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P-INJECTIVE GROUP RINGS

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Abstract. A ring R is called right P-injective if every homomorphism from a principal right ideal of R to R_R can be extended to a homomorphism from R_R to R_R . Let R be a ring and G a group. Based on a result of Nicholson and Yousif, we prove that the group ring RG is right P-injective if and only if (a) R is right P-injective; (b) G is locally finite; and (c) for any finite subgroup H of G and any principal right ideal I of RH, if $f \in \text{Hom}_R(I_R, R_R)$, then there exists $g \in \text{Hom}_R(\text{RH}_R, R_R)$ such that $g|_I = f$. Similarly, we also obtain equivalent characterizations of n-injective group rings and F-injective group rings.

Keywords: group ring; P-injective ring; n-injective ring; F-injective ring

MSC 2020: 16S34, 16D50

1. INTRODUCTION

Throughout this paper rings are associative with identity and modules are unitary modules. Let R be a ring, we use $\operatorname{Hom}_R(M_R, N_R)$ to denote the set of all R-homomorphisms between two right R-modules M_R and N_R . If G is a group, we use RG to denote the group ring of G over R. For $\alpha = \sum_{g \in G} a_g g \in \operatorname{RG}$, define $\operatorname{Supp}(\alpha) = \{g \in G : a_g \neq 0\}$ to be the support of α . If $h \in G$, the projection $\pi_h : \operatorname{RG} \to R$ given by $\pi_h\left(\sum_{g \in G} a_g g\right) = a_h$ is right and left R-linear. We write $\pi = \pi_{1_G}$. And $\pi(\alpha)$ is also called the trace of α . Note that, if $\alpha \in \operatorname{RG}$, then $\pi_h(\alpha) = \pi(\alpha h^{-1}) = \pi(h^{-1}\alpha)$, and hence

$$\alpha = \sum_{g \in G} \pi_g(\alpha)g = \sum_{g \in G} \pi(\alpha g^{-1})g = \sum_{g \in G} \pi(g^{-1}\alpha)g.$$

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Let *H* be a subgroup of *G*. The map $\pi_H \colon \mathrm{RG} \to \mathrm{RH}$ given by $\pi_H \left(\sum_{g \in G} a_g g \right) = \sum_{h \in H} a_h h$ is called the *projection* of RG onto RH. As RH is a subring of RG, RG is naturally a two-sided RH-module.

Recall that a ring R is called *right self-injective* if every homomorphism from a right ideal of R to R_R can be extended to an endomorphism of R_R . And R is called *right P-injective* if every homomorphism from a principal right ideal of Rto R_R can be extended to an endomorphism of R_R . Right P-injective rings were first introduced by Ikeda, see [3] in 1951. In 1963, Connell in [1] proved that for a finite group G, RG is right self-injective if and only if R is right self-injective. In 1971, Renault in [7] showed that G must be finite if RG is right self-injective. Thus RG is right self-injective if and only if R is right self-injective if and only if G is locally finite. In 1995, Nicholson and Yousif in [5], Theorem 4.1 proved the following result on P-injective group rings:

Let R be a ring and G a group.

(i) If RG is right P-injective, then R is right P-injective and G is locally finite.

(ii) If R is right self-injective and G is locally finite, then RG is right P-injective.

In this short paper, based on the above result of Nicholson and Yousif, an equivalent characterization of right P-injective group rings is given in Theorem 2.7. By a similar discussion, we also obtain an equivalent characterization of right *n*-injective group rings (see Theorem 2.9) and right F-injective group rings (see Corollary 2.10), respectively. Let *n* be a positive integer. Recall that a ring *R* is called *right n-injective* (*right F-injective*) if every homomorphism from an *n*-generated (finitely generated) right ideal of *R* to R_R can be extended to an endomorphism of R_R .

2. Results

Lemma 2.1 ([5], Theorem 4.1 (1)). Let R be a ring and G a group. If RG is right P-injective, then R is right P-injective and G is locally finite.

Let H be a subgroup of a group G. A complete set of representatives of left (right) cosets of H in G is called a *left* (*right*) *transversal* of H in G.

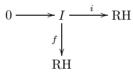
Proposition 2.2. Let R be a ring, H a subgroup of a group G and $\{g_i\}_{i \in A}$ a right transversal of H in G. Assume that I is a right ideal of the group ring RH and $\{\alpha_j \in \text{RH}\}_{j \in B}$ is a set of generators for the right RH-module I. Set $J = \sum_{i \in A} Ig_i$. Then J is a right ideal of RG and $\{\alpha_j \in \text{RH}\}_{j \in B}$ is also a set of generators for the right RG-module J.

Proof. As $I = \sum_{j \in B} \alpha_j(\text{RH})$, we have $J = \sum_{i \in A} \left(\sum_{j \in B} \alpha_j(\text{RH}) \right) g_i = \sum_{i \in A} \sum_{j \in B} (\alpha_j(\text{RH})g_i)$ $= \sum_{j \in B} \alpha_j \left(\sum_{i \in A} (\text{RH})g_i \right) = \sum_{j \in B} \alpha_j(\text{RG}).$

So J is a right ideal of RG generated by $\{\alpha_j \in RH\}_{j \in B}$.

Lemma 2.3. Let H be a subgroup of a group G and n a positive integer. If RG is right n-injective, then RH is also right n-injective.

Proof. Let $\{g_i\}_{i \in A}$ be a right transversal of H in G. It suffices to show that for any *n*-generated right ideal I of RH, the following diagram of RH-homomorphisms can be completed:



Set $J = \sum_{i} Ig_i$. By Proposition 2.2, J is an *n*-generated right ideal of RG. Define $\tilde{f}: J \to RG$ by

$$\tilde{f}\left(\sum \alpha_{i}g_{i}\right) = \sum f(\alpha_{i})g_{i}, \quad \alpha_{i} \in I.$$

If $u \in G$, then $g_i u = h_{ij}g_j$ for some $h_{ij} \in H$, and $j \in A$. So

$$\tilde{f}\left(\left(\sum \alpha_i g_i\right)u\right) = \tilde{f}\left(\sum \alpha_i h_{ij}g_j\right) = \sum f(\alpha_i h_{ij})g_j = \sum f(\alpha_i)h_{ij}g_j$$
$$= \sum f(\alpha_i)g_iu = \tilde{f}\left(\sum \alpha_i g_i\right)u.$$

Thus \tilde{f} is a well-defined right RG-linear map. Since RG is right *n*-injective, there exists a right RG-homomorphism $\tilde{\varphi}$ from RG to RG such that $\tilde{\varphi}|_J = \tilde{f}$. Now set $\varphi = \pi_H \tilde{\varphi}|_{\rm RH}$. Then φ is a right RH-linear map and

$$\varphi|_I = \pi_H \widetilde{\varphi}|_I = \pi_H \widehat{f}|_I = \pi_H f = f.$$

So RH is also right n-injective.

Taking $H = \{1_G\}$ in the above lemma and using Lemma 2.1, we have the following corollary.

Corollary 2.4. Let R be a ring and G a group. If the group ring RG is right *n*-injective (F-injective), then R is right *n*-injective (F-injective) and G is locally finite.

Lemma 2.5 ([1], Proposition 7). Let R be a ring and G a group. Assume that M is a right RG-module. Then there is a group monomorphism

$$t: \operatorname{Hom}_{\mathrm{RG}}(M_{\mathrm{RG}}, \mathrm{RG}_{\mathrm{RG}}) \to \operatorname{Hom}_{R}(M_{R}, R_{R})$$

such that $t(\varphi) = \pi \varphi$ for all $\varphi \in \operatorname{Hom}_{\operatorname{RG}}({}_{\operatorname{RG}}M, {}_{\operatorname{RG}}\operatorname{RG})$. In addition, if G is a finite group, then t is an isomorphism.

Let RG be the group ring of a group G over a ring R and let M be a right R-module. According to [4], elements of the group module MG are defined as follows:

$$\sum_{g \in G} m_g g, \quad \text{where } m_g \in M \text{ and } m_g = 0 \text{ for almost every } g.$$

The sum in MG is defined componentwise:

$$\sum_{g\in G}m_gg+\sum_{g\in G}n_gg=\sum_{g\in G}(m_g+n_g)g.$$

And the scalar product of $\sum\limits_{g\in G}m_gg$ by $\sum\limits_{g\in G}a_gg\in \mathrm{RG}$ is defined by

$$\left(\sum_{g\in G} m_g g\right) \left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} k_g g, \text{ where } k_g = \sum_{hh'=g} m_h a_{h'}.$$

With the above two operations, MG becomes a right RG-module. It is also clear that MG is a right R-module with the canonical scalar product

$$\left(\sum_{g\in G} m_g g\right)r = \sum_{g\in G} (m_g r)g, \quad r \in R.$$

The following result was given in [4] without proof. To be self-contained, we write down the proof.

Lemma 2.6 ([4], Lemma 5.1). Let M_R be a module and let H be a subgroup of a group G. Then

$$(MG)_{RG} \cong (MH \otimes_{RH} RG)_{RG}.$$

Proof. Let K be a right transversal of H in G. Then $\mathrm{RG} = \bigoplus_{k \in K} (\mathrm{RH})k$ is a free left RH-module with basis K. It is easy to see that every element of MH $\otimes_{\mathrm{RH}} \mathrm{RG}$ has the form $\sum_{k \in K} \alpha_k \otimes k, \, \alpha_k \in \mathrm{MH}$. Now define a map

$$\Phi \colon \operatorname{MH} \otimes_{\operatorname{RH}} \operatorname{RG} \to (\operatorname{MG})_{\operatorname{RG}}$$

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such that

$$\Phi\left(\sum_{k\in K}\alpha_k\otimes k\right) = \sum_{k\in K}\alpha_k k, \quad \alpha_k\in \mathrm{MH}.$$

It is clear that Φ is a right RG-homomorphism. Since MG is a direct sum of (MH)k, $k \in K$, Φ is an isomorphism.

Now we prove the main result of this paper.

Theorem 2.7. Let R be a ring and G a group. The following are equivalent:

- (i) RG is right P-injective;
- (ii) (a) *R* is right P-injective;
 - (b) G is a locally finite group;
 - (c) for each finite subgroup H of G and any principal right ideal I of RH, if $f \in \operatorname{Hom}_R(I_R, R_R)$, there exists $g \in \operatorname{Hom}_R(\operatorname{RH}_R, R_R)$, such that $g|_I = f$.

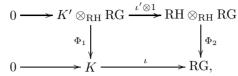
Proof. (i) \Rightarrow (ii). By Lemma 2.1, (a) and (b) are satisfied. For any finite subgroup H of G, by Lemma 2.3, RH is right P-injective. For (c), let H be a finite subgroup of G and I a principal right ideal of RH with $f \in \text{Hom}_R(I_R, R_R)$. Since H is finite, by Lemma 2.5, there exists $\varphi \in \text{Hom}_{\text{RH}}(I_{\text{RH}}, \text{RH}_{\text{RH}})$ such that $f = t(\varphi) = \pi \varphi$. As RH is right P-injective, there exists $\psi \in \text{Hom}_{\text{RH}}(\text{RH}_{\text{RH}}, \text{RH}_{\text{RH}})$ such that $\psi|_I = \varphi$. Take $g = \pi \psi$. Then g is a right R-linear map from RH_R to R_R and $g|_I = f$.

(ii) \Rightarrow (i). First, we show that for any finite subgroup H of G, RH is right P-injective. Assume $I = \alpha$ RH is a principal right ideal of RH and φ is a right RH-homomorphism from $I_{\rm RH}$ to RH_{RH}. We want to find an endomorphism ψ of RH_{RH} such that $\psi|_I = \varphi$. Let $f = \pi \varphi$. Then $f \in \text{Hom}_R(I_R, R_R)$. By the assumption, there exists $g \in \text{Hom}_R(\text{RH}_R, R_R)$ such that $g|_I = f$. Since H is finite, by Lemma 2.5 there exists $\psi \in \text{Hom}_{\rm RG}(\text{RH}_{\rm RH}, \text{RH}_{\rm RH})$ such that $\pi \psi = g$. Thus, $\pi \psi|_I = g_I = f = \pi \varphi$. For each $x \in I$, we have

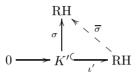
$$\begin{aligned} \varphi(x) &= \sum_{h \in H} \pi(\varphi(x)h^{-1})h = \sum_{h \in H} (\pi\varphi(xh^{-1}))h = \sum_{h \in H} (\pi\psi(xh^{-1}))h \\ &= \sum_{h \in H} \pi(\psi(x)h^{-1})h = \psi(x). \end{aligned}$$

Thus, $\psi|_I = \varphi$.

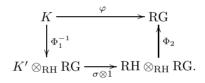
Next we show that RG is right P-injective; this needs to show that, for any principal right ideal $K = \alpha RG$ of RG, every right RG-homomorphism $\varphi \colon K \to RG$ can be extended to an endomorphism of RG_{RG}. Since G is locally finite, there exists a finite subgroup H of G such that $\varphi(\alpha) \in RH$ and $K' = \alpha(RH) \subseteq RH \subseteq RG$. Let $\iota \colon K \to RG$ and $\iota' \colon K' \to RH$ be the natural inclusions. If $\{g_i\}_{i \in A}$ is a right transversal of H, then $RG = \bigoplus_{i \in A} (RH)g_i$ is a free left RH-module. So RG is a free left RH-module. Hence we obtain the following commutative diagram with exact rows:



where Φ_2 is defined accordingly in Lemma 2.6. Since Φ_2 is a right RG-isomorphism by Lemma 2.6, it is clear that Φ_1 is also a right RG-isomorphism. As $\sigma = \pi_H \varphi|_{K'}$: $K' \to \text{RH}$ is a right RH-homomorphism and RH is right P-injective, there exists a right RH-homomorphism $\overline{\sigma}$: RH \to RH such that $\sigma = \overline{\sigma}\iota'$. Thus, we have the following diagrams:



and



Thus, $\varphi = \Phi_2(\sigma \otimes 1)\Phi_1^{-1} = \Phi_2(\overline{\sigma} \otimes 1)(\iota' \otimes 1)\Phi_1^{-1} = \Phi_2(\overline{\sigma} \otimes 1)\Phi_2^{-1}\iota$. So the right RG-homomorphism $\Phi_2(\overline{\sigma} \otimes 1)\Phi_2^{-1}$: RG \rightarrow RG extends φ .

Remark 2.8. By [6], Example 5.70, if R is right P-injective and G is locally finite (even finite), RG need not be right P-injective.

By Corollary 2.4, using discussions similar to those in Theorem 2.7, we have

Theorem 2.9. Let R be a ring, G a group and $n \ge 1$ an integer. The following are equivalent:

- (i) RG is right *n*-injective;
- (ii) (a) R is right n-injective;
 - (b) G is a locally finite group;
 - (c) for each finite subgroup H of G and any *n*-generated right ideal I of RH, if $f \in \operatorname{Hom}_R(I_R, R_R)$, then there exists $g \in \operatorname{Hom}_R(\operatorname{RH}_R, R_R)$ such that $g|_I = f$.

Corollary 2.10. Let R be a ring and G a group. The following are equivalent: (i) RG is right F-injective;

- (ii) (a) R is right F-injective;
 - (b) G is a locally finite group;
 - (c) for each finite subgroup H of G and any finitely right ideal I of RH, if $f \in \operatorname{Hom}_R(I_R, R_R)$, then there exists $g \in \operatorname{Hom}_R(\operatorname{RH}_R, R_R)$ such that $g|_I = f$.

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