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RAMSEY NUMBERS FOR TREES II

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Abstract. Let $r(G_1, G_2)$ be the Ramsey number of the two graphs G_1 and G_2 . For $n_1 \geq n_2 \geq 1$ let $S(n_1, n_2)$ be the double star given by $V(S(n_1, n_2)) = \{v_0, v_1, \dots, v_{n_1}, w_0, w_1, \dots, w_{n_2}\}$ and $E(S(n_1, n_2)) = \{v_0v_1, \dots, v_0v_{n_1}, v_0w_0, w_0w_1, \dots, w_0w_{n_2}\}$. We determine $r(K_{1, m-1}, S(n_1, n_2))$ under certain conditions. For $n \geq 6$ let $T_n^3 = S(n-5, 3)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$, where $V = \{v_0, v_1, \dots, v_{n-1}\}$, $E_2 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}$ and $E_3 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}$. We also obtain explicit formulas for $r(K_{1, m-1}, T_n)$, $r(T_m', T_n)$ ($n \geq m+3$), $r(T_n, T_n)$, $r(T_n', T_n)$ and $r(P_n, T_n)$, where $T_n \in \{T_n'', T_n''', T_n^3\}$, P_n is the path on n vertices and T_n' is the unique tree with n vertices and maximal degree $n-2$.

Keywords: Ramsey number; tree; Turán's problem

MSC 2020: 05C55, 05C05, 05C35

1. INTRODUCTION

In this paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G , and let $\Delta(G)$ and $\delta(G)$ denote the maximal degree and minimal degree of G , respectively.

For a graph G , as usual \overline{G} denotes the complement of G . Let G_1 and G_2 be two graphs. The Ramsey number $r(G_1, G_2)$ is the smallest positive integer n such that, for every graph G with n vertices, either G contains a copy of G_1 or \overline{G} contains a copy of G_2 .

Let \mathbb{N} be the set of positive integers. For $n \in \mathbb{N}$ with $n \geq 6$ let T_n be a tree on n vertices. As mentioned in [8], recently Zhao proved that $r(T_n, T_n) \leq 2n-2$, which was conjectured by Burr and Erdős, see [1].

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Let $m, n \in \mathbb{N}$. For $n \geq 3$ let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1}) = n - 1$, and for $n \geq 4$ let T'_n denote the unique tree on n vertices with $\Delta(T'_n) = n - 2$. In 1972, Harary in [6] showed that for $m, n \geq 3$,

$$(1.1) \quad r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m + n - 3 & \text{if } 2 \nmid mn, \\ m + n - 2 & \text{if } 2 \mid mn. \end{cases}$$

From [2], page 72, if G is a graph with $\delta(G) \geq n - 1$, then G contains every tree on n vertices. Using this fact, in 1995, Guo and Volkmann in [5] proved that for $n > m \geq 4$,

$$(1.2) \quad r(K_{1,m-1}, T'_n) = \begin{cases} m + n - 3 & \text{if } 2 \mid m(n - 1), \\ m + n - 4 & \text{if } 2 \nmid m(n - 1). \end{cases}$$

In 2012 the author in [9] evaluated the Ramsey number $r(T_m, T_n^*)$ for $T_m \in \{P_m, K_{1,m-1}, T'_m, T_m^*\}$, where P_m is a path on m vertices and T_n^* is the tree on n vertices with $V(T_n^*) = \{v_0, v_1, \dots, v_{n-1}\}$ and $E(T_n^*) = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In particular, he proved that for $n > m \geq 7$,

$$(1.3) \quad r(K_{1,m-1}, T_n^*) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid n - 3, \\ m + n - 4 & \text{if } m - 1 \nmid n - 3. \end{cases}$$

For $n \geq 5$ let $T_n^1 = (V, E_1)$ and $T_n^2 = (V, E_2)$ be the trees on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$, $E_1 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-4}v_{n-2}, v_{n-3}v_{n-1}\}$ and $E_2 = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-3}v_{n-1}\}$. Then $\Delta(T_n^1) = \Delta(T_n^2) = \Delta(T_n^*) = n - 3$. In [12], Sun, Wang and Wu proved that

$$(1.4) \quad r(K_{1,m-1}, T_n^1) = r(K_{1,m-1}, T_n^2) = m + n - 4 \quad \text{for } n > m \geq 7 \text{ and } 2 \mid mn.$$

For $n_1, n_2 \in \mathbb{N}$ with $n_1 \geq n_2$, let $S(n_1, n_2)$ be the double star given by

$$\begin{aligned} V(S(n_1, n_2)) &= \{v_0, v_1, \dots, v_{n_1}, w_0, w_1, \dots, w_{n_2}\}, \\ E(S(n_1, n_2)) &= \{v_0v_1, \dots, v_0v_{n_1}, v_0w_0, w_0w_1, \dots, w_0w_{n_2}\}. \end{aligned}$$

We say that v_0 and w_0 are centers of $S(n_1, n_2)$. In [4], Grossman, Harary and Klawe evaluated the Ramsey number $r(S(n_1, n_2), S(n_1, n_2))$ under certain conditions. In particular, they showed that for odd n_1 and $n_2 = 1, 2$,

$$(1.5) \quad r(S(n_1, n_2), S(n_1, n_2)) = \max\{2n_1 + 1, n_1 + 2n_2 + 2\}.$$

It is clear that $T'_n = S(n - 3, 1)$ and $T_n^2 = S(n - 4, 2)$. In this paper, we prove the following general result:

$$(1.6) \quad r(K_{1,m-1}, S(n_1, n_2)) = \begin{cases} m + n_1 & \text{if } 2 \mid mn_1, n_1 \geq m - 2 \geq n_2 \geq 2 \\ & \text{and } n_1 > m - 5 + n_2 + \frac{(n_2 - 1)(n_2 - 2)}{m - 1 - n_2}, \\ m - 1 + n_1 & \text{if } 2 \nmid mn_1, n_1 \geq m - 2 > n_2 \\ & \text{and } n_1 > m - 5 + n_2 + \frac{(n_2 - 1)^2}{m - 2 - n_2}. \end{cases}$$

Also,

$$(1.7) \quad r(K_{1,m-1}, T_n^1) = m + n - 5 \quad \text{for } n \geq m + 2 \geq 7 \text{ and } 2 \nmid mn.$$

For $n \geq 6$ let $T_n^3 = S(n - 5, 3)$, $T_n'' = (V, E_2)$ and $T_n''' = (V, E_3)$, where

$$V = \{v_0, v_1, \dots, v_{n-1}\}, \quad E_2 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_1v_{n-2}, v_2v_{n-1}\}, \\ E_3 = \{v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-3}, v_2v_{n-2}, v_3v_{n-1}\}.$$

Then $\Delta(T_n^3) = \Delta(T_n'') = \Delta(T_n''') = n - 4$. In this paper, we evaluate $r(K_{1,m-1}, T_n)$ and $r(T'_m, T_n)$ for $T_n \in \{T_n'', T_n''', T_n^3\}$. In particular, we show that

$$(1.8) \quad r(K_{1,m-1}, T_n'') = r(K_{1,m-1}, T_n''') = \begin{cases} m + n - 5 & \text{if } 2 \mid m(n - 1), m \geq 7, n \geq 15 \\ & \text{and } n > m + 1 + \frac{8}{m - 6}, \\ m + n - 6 & \text{if } 2 \nmid m(n - 1) \text{ and } n \geq m + 3 \geq 9, \end{cases}$$

and that for $m \geq 9$ and $n > m + 2 + \max\{0, (20 - m)/(m - 8)\}$,

$$(1.9) \quad r(T'_m, T_n'') = r(T'_m, T_n''') = r(T'_m, T_n^3) = \begin{cases} m + n - 5 & \text{if } m - 1 \mid n - 5, \\ m + n - 6 & \text{if } m - 1 \nmid n - 5. \end{cases}$$

We also prove that for $m \geq 11$, $n \geq (m - 3)^2 + 4$ and $m - 1 \nmid n - 5$,

$$(1.10) \quad r(G_m, T_n) = m + n - 6 \quad \text{for } G_m \in \{T_m^*, T_m^1, T_m^2\} \text{ and } T_n \in \{T_n'', T_n''', T_n^3\}.$$

In addition, we establish the following results:

$$\begin{aligned}
 r(T_n'', T_n'') &= r(T_n'', T_n''') = r(T_n''', T_n''') = \begin{cases} 2n - 9 & \text{if } 2 \mid n \text{ and } n > 29, \\ 2n - 8 & \text{if } 2 \nmid n \text{ and } n > 22, \end{cases} \\
 r(T_n^3, T_n'') &= r(T_n^3, T_n''') = r(T_n^3, T_n^3) = 2n - 8 \quad \text{for } n > 22, \\
 r(T_n'', T_n') &= r(T_n''', T_n') = r(T_n^3, T_n') = 2n - 5 \quad \text{for } n \geq 10, \\
 r(T_n'', T_n^i) &= r(T_n''', T_n^i) = r(T_n^3, T_n^i) = 2n - 7 \quad \text{for } n > 16 \text{ and } i = 1, 2, \\
 r(P_n, T_n'') &= r(P_n, T_n''') = r(P_n, T_n^3) = 2n - 9 \quad \text{for } n \geq 33.
 \end{aligned}$$

In addition to the above notation, throughout this paper, we use the following notation: $[x]$ —the greatest integer not exceeding x , $d(v)$ —the degree of the vertex v in a graph, $d(u, v)$ —the distance between the two vertices u and v in a graph, K_n —the complete graph on n vertices, $G[V_1]$ —the subgraph of G induced by vertices in the set V_1 , $G - V_1$ —the subgraph of G obtained by deleting vertices in V_1 and all edges incident with them, $\Gamma(v)$ —the set of vertices adjacent to the vertex v , $e(V_1V_1')$ —the number of edges with one endpoint in V_1 and the other endpoint in V_1' .

2. BASIC LEMMAS

For a forbidden graph L let $\text{ex}(p; L)$ be the maximal number of edges in a graph of order p not containing any copies of L . The corresponding Turán's problem is to evaluate $\text{ex}(p; L)$. Let $p, n \in \mathbb{N}$ with $p \geq n \geq 2$. For a given tree T_n on n vertices, it is difficult to determine the value of $\text{ex}(p; T_n)$. The famous Erdős-Sós conjecture asserts that $\text{ex}(p; T_n) \leq \frac{1}{2}(n-2)p$. Write $p = k(n-1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. In 1975 Faudree and Schelp in [3] showed that

$$(2.1) \quad \text{ex}(p; P_n) = k \binom{n-1}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2}.$$

In [10], [11], [12], the author and his coauthors determined $\text{ex}(p; T_n)$ for $T_n \in \{T_n', T_n^*, T_n^1, T_n^2, T_n^3, T_n'', T_n'''\}$.

Lemma 2.1 ([9], Lemma 2.1). *Let G_1 and G_2 be two graphs. Suppose that $p \in \mathbb{N}$, $p \geq \max\{|V(G_1)|, |V(G_2)|\}$ and $\text{ex}(p; G_1) + \text{ex}(p; G_2) < \binom{p}{2}$. Then $r(G_1, G_2) \leq p$.*

Proof. Let G be a graph of order p . If $e(G) \leq \text{ex}(p; G_1)$ and $e(\overline{G}) \leq \text{ex}(p; G_2)$, then $\text{ex}(p; G_1) + \text{ex}(p; G_2) \geq e(G) + e(\overline{G}) = \binom{p}{2}$. This contradicts the assumption. Hence, either $e(G) > \text{ex}(p; G_1)$ or $e(\overline{G}) > \text{ex}(p; G_2)$. Therefore, G contains a copy of G_1 or \overline{G} contains a copy of G_2 . This shows that $r(G_1, G_2) \leq |V(G)| = p$. So the lemma is proved. \square

Lemma 2.2. *Let $k, p \in \mathbb{N}$ with $p \geq k + 1$. Then there exists a k -regular graph of order p if and only if $2 \mid kp$.*

This is a known result. See for example [11], Corollary 2.1.

Lemma 2.3 ([9], Lemma 2.3). *Let G_1 and G_2 be two graphs with $\Delta(G_1) = d_1 \geq 2$ and $\Delta(G_2) = d_2 \geq 2$. Then:*

- (i) $r(G_1, G_2) \geq d_1 + d_2 - \frac{1}{2}(1 - (-1)^{(d_1-1)(d_2-1)})$.
- (ii) *Suppose that G_1 is a connected graph of order m and $d_1 < d_2 \leq m$. Then $r(G_1, G_2) \geq 2d_2 - 1 \geq d_1 + d_2$.*
- (iii) *Suppose that G_1 is a connected graph of order m and $d_2 > m$. If one of the conditions*
 - (1) $2 \mid (d_1 + d_2 - m)$,
 - (2) $d_1 \neq m - 1$,
 - (3) G_2 *has two vertices u and v such that $d(v) = \Delta(G_2)$ and $d(u, v) = 3$ holds, then $r(G_1, G_2) \geq d_1 + d_2$.*

Lemma 2.4. *Let $p, n \in \mathbb{N}$ with $p \geq n - 1 \geq 1$. Then $\text{ex}(p; K_{1, n-1}) = \lceil \frac{1}{2}(n - 2)p \rceil$.*

This is a known result. See for example [11], Theorem 2.1.

Lemma 2.5 ([11], Theorem 3.1). *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 5$, and let $r \in \{0, 1, \dots, n - 2\}$ be given by $p \equiv r \pmod{n - 1}$. Then*

$$\text{ex}(p; T_n^r) = \begin{cases} \left\lceil \frac{(n - 2)(p - 1) - r - 1}{2} \right\rceil & \text{if } n \geq 7 \text{ and } 2 \leq r \leq n - 4, \\ \frac{(n - 2)p - r(n - 1 - r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.6 ([12], Theorems 2.1 and 3.1). *Suppose that $p, n \in \mathbb{N}$, $p \geq n - 1 \geq 4$ and $p = k(n - 1) + r$, where $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n - 2\}$. For $i = 1$ or 2 ,*

$$\begin{aligned} \text{ex}(p; T_n^i) &= \max \left\{ \left\lceil \frac{(n - 2)p}{2} \right\rceil - (n - 1 + r), \frac{(n - 2)p - r(n - 1 - r)}{2} \right\} \\ &= \begin{cases} \left\lceil \frac{(n - 2)p}{2} \right\rceil - (n - 1 + r) & \text{if } n \geq 16 \text{ and } 3 \leq r \leq n - 6 \\ & \text{or if } 13 \leq n \leq 15 \text{ and } 4 \leq r \leq n - 7, \\ \frac{(n - 2)p - r(n - 1 - r)}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 2.7 ([10], Theorems 3.1 and 5.1). Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then

$$\begin{aligned} \text{ex}(p; T_n'') = \text{ex}(p; T_n''') &= \frac{(n-2)p - r(n-1-r)}{2} \\ &+ \max\left\{0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil\right\}. \end{aligned}$$

Lemma 2.8 ([10], Lemmas 4.6 and 4.7). Let $n \in \mathbb{N}$ with $n \geq 15$. Then

$$\text{ex}(2n-9; T_n^3) = n^2 - 10n + 24 + \max\left\{\left\lceil \frac{n}{2} \right\rceil, 13\right\}$$

and

$$\text{ex}(2n-8; T_n^3) = n^2 - 9n + 29 + \max\left\{\left\lceil \frac{n-37}{4} \right\rceil, 0\right\}.$$

Lemma 2.9 ([10], Theorems 4.1–4.5). Let $p, n \in \mathbb{N}$, $p \geq n \geq 10$, $p = k(n-1) + r$, $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$.

(i) If $r \in \{0, 1, 2, n-6, n-5, n-4, n-3, n-2\}$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p - r(n-1-r)}{2}.$$

(ii) If $n \geq 15$ and $r \in \{3, 4, \dots, n-9\}$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p - r(n-1-r)}{2} + \max\left\{0, \left\lceil \frac{r(n-4-r) - 3(n-1)}{2} \right\rceil\right\}.$$

(iii) If $n \geq 15$ and $r = n-8$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p - 7n + 30}{2} + \max\left\{\left\lceil \frac{n}{2} \right\rceil, 13\right\}.$$

(iv) If $n \geq 15$ and $r = n-7$, then

$$\text{ex}(p; T_n^3) = \frac{(n-2)p - 6(n-7)}{2} + \max\left\{\left\lceil \frac{n-37}{4} \right\rceil, 0\right\}.$$

Lemma 2.10. Let $n \in \mathbb{N}$, $n \geq 10$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume that $p = k(n-1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, \dots, n-2\}$. Then

$$\text{ex}(p; T_n) \leq \frac{(n-2)p}{2} - \min\left\{n-1+r, \frac{r(n-1-r)}{2}\right\}.$$

Proof. This is immediate from [10], Lemmas 2.8, 3.1, 4.1 and 5.1. □

Lemma 2.11 ([11], Theorems 4.1–4.3). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 6$ and $p = k(n-1) + r$ with $k \in \mathbb{N}$ and $r \in \{0, 1, n-5, n-4, n-3, n-2\}$. Then*

$$\text{ex}(p; T_n^*) = \begin{cases} \frac{(n-2)(p-2)}{2} + 1 & \text{if } n > 6 \text{ and } r = n-5, \\ \frac{(n-2)p - r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Lemma 2.12 ([11], Theorem 4.4). *Let $p, n \in \mathbb{N}$, $p \geq n \geq 11$, $r \in \{2, 3, \dots, n-6\}$ and $p \equiv r \pmod{n-1}$. Let $t \in \{0, 1, \dots, r+1\}$ be given by $n-3 \equiv t \pmod{r+2}$. Then*

$$\text{ex}(p; T_n^*) = \begin{cases} \left\lceil \frac{(n-2)(p-1) - 2r - t - 3}{2} \right\rceil & \text{if } r \geq 4 \text{ and } 2 \leq t \leq r-1, \\ \frac{(n-2)(p-1) - t(r+2-t) - r-1}{2} & \text{otherwise.} \end{cases}$$

3. FORMULAS FOR $r(T_n, T_n'')$, $r(T_n, T_n''')$ AND $r(T_n, T_n^3)$

Theorem 3.1. *Let $n \in \mathbb{N}$. Then*

$$r(T_n'', T_n'') = r(T_n'', T_n''') = r(T_n''', T_n''') = \begin{cases} 2n-9 & \text{if } 2 \mid n \text{ and } n > 29, \\ 2n-8 & \text{if } 2 \nmid n \text{ and } n > 22. \end{cases}$$

Proof. Suppose that $T_n, T_n^0 \in \{T_n'', T_n'''\}$. By Lemma 2.7,

$$\begin{aligned} \text{ex}(2n-9; T_n) &= \frac{(2n-9)(n-5) - (n-29)}{2} + \max\left\{0, \left\lceil \frac{n-29}{2} \right\rceil\right\} \\ &= \left\lceil \frac{(2n-9)(n-5)}{2} \right\rceil \quad \text{for } n \geq 29. \end{aligned}$$

Hence, for $n \in \{30, 32, 34, \dots\}$,

$$\text{ex}(2n-9; T_n) + \text{ex}(2n-9; T_n^0) = 2 \left\lceil \frac{(2n-9)(n-5)}{2} \right\rceil = (2n-9)(n-5) - 1 < \binom{2n-9}{2}.$$

Applying Lemma 2.1 yields $r(T_n, T_n^0) \leq 2n - 9$. On the other hand, appealing to Lemma 2.3 (i),

$$r(T_n, T_n^0) \geq n - 4 + n - 4 - \frac{1 - (-1)^{(n-5)(n-5)}}{2} = 2n - 9.$$

Therefore $r(T_n, T_n^0) = 2n - 9$ for $n \in \{30, 32, 34, \dots\}$.

Now assume that $2 \nmid n$ and $n > 22$. By Lemma 2.7,

$$\text{ex}(2n - 8; T_n) = \frac{(n - 2)(2n - 8) - 6(n - 7)}{2} = n^2 - 9n + 29.$$

Thus,

$$\text{ex}(2n - 8; T_n) + \text{ex}(2n - 8; T_n^0) = 2(n^2 - 9n + 29) < 2n^2 - 17n + 36 = \binom{2n - 8}{2}.$$

Hence, $r(T_n, T_n^0) \leq 2n - 8$ by Lemma 2.1. By Lemma 2.2, we may construct a regular graph G of order $2n - 9$ with degree $n - 5$. Clearly \overline{G} is also a regular graph with degree $n - 5$. Since $\Delta(T_n) = \Delta(T_n^0) = n - 4$, both G and \overline{G} do not contain any copies of T_n and T_n^0 . Therefore, $r(T_n, T_n^0) > 2n - 9$ and so $r(T_n, T_n^0) = 2n - 8$. This completes the proof. \square

Theorem 3.2. *Let $n \in \mathbb{N}$ with $n > 22$. Then*

$$r(T_n^3, T_n'') = r(T_n^3, T_n''') = r(T_n^3, T_n^3) = 2n - 8.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. When n is odd, using Lemma 2.3 (i) we see that $r(T_n^3, T_n) \geq n - 4 + n - 4 = 2n - 8$. When n is even, we may construct a regular graph H with degree $n - 10$ and $V(H) = \{v_1, \dots, v_{n-6}\}$. Let G_0 be a graph given by

$$V(G_0) = \{v_0, v_1, \dots, v_{n-4}, u_1, \dots, u_{n-6}\}$$

and

$$\begin{aligned} E(G_0) = E(H) \cup \{ & v_0v_1, \dots, v_0v_{n-4}, v_1v_{n-5}, \dots, v_{n-6}v_{n-5}, v_1v_{n-4}, \dots, v_{n-5}v_{n-4}, v_1u_1, \\ & v_1u_2, v_2u_1, v_2u_2, \dots, v_{n-7}u_{n-7}, v_{n-7}u_{n-6}, v_{n-6}u_{n-7}, v_{n-6}u_{n-6}, \\ & u_1u_2, \dots, u_1u_{n-6}, u_2u_3, \dots, u_2u_{n-6}, u_3u_{n-6}, \dots, u_{n-7}u_{n-6} \}. \end{aligned}$$

Then $d(v_0) = d(v_{n-5}) = d(v_{n-4}) = n - 4$ and $d(v_1) = \dots = d(v_{n-6}) = d(u_1) = \dots = d(u_{n-6}) = n - 5$. Clearly $|V(G_0)| = 2n - 9$ and G_0 does not contain any copies of T_n^3 . Since $\Delta(\overline{G_0}) = n - 5$ and $\Delta(T_n) = n - 4$, $\overline{G_0}$ does not contain any copies of T_n . Thus, $r(T_n^3, T_n) \geq |V(G_0)| + 1 = 2n - 8$.

From Lemma 2.7, $\text{ex}(2n-8; T_n'') = \text{ex}(2n-8; T_n''') = n^2 - 9n + 29$. By Lemma 2.8, $\text{ex}(2n-8; T_n^3) = n^2 - 9n + 29 + \max\{0, \lfloor \frac{1}{4}(n-37) \rfloor\}$. Thus,

$$\begin{aligned} \text{ex}(2n-8; T_n^3) + \text{ex}(2n-8; T_n) &\leq 2n^2 - 18n + 58 + 2 \max\left\{0, \left\lfloor \frac{n-37}{4} \right\rfloor\right\} \\ &< 2n^2 - 18n + 58 + n - 22 = \binom{2n-8}{2}. \end{aligned}$$

Hence, applying Lemma 2.1 gives $r(T_n^3, T_n) \leq 2n-8$ and so $r(T_n^3, T_n) = 2n-8$ as claimed. \square

Theorem 3.3. *Let $n \in \mathbb{N}$ with $n \geq 10$. Then*

$$r(T_n'', T_n') = r(T_n''', T_n') = r(T_n^3, T_n') = 2n-5.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(T_n) = n-4$ and $\Delta(T_n') = n-2$, using Lemma 2.3 (ii) we see that $r(T_n, T_n') \geq 2(n-2) - 1 = 2n-5$. By Lemmas 2.5, 2.7 and 2.9,

$$\begin{aligned} \text{ex}(2n-5; T_n) + \text{ex}(2n-5; T_n') &= \frac{(n-2)(2n-5) - 3(n-4)}{2} + \left\lfloor \frac{(n-2)(2n-6) - (n-3)}{2} \right\rfloor \\ &= \left\lfloor \frac{4n^2 - 23n + 37}{2} \right\rfloor < \frac{4n^2 - 22n + 30}{2} = \binom{2n-5}{2}. \end{aligned}$$

Hence, $r(T_n, T_n') \leq 2n-5$ by Lemma 2.1. Therefore, $r(T_n, T_n') = 2n-5$ as claimed. \square

Theorem 3.4. *Let $n \in \mathbb{N}$, $n > 16$ and $i \in \{1, 2\}$. Then*

$$r(T_n'', T_n^i) = r(T_n''', T_n^i) = r(T_n^3, T_n^i) = 2n-7.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(T_n) = n-4$ and $\Delta(T_n^i) = n-3$, using Lemma 2.3 (ii) we see that $r(T_n, T_n^i) \geq 2(n-3) - 1 = 2n-7$. From Lemmas 2.6, 2.7 and 2.9,

$$\begin{aligned} \text{ex}(2n-7; T_n) + \text{ex}(2n-7; T_n^i) &= \frac{(n-2)(2n-7) - 5(n-6)}{2} + \left\lfloor \frac{(n-2)(2n-7)}{2} \right\rfloor - (2n-7) \\ &= \left\lfloor \frac{4n^2 - 31n + 72}{2} \right\rfloor < \frac{4n^2 - 30n + 56}{2} = \binom{2n-7}{2}. \end{aligned}$$

Hence, $r(T_n, T_n^i) \leq 2n-7$ by Lemma 2.1. Therefore, $r(T_n, T_n^i) = 2n-7$ as claimed. \square

Theorem 3.5. Let $n \in \mathbb{N}$ with $n \geq 10$. Then $r(T_n, T_n^*) = 2n - 5$ for $T_n \in \{T_n'', T_n''', T_n^3\}$.

Proof. By Lemmas 2.7 and 2.9, $\text{ex}(2n - 5; T_n) = \frac{1}{2}((n - 2)(2n - 5) - 3(n - 4)) = n^2 - 6n + 11 < n^2 - 5n + 4$. Thus the result follows from [9], Lemma 3.1. \square

Remark 3.1. By [9], Theorem 6.3 with $m = n$ and $a = 2$, $r(T_n, K_{1, n-1}) = 2n - 3$ for $n \geq 6$ and $T_n \in \{T_n'', T_n''', T_n^3\}$.

Theorem 3.6. Let $n \in \mathbb{N}$. Then $r(P_n, T_n'') = r(P_n, T_n''') = 2n - 9$ for $n \geq 30$ and $r(P_n, T_n^3) = 2n - 9$ for $n \geq 33$.

Proof. Suppose that $n \geq 30$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(T_n) = n - 4$ and $\Delta(P_n) = 2$, appealing to Lemma 2.3 (ii) we obtain $r(P_n, T_n) \geq 2(n - 4) - 1 = 2n - 9$. By (2.1) and Lemma 2.7 for $T_n \in \{T_n'', T_n'''\}$,

$$\begin{aligned} \text{ex}(2n - 9; P_n) + \text{ex}(2n - 9; T_n) &= \frac{(n - 2)(2n - 9) - 7(n - 8)}{2} + \left\lceil \frac{(2n - 9)(n - 5)}{2} \right\rceil \\ &= \left\lceil \frac{4n^2 - 39n + 119}{2} \right\rceil < \frac{4n^2 - 38n + 90}{2} = \binom{2n - 9}{2}. \end{aligned}$$

Hence, applying Lemma 2.1 gives $r(P_n, T_n) \leq 2n - 9$ and so $r(P_n, T_n) = 2n - 9$.

Now assume that $n \geq 33$. From (2.1) and Lemma 2.8,

$$\begin{aligned} \text{ex}(2n - 9; P_n) + \text{ex}(2n - 9; T_n^3) &= \frac{(n - 2)(2n - 9) - 7(n - 8)}{2} + n^2 - 10n + 24 + \left\lceil \frac{n}{2} \right\rceil \\ &= 2n^2 - 20n + 61 + \left\lceil \frac{n}{2} \right\rceil < 2n^2 - 19n + 45 = \binom{2n - 9}{2}. \end{aligned}$$

Hence, $r(P_n, T_n^3) \leq 2n - 9$ by Lemma 2.1 and so $r(P_n, T_n^3) = 2n - 9$. \square

4. FORMULAS FOR $r(K_{1, m-1}, S(n_1, n_2))$, $r(K_{1, m-1}, T_n^1)$, $r(K_{1, m-1}, T_n'')$ AND $r(K_{1, m-1}, T_n''')$

Theorem 4.1. Let $m, n_1, n_2 \in \mathbb{N}$ with $n_1 \geq m - 2 \geq n_2 \geq 2$ and $2 \mid mn_1$. If $n_1 > m - 5 + n_2 + (n_2 - 1)(n_2 - 2)/(m - 1 - n_2)$, then $r(K_{1, m-1}, S(n_1, n_2)) = m + n_1$.

Proof. Since $\Delta(S(n_1, n_2)) = n_1 + 1$, from Lemma 2.3 (i) we see that

$$r(K_{1, m-1}, S(n_1, n_2)) \geq m - 1 + n_1 + 1 - \frac{1 - (-1)^{(m-2)n_1}}{2} = m + n_1.$$

Now we show that $r(K_{1,m-1}, S(n_1, n_2)) \leq m + n_1$. Let G be a graph of order $m + n_1$ such that \overline{G} does not contain any copies of $K_{1,m-1}$. That is, $\Delta(\overline{G}) \leq m - 2$. We show that G contains a copy of $S(n_1, n_2)$. Clearly

$$\delta(G) = m + n_1 - 1 - \Delta(\overline{G}) \geq m + n_1 - 1 - (m - 2) = n_1 + 1.$$

Suppose that $\Delta(G) = n_1 + 1 + s$, $v_0 \in V(G)$, $d(v_0) = \Delta(G)$, $\Gamma(v_0) = \{v_1, \dots, v_{n_1+1+s}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1$. Then $|V_1'| = m - 2 - s$. For $i = 1, 2, \dots, n_1 + 1 + s$, we have

$$|V_1'| + 1 + |\Gamma(v_i) \cap \Gamma(v_0)| \geq d(v_i) \geq \delta(G) \geq n_1 + 1$$

and so

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geq n_1 - |V_1'| = n_1 - (m - 2) + s \geq s.$$

For $s \geq n_2$ we have $|\Gamma(v_i) \cap \Gamma(v_0)| \geq s \geq n_2$ and $|\Gamma(v_0)| - n_2 = n_1 + 1 + s - n_2 \geq n_1 + 1$. Hence $G[V_1]$ contains a copy of $S(n_1, n_2)$ with centers v_0 and v_i .

Now assume that $s < n_2$ and $V_1' = V(G) - V_1 = \{u_1, \dots, u_{m-2-s}\}$. It is clear that for $i = 1, 2, \dots, m - 2 - s$,

$$m - 3 - s + |\Gamma(u_i) \cap \Gamma(v_0)| = |V_1'| - 1 + |\Gamma(u_i) \cap \Gamma(v_0)| \geq d(u_i) \geq \delta(G) \geq n_1 + 1$$

and so $|\Gamma(u_i) \cap \Gamma(v_0)| \geq n_1 - (m - 4 - s)$. It then follows that $e(V_1 V_1') \geq (m - 2 - s) \times (n_1 - (m - 4 - s))$. By the assumption,

$$n_1 > m - 5 + n_2 + \frac{(n_2 - 2)(n_2 - 1)}{m - 1 - n_2} \geq m - 5 + n_2 - 2s + \frac{(n_2 - 2)(n_2 - 1 - s)}{m - 1 - n_2}.$$

Thus, $(m - 1 - n_2)n_1 > (m - 2 - s)(m - 4 - s) + (s + 1)(n_2 - s - 1)$ and so $e(V_1 V_1') \geq (m - 2 - s)(n_1 - (m - 4 - s)) > (n_1 + 1 + s)(n_2 - s - 1)$. Therefore, $|\Gamma(v_i) \cap V_1'| \geq n_2 - s$ for some $v_i \in \Gamma(v_0)$. From the above, $|\Gamma(v_i) \cap \Gamma(v_0)| \geq s$. Thus, G contains a copy of $S(n_1, n_2)$ with centers v_0 and v_i . Therefore $r(K_{1,m-1}, S(n_1, n_2)) \leq m + n_1$ and so the theorem is proved. \square

Corollary 4.1. *Let $m, n \in \mathbb{N}$, $n - 2 \geq m \geq 4$ and $2 \mid mn$. Then $r(K_{1,m-1}, T_n^2) = m + n - 4$.*

Proof. Since $T_n^2 = S(n - 4, 2)$, putting $n_1 = n - 4$ and $n_2 = 2$ in Theorem 4.1 yields the result. \square

Corollary 4.2. *Let $m, n \in \mathbb{N}$, $m \geq 5$, $n > m + 3 + 2/(m - 4)$ and $2 \mid m(n - 1)$. Then $r(K_{1,m-1}, T_n^3) = m + n - 5$.*

P r o o f. Since $T_n^3 = S(n - 5, 3)$, taking $n_1 = n - 5$ and $n_2 = 3$ in Theorem 4.1 gives the result. \square

Theorem 4.2. *Let $m, n_1, n_2 \in \mathbb{N}$, $n_1 \geq m - 2 > n_2$ and $2 \nmid mn_1$. If $n_1 > m - 5 + n_2 + (n_2 - 1)^2 / (m - 2 - n_2)$, then $r(K_{1, m-1}, S(n_1, n_2)) = m - 1 + n_1$.*

P r o o f. Since $\Delta(S(n_1, n_2)) = n_1 + 1$, from Lemma 2.3 (i) we see that

$$r(K_{1, m-1}, S(n_1, n_2)) \geq m - 1 + n_1 + 1 - \frac{1 - (-1)^{(m-2)n_1}}{2} = m - 1 + n_1.$$

Now we show that $r(K_{1, m-1}, S(n_1, n_2)) \leq m - 1 + n_1$. Let G be a graph of order $m - 1 + n_1$ such that \overline{G} does not contain any copies of $K_{1, m-1}$. We need to show that G contains a copy of $S(n_1, n_2)$. Clearly $\Delta(\overline{G}) \leq m - 2$ and so $\delta(G) = m - 2 + n_1 - \Delta(\overline{G}) \geq n_1$. Since $2 \nmid mn_1$, there is no regular graph of order $m - 1 + n_1$ with degree n_1 by Euler's theorem. Hence $\Delta(G) \geq \delta(G) + 1 \geq n_1 + 1$. Suppose that $\Delta(G) = n_1 + 1 + s$, $v_0 \in V(G)$, $d(v_0) = \Delta(G)$, $\Gamma(v_0) = \{v_1, \dots, v_{n_1+1+s}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V'_1 = V(G) - V_1$. Then $|V'_1| = m - 3 - s$. For $v_i \in \Gamma(v_0)$, $d(v_i) \geq \delta(G) \geq n_1$ and so $|\Gamma(v_i) \cap \Gamma(v_0)| + 1 + |V'_1| \geq d(v_i) \geq n_1$. Thus,

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geq n_1 - 1 - |V'_1| = n_1 - (m - 2) + s \geq s.$$

Hence, $G[V_1]$ contains a copy of $S(n_1, n_2)$ with centers v_0 and v_i for $s \geq n_2$.

Now assume that $s < n_2$ and $V'_1 = \{u_1, \dots, u_{m-3-s}\}$. As $d(u_i) \geq n_1$, we see that $|\Gamma(u_i) \cap \Gamma(v_0)| \geq n_1 - (m - 4 - s)$ and so $e(V_1 V'_1) \geq (m - 3 - s)(n_1 - (m - 4 - s))$. Since

$$n_1 > m - 5 + n_2 + \frac{(n_2 - 1)^2}{m - 2 - n_2} \geq m - 5 + n_2 - 2s + \frac{(n_2 - 1)(n_2 - 1 - s)}{m - 2 - n_2},$$

we get $(m - 2 - n_2)n_1 > (m - 3 - s)(m - 4 - s) + (s + 1)(n_2 - s - 1)$. Hence,

$$\begin{aligned} e(V_1 V'_1) &\geq (m - 3 - s)(n_1 - (m - 4 - s)) \\ &> (m - 3 - s)n_1 - (m - 2 - n_2)n_1 + (s + 1)(n_2 - s - 1) \\ &= (n_1 + 1 + s)(n_2 - s - 1). \end{aligned}$$

Therefore, we have $|\Gamma(v_i) \cap V'_1| \geq n_2 - s$ for some $v_i \in \Gamma(v_0)$. From the above, $|\Gamma(v_i) \cap \Gamma(v_0)| \geq s$. Thus, G contains a copy of $S(n_1, n_2)$ with centers v_0 and v_i . Consequently, $r(K_{1, m-1}, S(n_1, n_2)) \leq m - 1 + n_1$ and so the theorem is proved. \square

Corollary 4.3. *Let $m, n \in \mathbb{N}$, $m \geq 5$, $n > m + 1 + 1 / (m - 4)$ and $2 \nmid mn$. Then $r(K_{1, m-1}, T_n^2) = m + n - 5$.*

Proof. Since $T_n^2 = S(n-4, 2)$, putting $n_1 = n-4$ and $n_2 = 2$ in Theorem 4.2 yields the result. \square

Corollary 4.4. *Let $m, n \in \mathbb{N}$, $m \geq 6$, $n > m + 3 + 4/(m-5)$ and $2 \nmid m(n-1)$. Then $r(K_{1,m-1}, T_n^3) = m + n - 6$.*

Proof. Since $T_n^3 = S(n-5, 3)$, putting $n_1 = n-5$ and $n_2 = 3$ in Theorem 4.2 we deduce the result. \square

Theorem 4.3. *Let $m, n \in \mathbb{N}$, $n \geq m + 2 \geq 7$ and $2 \nmid mn$. Then $r(K_{1,m-1}, T_n^1) = m + n - 5$.*

Proof. Since $n > m$ and $2 \nmid mn$, we have $n \geq m + 2$. Let G be a graph of order $m+n-5$ such that \overline{G} does not contain any copies of $K_{1,m-1}$. We show that G contains a copy of T_n^1 . Clearly $\Delta(\overline{G}) \leq m-2$ and so $\delta(G) = m+n-6-\Delta(\overline{G}) \geq n-4$. If $\Delta(G) = n-4$, then G is a regular graph of order $m+n-5$ with degree $n-4$ and so $(m+n-5)(n-4) = 2e(G)$. Since $m+n-5$ and $n-4$ are odd, we get a contradiction. Thus, $\Delta(G) \geq n-3$. Assume that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n-3+c$, $\Gamma(v_0) = \{v_1, \dots, v_{n-3+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V_1' = V(G) - V_1 = \{u_1, u_2, \dots, u_{m-3-c}\}$. Since $\delta(G) \geq n-4$ for $v_i \in \Gamma(v_0)$ we have $1 + |\Gamma(v_i) \cap \Gamma(v_0)| + |V_1'| \geq d(v_i) \geq n-4$ and so $|\Gamma(v_i) \cap \Gamma(v_0)| \geq n-5 - (m-3-c) = n-m-2+c \geq c$.

We first assume that $|V_1'| = m-3-c \geq 2$. For $i = 1, 2$ we have $|\Gamma(u_i) \cap \Gamma(v_0)| + |V_1'| - 1 \geq d(u_i) \geq \delta(G) \geq n-4$ and so $|\Gamma(u_i) \cap \Gamma(v_0)| \geq n-4+1 - (m-3-c) = n-m+c \geq 2$. Hence G contains a copy of T_n^1 . If $|V_1'| = 1$, then $c = m-4 \geq 1$. Since $d(u_1) \geq n-4 > 1$, we have $u_1 v_j \in E(G)$ for some $v_j \in \Gamma(v_0)$. Recall that $|\Gamma(v_i) \cap \Gamma(v_0)| \geq c \geq 1$ for $v_i \in \Gamma(v_0)$. G must contain a copy of T_n^1 . Now assume that $|V_1'| = 0$. That is, $c = m-3$ and $G = G[V_1]$. Since $d(v_0) = n-3+m-3 \geq n-3+2$ and $d(v_i) \geq n-4 \geq 3$ for $v_i \in \Gamma(v_0)$, we see that $G[\Gamma(v_0)]$ contains a copy of $2K_2$ and so G contains a copy of T_n^1 .

By the above, G contains a copy of T_n^1 . Therefore $r(K_{1,m-1}, T_n^1) \leq m+n-5$. From Lemma 2.3,

$$r(K_{1,m-1}, T_n^j) \geq m-1+n-3 - \frac{1 - (-1)^{(m-2)(n-4)}}{2} = m+n-5.$$

Hence $r(K_{1,m-1}, T_n^1) = m+n-5$ as claimed. \square

Lemma 4.1. *Let $m, n \in \mathbb{N}$, $n \geq 15$, $m \geq 7$, $n > m + 1 + 8/(m-6)$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Let G_m be a connected graph of order m such that $\text{ex}(m+n-5; G_m) \leq \frac{1}{2}(m-2)(m+n-5)$. Then $r(G_m, T_n) \leq m+n-5$. Moreover, if $m-1 \mid n-5$, then $r(G_m, T_n) = m+n-5$.*

Proof. If $T_n \neq T_n^3$ or $m \notin \{n-3, n-4\}$, appealing to Lemmas 2.7 and 2.9 we have

$$\begin{aligned} \text{ex}(m+n-5; T_n) &= \frac{(n-2)(m+n-5) - (m-4)(n-m+3)}{2} \\ &\quad + \max\left\{0, \left\lceil \frac{(m-4)(n-m) - 3(n-1)}{2} \right\rceil\right\} \\ &= \frac{(n-2)(m+n-5) - (m-4)(n-m+3)}{2} \\ &\quad + \max\left\{0, \left\lceil \frac{(m-7)(n-m-3) - 18}{2} \right\rceil\right\}. \end{aligned}$$

Thus, if $(m-7)(n-m-3) \geq 18$, then

$$\begin{aligned} &\text{ex}(m+n-5; G_m) + \text{ex}(m+n-5; T_n) \\ &\leq \frac{(m-2)(m+n-5)}{2} + \frac{(n-2)(m+n-5) - (m-4)(n-m+3)}{2} \\ &\quad + \frac{(m-7)(n-m-3) - 18}{2} \\ &= \frac{(m+n-5)(m+n-7)}{2} \\ &< \binom{m+n-5}{2}. \end{aligned}$$

If $(m-7)(n-m-3) < 18$, since $n > m+1+8/(m-6)$ we see that $(m-6)n > m^2 - 5m + 2$, $(m-4)(n-m+3) > 2(m+n-5)$ and so

$$\begin{aligned} &\text{ex}(m+n-5; G_m) + \text{ex}(m+n-5; T_n) \\ &\leq \frac{(m-2+n-2)(m+n-5) - (m-4)(n-m+3)}{2} < \binom{m+n-5}{2}. \end{aligned}$$

Hence, $r(G_m, T_n) \leq m+n-5$ by Lemma 2.1.

For $m = n-3$, using Lemma 2.8 we see that

$$\begin{aligned} \text{ex}(m+n-5; G_m) + \text{ex}(m+n-5; T_n^3) &= \text{ex}(2n-8; G_{n-3}) + \text{ex}(2n-8; T_n^3) \\ &\leq \frac{(2n-8)(n-5)}{2} + n^2 - 9n + 29 + \max\left\{0, \left\lceil \frac{n-37}{4} \right\rceil\right\} \\ &= 2n^2 - 18n + 49 + \max\left\{0, \left\lceil \frac{n-37}{4} \right\rceil\right\} \\ &< 2n^2 - 17n + 36 = \binom{m+n-5}{2}. \end{aligned}$$

For $m = n - 4$, appealing to Lemma 2.8,

$$\begin{aligned} \text{ex}(m + n - 5; G_m) + \text{ex}(m + n - 5; T_n^3) &= \text{ex}(2n - 9; G_{n-4}) + \text{ex}(2n - 9; T_n^3) \\ &\leq \frac{(2n - 9)(n - 6)}{2} + n^2 - 10n + 24 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} \\ &= 2n^2 - 20n + 51 - \frac{n}{2} + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} \\ &< 2n^2 - 19n + 45 = \binom{m + n - 5}{2}. \end{aligned}$$

Thus, $r(G_m, T_n^3) \leq m + n - 5$ for $m = n - 4, n - 3$ by Lemma 2.1.

Now assume that $m - 1 \mid n - 5$. Then $m + n - 6 = k(m - 1)$ for $k \in \{2, 3, \dots\}$. Since $\Delta(\overline{kK_{m-1}}) = n - 5$ we see that kK_{m-1} does not contain G_m as a subgraph and $\overline{kK_{m-1}}$ does not contain T_n as a subgraph. Hence $r(G_m, T_n) > k(m - 1) = m + n - 6$ and so $r(G_m, T_n) = m + n - 5$. The proof is now complete. \square

Theorem 4.4. *Let $m, n \in \mathbb{N}$, $n \geq 15$, $m \geq 7$, $n > m + 1 + 8/(m - 6)$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. If $2 \mid m(n - 1)$, then $r(K_{1, m-1}, T_n) = m + n - 5$.*

Proof. By Euler's theorem or Lemma 2.4, $\text{ex}(m + n - 5; K_{1, m-1}) \leq \frac{1}{2}(m - 2) \times (m + n - 5)$. Thus, applying Lemma 4.1 we obtain $r(K_{1, m-1}, T_n) \leq m + n - 5$. Suppose that $2 \nmid m(n - 1)$. By Lemma 2.3,

$$r(K_{1, m-1}, T_n) \geq m - 1 + n - 4 - \frac{1 - (-1)^{(m-2)(n-5)}}{2} = m + n - 5.$$

Thus the result follows. \square

Corollary 4.5. *Let $n \in \mathbb{N}$, $n \geq 17$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Then $r(K_{1, n-3}, T_n) = 2n - 7$.*

Proof. Taking $m = n - 2$ in Theorem 4.4 gives the result. \square

Theorem 4.5. *Let $m, n \in \mathbb{N}$, $m \geq 6$, $n \geq m + 3$ and $2 \nmid m(n - 1)$. Then*

$$r(K_{1, m-1}, T_n'') = r(K_{1, m-1}, T_n''') = m + n - 6.$$

Proof. Let G be a graph of order $m + n - 6$ such that \overline{G} does not contain any copies of $K_{1, m-1}$. That is, $\Delta(\overline{G}) \leq m - 2$. Thus, $\delta(G) = m + n - 7 - \Delta(\overline{G}) \geq n - 5$. If $\Delta(G) = n - 5$, then G is a regular graph of order $m + n - 6$ with degree $n - 5$ and so $(m + n - 6)(n - 5) = 2e(G)$. Since $m + n - 6$ and $n - 5$ are odd, we get a contradiction. Thus, $\Delta(G) \geq n - 4$. Assume that $v_0 \in V(G)$, $d(v_0) = \Delta(G) = n - 4 + c$, $\Gamma(v_0) =$

$\{v_1, \dots, v_{n-4+c}\}$, $V_1 = \{v_0\} \cup \Gamma(v_0)$ and $V'_1 = V(G) - V_1 = \{u_1, \dots, u_{m-3-c}\}$. Since $\delta(G) \geq n - 5$, we see that for $v_i \in \Gamma(v_0)$, $|\Gamma(v_i) \cap \Gamma(v_0)| + 1 + |V'_1| \geq d(v_i) \geq n - 5$ and so

$$|\Gamma(v_i) \cap \Gamma(v_0)| \geq n - 5 - 1 - (m - 3 - c) = n - m - 3 + c \geq c.$$

For $u_i \in V'_1$, we see that $|\Gamma(u_i) \cap \Gamma(v_0)| + |V'_1| - 1 \geq d(u_i) \geq n - 5$ and so

$$|\Gamma(u_i) \cap \Gamma(v_0)| \geq n - 5 - (m - 4 - c) = n - m - 1 + c \geq 2 + c.$$

We first assume that $c = 0$. Since $|V'_1| = m - 3 \geq 3$ and $\delta(G) \geq n - 5$, we see that $|\Gamma(u_i) \cap \{v_1, \dots, v_{n-4}\}| \geq n - 5 - (m - 4) = n - m - 1 \geq 2$ for $u_i \in V'_1$ and so $e(V_1 V'_1) \geq (m-3)(n-m-1)$. Since $n \geq m+3$ we see that $(m-4)n \geq (m-4)(m+3) = m^2 - m - 12 > m^2 - 2m - 7$ and so $e(V_1 V'_1) \geq (m-3)(n-m-1) > n - 4$. Therefore, $|\Gamma(v_i) \cap V'_1| \geq 2$ for some $i \in \{1, 2, \dots, n-4\}$. With no loss of generality, we may suppose that $u_1 v_i, u_2 v_i, u_2 v_j, u_3 v_k \in E(G)$, where v_i, v_j, v_k are distinct vertices in $\Gamma(v_0)$. Thus G contains a copy of T''_n and a copy of T'''_n .

Next we assume that $|V'_1| = m - 3 - c \geq 3$ and $c \geq 1$. Then $|\Gamma(u_i) \cap \Gamma(v_0)| \geq 3$ for $i = 1, 2, 3$. Hence there are distinct vertices $v_j, v_k, v_l \in \Gamma(v_0)$ such that $u_1 v_j, u_2 v_k, u_3 v_l \in E(G)$ and so G contains a copy of T'''_n . Since $d(v_j) \geq n - 5 > 2$, v_j is adjacent to some vertex w different from v_0 and u_1 . Hence, G contains a copy of T''_n .

Now assume that $|V'_1| = 2$. That is, $c = m - 5$. Since $|\Gamma(u_i) \cap \Gamma(v_0)| \geq \delta(G) - 1 \geq n - 6 \geq 3$ for $i = 1, 2$, and $|\Gamma(v_i) \cap \Gamma(v_0)| \geq n - m - 3 + c = n - 8 \geq 1$ for $v_i \in \Gamma(v_0)$, it is easy to see that G contains a copy of T''_n and a copy of T'''_n .

Suppose that $|V'_1| = 1$. Then $c = m - 4 \geq 2$, $d(u_1) \geq \delta(G) \geq n - 5 \geq 4$ and $d(v_i) \geq \delta(G) \geq n - 5 \geq 4$ for $i = 1, 2, \dots, n - 4 + m - 4$. Hence G contains a copy of T''_n and a copy of T'''_n .

Finally we assume that $|V'_1| = 0$. That is, $c = m - 3$. Since $d(v_i) \geq \delta(G) \geq n - 5 \geq 4$ for $i = 1, 2, \dots, n - 4 + m - 3$, it is easy to see that G contains a copy of T''_n and a copy of T'''_n .

Suppose that $T_n \in \{T''_n, T'''_n\}$. By the above, G contains a copy of T_n . Hence $r(K_{1,m-1}, T_n) \leq m + n - 6$. By Lemma 2.3, $r(K_{1,m-1}, T_n) \geq m - 1 + n - 4 - \frac{1}{2}(1 - (-1)^{(m-2)(n-5)}) = m + n - 6$. Thus $r(K_{1,m-1}, T_n) = m + n - 6$ as asserted. \square

Theorem 4.6. *Let $n \in \mathbb{N}$ with $n \geq 15$. Then $r(K_{1,n-4}, T_n^3) = 2n - 8$.*

Proof. By Euler's theorem, $\text{ex}(2n - 8; K_{1,n-4}) \leq \frac{1}{2}(n - 5)(2n - 8)$. Thus, $r(K_{1,n-4}, T_n^3) \leq 2n - 8$ by taking $G_m = K_{1,n-4}$ in Lemma 4.1. If $2 \nmid n$, from Lemma 2.3 we have $r(K_{1,n-4}, T_n^3) \geq n - 4 + n - 4 = 2n - 8$. Thus the result is true for odd n . Now assume that $2 \mid n$. Let G_0 be the graph of order $2n - 9$ constructed

in Theorem 3.2. Then G_0 does not contain T_n^3 as a subgraph. As $\delta(G_0) = n - 5$, we have $\Delta(\overline{G}_0) = 2n - 10 - (n - 5) = n - 5$ and so \overline{G}_0 does not contain $K_{1,n-4}$ as a subgraph. Hence $r(K_{1,n-4}, T_n^3) > |V(G_0)| = 2n - 9$ and so $r(K_{1,n-4}, T_n^3) = 2n - 8$ as claimed. \square

Theorem 4.7. *Let $n \in \mathbb{N}$ with $n \geq 10$. Then*

$$r(K_{1,n-2}, T_n^3) = r(K_{1,n-2}, T_n'') = r(K_{1,n-2}, T_n''') = 2n - 5.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(K_{1,n-2}) = n - 2$ and $\Delta(T_n) = n - 4$, we have $r(K_{1,n-2}, T_n) \geq 2(n - 2) - 1 = 2n - 5$ by Lemma 2.3 (ii). By Lemmas 2.4, 2.7 and 2.9,

$$\begin{aligned} \text{ex}(2n - 5; K_{1,n-2}) &= \left\lceil \frac{(n-3)(2n-5)}{2} \right\rceil = n^2 - 6n + 8 + \left\lceil \frac{n-1}{2} \right\rceil, \\ \text{ex}(2n - 5; T_n) &= \frac{(n-2)(2n-5) - 3(n-4)}{2} = n^2 - 6n + 11. \end{aligned}$$

Thus,

$$\begin{aligned} \text{ex}(2n - 5; K_{1,n-2}) + \text{ex}(2n - 5; T_n) &= n^2 - 6n + 8 + \left\lceil \frac{n-1}{2} \right\rceil + n^2 - 6n + 11 \\ &< 2n^2 - 11n + 15 = \binom{2n-5}{2}. \end{aligned}$$

Now, applying Lemma 2.1 yields $r(K_{1,n-2}, T_n) \leq 2n - 5$ and so $r(K_{1,n-2}, T_n) = 2n - 5$, which proves the theorem. \square

5. FORMULAS FOR $r(T'_m, T''_n)$, $r(T'_m, T'''_n)$ AND $r(T'_m, T_n^3)$

Theorem 5.1. *Let $m, n \in \mathbb{N}$, $n \geq 15$, $m \geq 7$ and $m - 1 \mid n - 5$. Suppose that $G_m \in \{P_m, T'_m, T_m^*, T_m^1, T_m^2, T_m^3, T_m'', T_m'''\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume that $m \geq 10$ or $G_m \notin \{T_m^3, T_m'', T_m'''\}$. Then $r(G_m, T_n) = m + n - 5$.*

Proof. Note that $m + n - 5 \equiv 1 \pmod{m-1}$. By (2.1) and Lemmas 2.5, 2.6, 2.10 and 2.11, $\text{ex}(m + n - 5; G_m) \leq \frac{1}{2}(m-2)(m+n-5)$. Thus, applying Lemma 4.1 and the fact $n \geq m + 4$ gives the result. \square

Theorem 5.2. *Let $m, n \in \mathbb{N}$, $m \geq 9$, $n > m + 2 + \max\{0, (20 - m)/(m - 8)\}$ and $m - 1 \nmid n - 5$. Then*

$$r(T'_m, T''_n) = r(T'_m, T'''_n) = r(T'_m, T_n^3) = m + n - 6.$$

Proof. Let $T_n \in \{T_n'', T_n''', T_n^3\}$. Since $\Delta(T_m') = m - 2 < m - 1$ and $\Delta(T_n) = n - 4 > m - 2$, we have $r(T_m', T_n) \geq m - 2 + n - 4 = m + n - 6$ by Lemma 2.3 (ii)–(iii). Note that $m \geq 9$ and so $n \geq 15$. Since $n > m + 2 + (20 - m)/(m - 8)$, we see that $(m - 8)n > m^2 - 7m + 4$ and so $(m - 5)(n - m + 4) > 3(m + n - 6) - (m - 2)$.

Suppose that $T_n \neq T_n^3$ or $n \neq m + 3$. From Lemmas 2.7 and 2.9, if $(m - 5)(n - m + 1) \geq 3(n - 1)$, then

$$\begin{aligned} \text{ex}(m + n - 6; T_n) &\leq \frac{(n - 2)(m + n - 6) - (m - 5)(n - m + 4)}{2} \\ &\quad + \frac{(m - 5)(n - m + 1) - 3(n - 1)}{2} = \frac{(n - 5)(m + n - 6)}{2}; \end{aligned}$$

if $(m - 5)(n - m + 1) < 3(n - 1)$, then

$$\begin{aligned} \text{ex}(m + n - 6; T_n) &= \frac{(n - 2)(m + n - 6) - (m - 5)(n - m + 4)}{2} \\ &< \frac{(n - 2)(m + n - 6) - 3(m + n - 6) + m - 2}{2} \\ &= \frac{(n - 5)(m + n - 6) + m - 2}{2}. \end{aligned}$$

Recall that $m - 1 \nmid n - 5$. By Lemma 2.5, $\text{ex}(m + n - 6; T_m') \leq \frac{1}{2}((m - 2)(m + n - 6) - (m - 2))$. Thus,

$$\begin{aligned} \text{ex}(m + n - 6; T_m') + \text{ex}(m + n - 6; T_n) \\ &< \frac{(m - 2)(m + n - 6) - (m - 2)}{2} + \frac{(n - 5)(m + n - 6) + m - 2}{2} = \binom{m + n - 6}{2}. \end{aligned}$$

Now applying Lemma 2.1 yields $r(T_m', T_n) \leq m + n - 6$ and so $r(T_m', T_n) = m + n - 6$.

Now assume that $T_n = T_n^3$ and $n = m + 3$. Then $\max\{0, (20 - m)/(m - 8)\} < 1$ and so $m = n - 3 \geq 15$. Also, $m + n - 6 = 2n - 9 = n - 1 + n - 8 = 2m - 3 = m - 1 + m - 2$. From Lemma 2.9 (iii),

$$\text{ex}(2n - 9; T_n^3) = n^2 - 10n + 24 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\}.$$

By Lemma 2.5, $\text{ex}(2m - 3; T_m') = \frac{1}{2}((m - 2)(2m - 3) - (m - 2)) = (m - 2)^2 = (n - 5)^2$. Thus,

$$\begin{aligned} \text{ex}(m + n - 6; T_m') + \text{ex}(m + n - 6; T_n^3) \\ &= (n - 5)^2 + n^2 - 10n + 24 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} \\ &= 2n^2 - 20n + 49 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} < 2n^2 - 19n + 45 = \binom{2n - 9}{2}. \end{aligned}$$

Applying Lemma 2.1 gives $r(T_m', T_n^3) \leq m + n - 6$ and so $r(T_m', T_n^3) = m + n - 6$ for $n = m + 3$. This completes the proof. \square

Theorem 5.3. *Let $n \in \mathbb{N}$ with $n \geq 18$. Then*

$$r(T'_{n-3}, T''_n) = r(T'_{n-3}, T'''_n) = r(T'_{n-3}, T_n^3) = 2n - 9.$$

Proof. Suppose that $T_n \in \{T''_n, T'''_n, T_n^3\}$. Since $\Delta(T_n) = n - 4 > n - 5 = \Delta(T'_{n-3})$, from Lemma 2.3 (ii) we have $r(T'_{n-3}, T_n) \geq 2(n - 4) - 1 = 2n - 9$. By Lemma 2.5, $\text{ex}(2n - 9; T'_{n-3}) = \frac{1}{2}(n - 5)(2n - 10) = n^2 - 10n + 25$. From Lemma 2.7 for $T_n \in \{T''_n, T'''_n\}$,

$$\begin{aligned} \text{ex}(2n - 9; T_n) &= \frac{(n - 2)(2n - 9) - 7(n - 8)}{2} + \max\left\{0, \left\lfloor \frac{4(n - 8) - 3(n - 1)}{2} \right\rfloor\right\} \\ &= n^2 - 10n + 37 + \max\left\{0, \left\lfloor \frac{n - 29}{2} \right\rfloor\right\} < n^2 - 9n + 20 \end{aligned}$$

and so

$$\text{ex}(2n - 9; T'_{n-3}) + \text{ex}(2n - 9; T_n) < n^2 - 10n + 25 + n^2 - 9n + 20 = \binom{2n - 9}{2}.$$

Now, applying Lemma 2.1 yields $r(T'_{n-3}, T_n) \leq 2n - 9$ and so $r(T'_{n-3}, T_n) = 2n - 9$. On the other hand, from Lemma 2.8 we have

$$\text{ex}(2n - 9; T_n^3) = n^2 - 10n + 24 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} < n^2 - 9n + 20.$$

Thus,

$$\text{ex}(2n - 9; T'_{n-3}) + \text{ex}(2n - 9; T_n^3) < n^2 - 10n + 25 + n^2 - 9n + 20 = \binom{2n - 9}{2}.$$

Applying Lemma 2.1, $r(T'_{n-3}, T_n^3) \leq 2n - 9$ and so $r(T'_{n-3}, T_n^3) = 2n - 9$, which completes the proof. \square

Theorem 5.4. *Let $m, n \in \mathbb{N}$ with $n > m \geq 10$, and $T_m \in \{T''_m, T'''_m, T_m^3\}$. Then*

$$r(T_m, T'_n) = r(T_m, T_n^*) = \begin{cases} m + n - 3 & \text{if } m - 1 \mid n - 3, \\ m + n - 4 & \text{if } m - 1 \nmid n - 3 \text{ and } n \geq (m - 3)^2 + 2. \end{cases}$$

Proof. If $m - 1 \mid n - 3$, then $\text{ex}(m + n - 3; T_m) = \frac{1}{2}((m - 2)(m + n - 3) - (m - 2))$ by Lemmas 2.7 and 2.9. Thus, the result follows from [9], Theorems 4.1 and 5.1.

Now assume that $m - 1 \nmid n - 3$. By Lemma 2.10, $\text{ex}(m + n - 4; T_m) < \frac{1}{2}(m - 2) \times (m + n - 4)$. Applying [9], Theorems 4.4 and 5.4 deduces the result. The proof is now complete. \square

6. EVALUATION OF $r(T_m^0, T_n)$ WITH $T_m^0 \in \{T_m^*, T_m^1, T_m^2\}$ AND $T_n \in \{T_n'', T_n''', T_n^3\}$

Lemma 6.1 ([7], Theorem 8.3, pages 11–12). *Let $a, b, n \in \mathbb{N}$. If a is coprime to b and $n \geq (a-1)(b-1)$, then there are two nonnegative integers x and y such that $n = ax + by$.*

Theorem 6.1. *Let $m, n \in \mathbb{N}$ with $m \geq 9$, $n > m+1+12/(m-8)$ and $m-1 \nmid n-5$. Suppose that $T_m^0 \in \{T_m^*, T_m^1, T_m^2\}$ and $T_n \in \{T_n'', T_n''', T_n^3\}$. Assume that $T_m^0 \neq T_m^*$ or $m \geq 11$. Then $r(T_m^0, T_n) = m+n-7$ or $m+n-6$. If $n \geq (m-3)^2+4$ or $m+n-7 = (m-1)x + (m-2)y$ for some nonnegative integers x and y , then $r(T_m^0, T_n) = m+n-6$.*

Proof. Note that $\Delta(T_m^0) = m-3 < n-4 = \Delta(T_n)$. Using Lemma 2.3 (ii)–(iii), $r(T_m^0, T_n) \geq m-3+n-4 = m+n-7$. Since $m-1 \nmid n-5$, from Lemmas 2.6, 2.11 and 2.12 we have $\text{ex}(m+n-6; T_m^0) \leq \frac{1}{2}((m-2)(m+n-6) - (m-2))$.

We first assume that $T_n \neq T_n^3$ or $n \neq m+2, m+3$. By the proof of Theorem 5.2, $\text{ex}(m+n-6; T_n) < \frac{1}{2}((n-5)(m+n-6) + m-2)$. Thus,

$$\begin{aligned} & \text{ex}(m+n-6; T_m^0) + \text{ex}(m+n-6; T_n) \\ & < \frac{(m-2)(m+n-6) - (m-2)}{2} + \frac{(n-5)(m+n-6) + m-2}{2} = \binom{m+n-6}{2}. \end{aligned}$$

Hence, $r(T_m^0, T_n) \leq m+n-6$ by Lemma 2.1 and so $r(T_m^0, T_n) = m+n-6$ or $m+n-7$.

We next assume that $T_n = T_n^3$ and $n = m+2$. Then $m+n-6 = 2n-8 = n-1+n-7$, $m+2 > m+1+12/(m-8)$ and so $n-2 = m > 20$. By Lemma 2.9 (iv),

$$\begin{aligned} \text{ex}(m+n-6; T_n^3) &= \text{ex}(2n-8; T_n^3) \\ &= \frac{(n-2)(2n-8) - 6(n-7)}{2} + \max\left\{\left\lceil \frac{n-37}{4} \right\rceil, 0\right\} \\ &= n^2 - 9n + 29 + \max\left\{\left\lceil \frac{n-37}{4} \right\rceil, 0\right\} < n^2 - 9n + 29 + \frac{n-22}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} \text{ex}(m+n-6; T_m^0) + \text{ex}(m+n-6; T_n) &< \frac{(n-4)(2n-9)}{2} + n^2 - 9n + 29 + \frac{n-22}{2} \\ &= (n-4)(2n-9) = \binom{2n-8}{2}. \end{aligned}$$

Hence $r(T_m^0, T_n^3) \leq m+n-6$ by Lemma 2.1 and so $r(T_m^0, T_n^3) = m+n-6$ or $m+n-7$.

Finally, we assume that $T_n = T_n^3$ and $n = m + 3$. Then $m + n - 6 = 2n - 9 = n - 1 + n - 8$, $m + 3 > m + 1 + 12 / (m - 8)$ and so $n - 3 = m \geq 15$. From Lemma 2.9 (iii),

$$\begin{aligned} \text{ex}(m + n - 6; T_n^3) &= \text{ex}(2n - 9; T_n^3) = \frac{(n - 2)(2n - 9) - 7n + 30}{2} + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} \\ &= n^2 - 10n + 24 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\}. \end{aligned}$$

Recall that

$$\begin{aligned} \text{ex}(m + n - 6; T_m^0) &= \text{ex}(2m - 3; T_m^0) \leq \frac{(m - 2)(2m - 3) - (m - 2)}{2} \\ &= (m - 2)^2 = (n - 5)^2. \end{aligned}$$

We then obtain

$$\begin{aligned} \text{ex}(m + n - 6; T_m^0) + \text{ex}(m + n - 6; T_n^3) &= (n - 5)^2 + n^2 - 10n + 24 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} \\ &= 2n^2 - 20n + 49 + \max\left\{\left\lfloor \frac{n}{2} \right\rfloor, 13\right\} \\ &< 2n^2 - 19n + 45 = \binom{2n - 9}{2}. \end{aligned}$$













Applying Lemma 2.1 gives $r(T_m^0, T_n^3) \leq m + n - 6$ and so $r(T_m^0, T_n^3) = m + n - 6$ or $m + n - 7$ for $n = m + 3$.

If $m + n - 7 = (m - 1)x + (m - 2)y$ for some nonnegative integers x and y , setting $G = xK_{m-1} \cup yK_{m-2}$ we find that G does not contain any copies of T_m^0 . Observe that $\Delta(\overline{G}) = n - 5$ or $n - 6$. We see that \overline{G} does not contain any copies of T_n . Hence $r(T_m^0, T_n) > |V(G)| = m + n - 7$ and so $r(T_m^0, T_n) = m + n - 6$. If $n \geq (m - 3)^2 + 4$, then $m + n - 7 \geq (m - 2)(m - 3)$. By Lemma 6.1, $m + n - 7 = (m - 1)x + (m - 2)y$ for some nonnegative integers x and y and so $r(T_m^0, T_n) = m + n - 6$ as claimed.

Summarizing the above proves the theorem. \square

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