_{10}, x_{20}) = \Big(\frac{3}{3g_D - 2l_{1D}} x_{10}, \frac{3}{3g_D - 2l_{1D}} x_{20}, \frac{3}{3g_D - 2l_{2D}} x_{10}, \frac{3}{3g_D - 2l_{2D}} x_{20}\Big),
$$

where

$$V_0 = \Bigg\{ (x_{10}, x_{20});\ \sqrt{x_{10}^2 + x_{20}^2} < r,$$

$$r > \overline{F}_D \sqrt{\left[1 + 9\sum_{i=1}^{2} \frac{1}{(2l_{iD} - 3g_D)^2}\right](K_1^2 + K_2^2)} \Bigg\},$$

and construct a set

$$Z = \{z_\alpha = (\alpha, \beta_0(\alpha));\ \alpha \in \overline{V}_0\}.$$

We obtain the solution of the unperturbed system (3.16) with the initial value $z_\alpha \in Z$ ($\alpha = (x_{10}, x_{20})$) as follows:

$$X(\tau; z_\alpha, 0) = \begin{pmatrix} x_{10}\cos\tau + x_{20}\sin\tau \\ -x_{10}\sin\tau + x_{20}\cos\tau \\ -\dfrac{3}{2l_{1D} - 3g_D}(x_{10}\cos\tau + x_{20}\sin\tau) \\ \dfrac{3}{2l_{1D} - 3g_D}(x_{10}\sin\tau - x_{20}\cos\tau) \\ -\dfrac{3}{2l_{2D} - 3g_D}(x_{10}\cos\tau + x_{20}\sin\tau) \\ \dfrac{3}{2l_{2D} - 3g_D}(x_{10}\sin\tau - x_{20}\cos\tau) \end{pmatrix}.$$

It is easy to obtain that some fundamental solution matrix of the corresponding linearized system along the periodic solution $X(\tau, z_\alpha, 0)$ of the unperturbed system (3.16) is

$$M_{z_\alpha}(\tau) = \begin{pmatrix} M_\alpha(\tau) & 0 \\ C_\alpha(\tau) & \Phi_\alpha(\tau) \end{pmatrix}, \tag{3.17}$$

where

$$M_\alpha(\tau) = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix},\ \ C_\alpha(\tau) = \begin{pmatrix} -\dfrac{3}{2l_{1D} - 3g_D}\cos\tau & -\dfrac{3}{2l_{1D} - 3g_D}\sin\tau \\ \dfrac{3}{2l_{1D} - 3g_D}\sin\tau & -\dfrac{3}{2l_{1D} - 3g_D}\cos\tau \\ -\dfrac{3}{2l_{2D} - 3g_D}\cos\tau & -\dfrac{3}{2l_{2D} - 3g_D}\sin\tau \\ \dfrac{3}{2l_{2D} - 3g_D}\sin\tau & -\dfrac{3}{2l_{2D} - 3g_D}\cos\tau \end{pmatrix},$$

$$\Phi_\alpha(\tau) = \begin{pmatrix} \cos\sqrt{\frac{3g_D}{2l_{1D}}}\tau & \sqrt{\frac{2l_{1D}}{3g_D}}\sin\sqrt{\frac{3g_D}{2l_{1D}}}\tau & 0 & 0 \\ -\sqrt{\frac{3g_D}{2l_{1D}}}\sin\sqrt{\frac{3g_D}{2l_{1D}}}\tau & \cos\sqrt{\frac{3g_D}{2l_{1D}}}\tau & 0 & 0 \\ 0 & 0 & \cos\sqrt{\frac{3g_D}{2l_{2D}}}\tau & \sqrt{\frac{2l_{2D}}{3g_D}}\sin\sqrt{\frac{3g_D}{2l_{2D}}}\tau \\ 0 & 0 & -\sqrt{\frac{3g_D}{2l_{2D}}}\sin\sqrt{\frac{3g_D}{2l_{2D}}}\tau & \cos\sqrt{\frac{3g_D}{2l_{2D}}}\tau \end{pmatrix}.$$

According to $p\sqrt{3g_D/(2l_{iD})} \notin \mathbb{N}_+$ $(i = 1, 2)$ and Theorem 1.2, we obtain that $\det(\Phi_\alpha(T) - \Phi_\alpha(0)) \neq 0$. Now, we construct the function

$$F(\alpha) = \int_0^T M_\alpha^{-1}(\tau) f(\tau, z_\alpha)\,\mathrm{d}\tau,$$

where

$$f(\tau, x_1, x_2, y_{11}, y_{12}, y_{21}, y_{22}) = \begin{pmatrix} 0 \\ \overline{F}_D\cos(w\tau) + \frac{1}{4}\sum_{i=1}^{2}\overline{m}_{iD}x_1 - \overline{c}_D x_2 + \frac{3g_D}{4}\sum_{i=1}^{2}\overline{m}_{iD}y_{i1} \end{pmatrix}.$$

If $w \neq 1$,

$$F(x_{10}, x_{20}) = p\pi \begin{pmatrix} -\overline{c}_D x_{10} - \Big(\frac{1}{4}\sum_{i=1}^{2}\overline{m}_{iD} - \frac{9g_D}{4}\sum_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD} - 3g_D}\Big)x_{20} \\ \Big(\frac{1}{4}\sum_{i=1}^{2}\overline{m}_{iD} - \frac{9g_D}{4}\sum_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD} - 3g_D}\Big)x_{10} - \overline{c}_D x_{20} \end{pmatrix}.$$

Solving the equation $F(x_{10}, x_{20}) = 0$, we obtain $(x_{10}, x_{20}) = (0, 0) \in V$ such that

$$\det\frac{\partial F(x_{10}, x_{20})}{\partial(x_{10}, x_{20})}\Big|_{(0,0)} \neq 0.$$

According to the local averaging theorem [4], for every sufficiently small $\varepsilon > 0$, system (3.15) has at least a $T = 2p\pi$-periodic solution

$$(x_1(\tau,\varepsilon), x_2(\tau,\varepsilon), y_{11}(\tau,\varepsilon), y_{12}(\tau,\varepsilon), y_{21}(\tau,\varepsilon), y_{22}(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(x_1(0,\varepsilon), x_2(0,\varepsilon), y_{11}(0,\varepsilon), y_{12}(0,\varepsilon), y_{21}(0,\varepsilon), y_{22}(0,\varepsilon)) = (0,0,0,0,0,0),$$

that is, system (3.14) has at least a $T = 2p\pi$-periodic solution

$$(\overline{x}(\tau,\varepsilon), \overline{\varphi}_1(\tau,\varepsilon), \overline{\varphi}_2(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\overline{\varphi}_2(0,\varepsilon),\overline{\varphi}_2'(0,\varepsilon)) = (0,0,0,0,0,0);$$

if $w=1$,

$$F(x_{10},x_{20}) = \pi\begin{pmatrix} -\overline{c}_D x_{10} - \Big(\dfrac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD} - \dfrac{9g_D}{4}\sum\limits_{i=1}^{2}\dfrac{\overline{m}_{iD}}{2l_{iD}-3g_D}\Big)x_{20} \\ \overline{F}_D + \Big(\dfrac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD} - \dfrac{9g_D}{4}\sum\limits_{i=1}^{2}\dfrac{\overline{m}_{iD}}{2l_{iD}-3g_D}\Big)x_{10} - \overline{c}_D x_{20} \end{pmatrix}.$$

Solving the equation $F(x_{10},x_{20})=0$, we obtain that $(x_{10},x_{20}) = (-K_2\overline{F}_D, K_1\overline{F}_D) \in V$ such that

$$\det\Big(\frac{\partial F(x_{10},x_{20})}{\partial(x_{10},x_{20})}\Big|_{(-K_2\overline{F}_D,K_1\overline{F}_D)}\Big) \neq 0.$$

According to the local averaging theorem [4], for every sufficiently small $\varepsilon>0$, system (3.15) has at least a $T=2\pi$-periodic solution

$$(x_1(\tau,\varepsilon),x_2(\tau,\varepsilon),y_{11}(\tau,\varepsilon),y_{12}(\tau,\varepsilon),y_{21}(\tau,\varepsilon),y_{22}(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(x_1(0,\varepsilon),x_2(0,\varepsilon),y_{11}(0,\varepsilon),y_{12}(0,\varepsilon),y_{21}(0,\varepsilon),y_{22}(0,\varepsilon))$$
$$= \Big(-K_2\overline{F}_D, K_1\overline{F}_D, \frac{3K_2\overline{F}_D}{2l_{1D}-3g_D}, -\frac{3K_1\overline{F}_D}{2l_{1D}-3g_D}, \frac{3K_2\overline{F}_D}{2l_{2D}-3g_D}, -\frac{3K_1\overline{F}_D}{2l_{2D}-3g_D}\Big),$$

that is, system (3.14) has at least a $T=2\pi$-periodic solution

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\overline{\varphi}_2(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\overline{\varphi}_2(0,\varepsilon),\overline{\varphi}_2'(0,\varepsilon))$$
$$= \Big(-K_2\overline{F}_D, K_1\overline{F}_D, \frac{3K_2\overline{F}_D}{2l_{1D}-3g_D}, -\frac{3K_1\overline{F}_D}{2l_{1D}-3g_D}, \frac{3K_2\overline{F}_D}{2l_{2D}-3g_D}, -\frac{3K_1\overline{F}_D}{2l_{2D}-3g_D}\Big).$$

This completes the proof of Theorem 3.3. □

Proof of Theorem 3.4. Let V be a enough big bounded open subset around $0\in\mathbb{R}^6$. The solution of the unperturbed system (3.16) with the initial value $\alpha =$

$(x_{10}, x_{20}, y_{110}, y_{120}, y_{210}, y_{220}) \in \overline{V}$ is as follows:

$$X(\tau,\alpha,0) = \begin{pmatrix} x_{10}\cos\tau + x_{20}\sin\tau \\ -x_{10}\sin\tau + x_{20}\cos\tau \\ -\dfrac{3(x_{10}\cos\tau + x_{20}\sin\tau)}{2l_{1D}-3g_D} + y_{110}\cos\sqrt{\dfrac{3g_D}{2l_{1D}}}\tau + y_{120}\sqrt{\dfrac{2l_{1D}}{3g_D}}\sin\sqrt{\dfrac{3g_D}{2l_{1D}}}\tau \\ \dfrac{3(x_{10}\sin\tau - x_{20}\cos\tau)}{2l_{1D}-3g_D} - y_{110}\sqrt{\dfrac{3g_D}{2l_{1D}}}\sin\sqrt{\dfrac{3g_D}{2l_{1D}}}\tau + y_{120}\cos\sqrt{\dfrac{3g_D}{2l_{1D}}}\tau \\ -\dfrac{3(x_{10}\cos\tau + x_{20}\sin\tau)}{2l_{2D}-3g_D} + y_{210}\cos\sqrt{\dfrac{3g_D}{2l_{2D}}}\tau + y_{220}\sqrt{\dfrac{2l_{2D}}{3g_D}}\sin\sqrt{\dfrac{3g_D}{2l_{2D}}}\tau \\ \dfrac{3(x_{10}\sin\tau - x_{20}\cos\tau)}{2l_{2D}-3g_D} - y_{210}\sqrt{\dfrac{3g_D}{2l_{2D}}}\sin\sqrt{\dfrac{3g_D}{2l_{2D}}}\tau + y_{220}\cos\sqrt{\dfrac{3g_D}{2l_{2D}}}\tau \end{pmatrix},$$

where $X(\tau,\alpha,0)$ is a $T = 2pp_1p_2\pi$-periodic solution of the unperturbed system (3.16). It is easy to see that some fundamental solution matrix of the corresponding linearized system along the periodic solution $X(\tau,\alpha,0)$ of the unperturbed system (3.16) is still (3.17), and we denote (3.17) as $H_\alpha(\tau)$. Now, we construct the function

$$F(\alpha) = \int_0^T H_\alpha^{-1}(\tau) f(\tau, X(\tau,\alpha,0))\,\mathrm{d}\tau,$$

where

$$f(\tau, x_1, x_2, y_{11}, y_{12}, y_{21}, y_{22}) = \begin{pmatrix} 0 \\ \overline{F}_D\cos(w\tau) + \frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD}x_1 - \overline{c}_D x_2 + \frac{3g_D}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD}y_{i1} \\ 0 \\ \frac{3\overline{F}_D}{2l_{1D}}\cos(w\tau) + \frac{3}{8l_{1D}}\sum\limits_{i=1}^{2}\overline{m}_{iD}x_1 - \frac{3\overline{c}_D}{2l_{1D}}x_2 + \frac{9g_D}{8l_{1D}}\sum\limits_{i=1}^{2}\overline{m}_{iD}y_{i1} + \frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{12} \\ 0 \\ \frac{3\overline{F}_D}{2l_{2D}}\cos(w\tau) + \frac{3}{8l_{2D}}\sum\limits_{i=1}^{2}\overline{m}_{iD}x_1 - \frac{3\overline{c}_D}{2l_{2D}}x_2 + \frac{9g_D}{8l_{2D}}\sum\limits_{i=1}^{2}\overline{m}_{iD}y_{i1} + \frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{22} \end{pmatrix}.$$

If $w = 1$, we obtain that

$$F(\alpha) = p_1p_2\pi \begin{pmatrix} -\overline{c}_D x_{10} - \left(\frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD} - \frac{9g_D}{4}\sum\limits_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD}-3g_D}\right)x_{20} \\ \overline{F}_D + \left(\frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD} - \frac{9g_D}{4}\sum\limits_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD}-3g_D}\right)x_{10} - \overline{c}_D x_{20} \\ \frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{110} + \frac{9g_D\overline{m}_{1D}}{4(2l_{1D}-3g_D)}y_{120} - \frac{27g_D^2\overline{m}_{1D}}{8l_{1D}(2l_{1D}-3g_D)}y_{110} + \frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{120} \\ \frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{210} + \frac{9g_D\overline{m}_{2D}}{4(2l_{2D}-3g_D)}y_{220} - \frac{27g_D^2\overline{m}_{2D}}{8l_{2D}(2l_{2D}-3g_D)}y_{210} + \frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{220} \end{pmatrix}.$$

Solving the equation $F(\alpha)=0$, we obtain that $\alpha_0=(-K_2\overline{F}_D, K_1\overline{F}_D,0,0,0,0)\in V$ such that

$$\det\frac{\partial F(\alpha)}{\partial\alpha}\Big|_{\alpha=\alpha_0}\neq 0.$$

According to Theorem 1.3, for every sufficiently small $\varepsilon>0$, system (3.15) has at least a $T=2p_1p_2\pi$-periodic solution

$$(x_1(\tau,\varepsilon),x_2(\tau,\varepsilon),y_{11}(\tau,\varepsilon),y_{12}(\tau,\varepsilon),y_{21}(\tau,\varepsilon),y_{22}(\tau,\varepsilon))$$

such that

$$\begin{aligned}&\lim_{\varepsilon\to 0}(x_1(0,\varepsilon),x_2(0,\varepsilon),y_{11}(0,\varepsilon),y_{12}(0,\varepsilon),y_{21}(0,\varepsilon),y_{22}(0,\varepsilon))\\&\qquad=(-K_2\overline{F}_D,K_1\overline{F}_D,0,0,0,0),\end{aligned}$$

that is, system (3.14) has at least a $T=2p_1p_2\pi$-periodic solution

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\overline{\varphi}_2(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\overline{\varphi}_2(0,\varepsilon),\overline{\varphi}_2'(0,\varepsilon))=(-K_2\overline{F}_D,K_1\overline{F}_D,0,0,0,0);$$

if $w\neq 1$, $\sqrt{3g_D/(2l_{iD})}\neq w$ $(i=1,2)$, we obtain that

$$F(\alpha)=pp_1p_2\pi\begin{pmatrix}-\overline{c}_Dx_{10}-\left(\frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD}-\frac{9g_D}{4}\sum\limits_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD}-3g_D}\right)x_{20}\\ \left(\frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD}-\frac{9g_D}{4}\sum\limits_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD}-3g_D}\right)x_{10}-\overline{c}_Dx_{20}\\ \frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{110}+\frac{9g_D\overline{m}_{1D}}{4(2l_{1D}-3g_D)}y_{120}-\frac{27g_D^2\overline{m}_{1D}}{8l_{1D}(2l_{1D}-3g_D)}y_{110}+\frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{120}\\ \frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{210}+\frac{9g_D\overline{m}_{2D}}{4(2l_{2D}-3g_D)}y_{220}-\frac{27g_D^2\overline{m}_{2D}}{8l_{2D}(2l_{2D}-3g_D)}y_{210}+\frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{220}\end{pmatrix}.$$

Solving the equation $F(\alpha)=0$, we obtain that $\alpha_0=(0,0,0,0,0,0)\in V$ such that

$$\det\frac{\partial F(\alpha)}{\partial\alpha}\Big|_{\alpha=\alpha_0}\neq 0.$$

According to Theorem 1.3, for every sufficiently small $\varepsilon>0$, system (3.15) has at least a $T=2pp_1p_2\pi$-periodic solution

$$(x_1(\tau,\varepsilon),x_2(\tau,\varepsilon),y_{11}(\tau,\varepsilon),y_{12}(\tau,\varepsilon),y_{21}(\tau,\varepsilon),y_{22}(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(x_1(0,\varepsilon),x_2(0,\varepsilon),y_{11}(0,\varepsilon),y_{12}(0,\varepsilon),y_{21}(0,\varepsilon),y_{22}(0,\varepsilon))=(0,0,0,0,0,0),$$

that is, system (3.14) has at least a $T=2pp_1p_2\pi$-periodic solution

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\overline{\varphi}_2(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\overline{\varphi}_2(0,\varepsilon),\overline{\varphi}_2'(0,\varepsilon))=(0,0,0,0,0,0);$$

if $\sqrt{\frac{3}{2}g_D/l_{1D}}=w$, we obtain that

$$F(\alpha)=pp_1p_2\pi$$
$$\times\begin{pmatrix} -\overline{c}_D x_{10}-\left(\frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD}-\frac{9g_D}{4}\sum\limits_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD}-3g_D}\right)x_{20}\\ \left(\frac{1}{4}\sum\limits_{i=1}^{2}\overline{m}_{iD}-\frac{9g_D}{4}\sum\limits_{i=1}^{2}\frac{\overline{m}_{iD}}{2l_{iD}-3g_D}\right)x_{10}-\overline{c}_D x_{20}\\ \frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{110}+\frac{9g_D\overline{m}_{1D}}{4(2l_{1D}-3g_D)}y_{120}-\frac{9g_D\overline{F}_D}{2l_{1D}(2l_{1D}-3g_D)}-\frac{27g_D^2\overline{m}_{1D}}{8l_{1D}(2l_{1D}-3g_D)}y_{110}+\frac{3\overline{c}_{P1D}}{\overline{m}_{1D}l_{1D}^2}y_{120}\\ \frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{210}+\frac{9g_D\overline{m}_{2D}}{4(2l_{2D}-3g_D)}y_{220}-\frac{27g_D^2\overline{m}_{2D}}{8l_{2D}(2l_{2D}-3g_D)}y_{210}+\frac{3\overline{c}_{P2D}}{\overline{m}_{2D}l_{2D}^2}y_{220} \end{pmatrix}.$$

Solving the equation $F(\alpha)=0$, one obtains that $\alpha_0=(0,0,a_1,a_2,0,0)\in V$ such that

$$\det\frac{\partial F(\alpha)}{\partial\alpha}\Big|_{\alpha=\alpha_0}\neq 0.$$

According to Theorem 1.3, for every small enough $\varepsilon>0$, system (3.15) has at least a $T=2pp_1p_2\pi$-periodic solution

$$(x_1(\tau,\varepsilon),x_2(\tau,\varepsilon),y_{11}(\tau,\varepsilon),y_{12}(\tau,\varepsilon),y_{21}(\tau,\varepsilon),y_{22}(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(x_1(0,\varepsilon),x_2(0,\varepsilon),y_{11}(0,\varepsilon),y_{12}(0,\varepsilon),y_{21}(0,\varepsilon),y_{22}(0,\varepsilon))=(0,0,a_1,a_2,0,0),$$

that is, the system (3.14) has at least a $T=2pp_1p_2\pi$ -periodic solution

$$(\overline{x}(\tau,\varepsilon),\overline{\varphi}_1(\tau,\varepsilon),\overline{\varphi}_2(\tau,\varepsilon))$$

such that

$$\lim_{\varepsilon\to 0}(\overline{x}(0,\varepsilon),\overline{x}'(0,\varepsilon),\overline{\varphi}_1(0,\varepsilon),\overline{\varphi}_1'(0,\varepsilon),\overline{\varphi}_2(0,\varepsilon),\overline{\varphi}_2'(0,\varepsilon))=(0,0,a_1,a_2,0,0).$$

This completes the proof of Theorem 3.4. □

Above, we use the local averaging theorem [4] to prove Theorem 3.3 which solves the existence of a periodic solution of TMA with multiple pendulums, and prove Theorem 3.4 by the supplemental Theorem 1.3. In some way, Theorem 3.4 supplements Theorem 3.3 so that the existence of a periodic solution of TMA with multiple pendulums has been extended.

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