Martin Dzúrik
An upper bound of a generalized upper Hamiltonian number of a graph

*Archivum Mathematicum*, Vol. 57 (2021), No. 5, 299–311

Persistent URL: [http://dml.cz/dmlcz/149135](http://dml.cz/dmlcz/149135)

**Terms of use:**

© Masaryk University, 2021

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
AN UPPER BOUND OF A GENERALIZED UPPER HAMILTONIAN NUMBER OF A GRAPH

MARTIN DZÚRIK

Abstract. In this article we study graphs with ordering of vertices, we define a generalization called a pseudoordering, and for a graph $H$ we define the $H$-Hamiltonian number of a graph $G$. We will show that this concept is a generalization of both the Hamiltonian number and the traceable number. We will prove equivalent characteristics of an isomorphism of graphs $G$ and $H$ using $H$-Hamiltonian number of $G$. Furthermore, we will show that for a fixed number of vertices, each path has a maximal upper $H$-Hamiltonian number, which is a generalization of the same claim for upper Hamiltonian numbers and upper traceable numbers. Finally we will show that for every connected graph $H$ only paths have maximal $H$-Hamiltonian number.

1. Introduction

In this article we study a part of graph theory based on an ordering of vertices. We define a generalization called a pseudoordering of a graph. We will show how to generalize a Hamiltonian number, for a graph $H$ we define the $H$-Hamiltonian number of a graph $G$ and we will show that this concept is a generalization of both the Hamiltonian number and the traceable number. We get them by a special choice of graph $H$. Furthermore, we will study a maximalization of upper $H$-Hamiltonian number for a fixed number of vertices. We will show that, for a fixed number of vertices, each path has a maximal upper $H$-Hamiltonian number. From the definition it will be obvious that a lower bound of the $H$-Hamiltonian number is the number of edges $|E(H)|$ and the graph $G$ has a minimal lower $H$-Hamiltonian number if and only if $H$ is a subgraph of $G$. Now we can say that $G$ having a maximal upper $H$-Hamiltonian number is dual to $H$ being a subgraph of $G$. Furthermore, by above for every two finite graphs $G$ and $H$ such that $G$ is connected satisfying $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, we get that $G \cong H$ if and only if the lower $H$-Hamiltonian number of $G$ is $|E(H)|$.

In [2] it is proved that $G$ has a maximal upper traceable number if and only if $G$ is a path. The same is proved for Hamiltonian number. We will show that for

2020 Mathematics Subject Classification: primary 05C12; secondary 05C45.

Key words and phrases: graph, vertices, ordering, pseudoordering, upper Hamiltonian number, upper traceable number, upper $H$-Hamiltonian number, Hamiltonian spectra.

Received March 23, 2021, revised June 2021. Editor J. Nešetřil.

DOI: 10.5817/AM2021-5-299
$H$ connected $G$ has a maximal $H$-Hamiltonian number if and only if $G$ is a path. This shows that this generalization of ordering of vertices is natural.

This article is based on the bachelor thesis [1]. The author would like to thank Jiří Rosický for many helpful discussions.

In this article we will study a generalization of Hamiltonian spectra of undirected finite graphs. Recall that, a graph $G$ is a pair

$$G = (V(G), E(G)),$$

where $V(G)$ is a finite set of vertices of $G$ and $E(G) \subseteq V(G) \times V(G)$, a symmetric Antireflexive relation, is a set of edges. We will denote an edge between $v$ and $u$ by \{v, u\}.

Recall that, an ordering on the graph $G$ is a bijection $f : \{1, 2, \ldots, |V(G)|\} \rightarrow V(G),$ we denote

$$s(f, G) = \sum_{i=1}^{|V(G)|} \rho_G(f(i), f(i+1)),$$

$$\bar{s}(f, G) = \sum_{i=1}^{|V(G)|-1} \rho_G(f(i), f(i+1)),$$

where $\rho_G(x, y)$ is the distance of $x, y$ in the graph $G$ and $f(|V(G)| + 1) := f(1)$, for better notation. We will write only $s(f)$, $\bar{s}(f)$ if the graph is clear from context. Then

$$\{s(f, G) \mid f \text{ ordering on } G\}$$

$$\{\bar{s}(f, G) \mid f \text{ ordering on } G\}$$

are the Hamiltonian spectrum of the graph $G$ and the traceable spectrum of the graph $G$, respectively.

We want to generalize the notion of an ordering of a graph.

**Definition 1.1.** Let $G, H$ be graphs such that $|V(G)| = |V(H)|$ and $f : V(H) \rightarrow V(G)$ is a bijection, then we call $f$ a pseudoordering on the graph $G$ (by $H$), denote

$$s_H(f, G) = \sum_{\{x, y\} \in E(H)} \rho_G(f(x), f(y)),$$

where $\rho_G(x, y)$ is the distance of $x, y$ in the graph $G$. We will call $s_H(f, G)$ the sum of the pseudoordering $f$. Then

$$\{s_H(f, G) \mid f \text{ pseudoordering on } G \text{ by } H\}$$

is the $H$-Hamiltonian spectrum of the graph $G$.

The minimum and the maximum of a Hamiltonian spectrum and of a traceable spectrum are called the (lower) Hamiltonian number and the upper Hamiltonian number, respectively. Furthermore, the (lower) traceable number and the upper traceable number of a graph $G$ are denoted by
### Definition 1.2.

\[
\begin{align*}
    h_H(G) &= \min\{s_H(f, G) \mid f \text{ pseudoordering on } G\}, \\
    h_H^+(G) &= \max\{s_H(f, G) \mid f \text{ pseudoordering on } G\},
\end{align*}
\]

We will call them the \textit{lower $H$-Hamiltonian number} and the \textit{upper $H$-Hamiltonian number} of a graph $G$, respectively.

Now take $H = C_{|V(G)|}$, where $C_n$ is the cycle with $n$ vertices. When we denote the vertices of $C_{|V(G)|}$ by $\{1, 2, \ldots, |V(G)|\}$ we can see that

\[
s(f, G) = s_{C_{|V(G)|}}(f, G).
\]

Analogously for $H = P_{|V(G)|-1}$, where $P_{n-1}$ is the path of length $n - 1$, we get that

\[
s(f, G) = s_{P_{|V(G)|-1}}(f, G).
\]

### Remark 1.3.

The $C_{|V(G)|}$-Hamiltonian spectrum of a graph $G$ is equal to the Hamiltonian spectrum of $G$ for $|V(G)| \geq 3$, and the $P_{|V(G)|-1}$-Hamiltonian spectrum of $G$ is equal to the traceable spectrum of $G$ for $|V(G)| \geq 2$.

### Lemma 1.4.

Let $G$ be a connected finite graph and $H$ be a graph such that $|V(G)| = |V(H)|$, then $h_H(G) = |E(H)|$ if and only if $H$ is isomorphic to some subgraph of $G$.

**Proof.** Let $f : V(H) \to V(G)$ be a pseudoordering satisfying $s(f, G) = |E(H)|$, then $f$ is an injective graph homomorphism. The opposite implication is obvious. $\square$

### Lemma 1.5.

Let $G$ be a connected finite graph and $H$ be a graph such that $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$, then $h_H(G) = |E(H)|$ if and only if $H$ is isomorphic to the graph $G$.

**Proof.** The graph $H$ is isomorphic to a subgraph of $G$ and furthermore $|V(G)| = |V(H)|$, $|E(G)| = |E(H)|$, hence $H \cong G$. The opposite implication is obvious. $\square$

#### 2. Maximalization of the upper $H$-Hamiltonian number of a graph $G$

In this section we will prove that for every pair of connected graphs $H, G$ and each pseudoordering $f$ there exists a pseudoordering

\[
g : V(H) \to \{1, 2, \ldots, |V(G)|\}
\]

such that

\[
s_H(f, G) \leq s_H(g, P_{|V(G)|-1}).
\]
At first, let $G$ be a tree. We will only work with graphs which have at least 2 vertices.

**Definition 2.1.** Let $G$ and $H$ be graphs such that $G$ is connected, $|V(G)| = |V(H)|$ and $f : V(H) \to V(G)$ is a pseudoordering. Furthermore, let $a, b \in V(G)$, we define $a \sim_{H,f} b$ if and only if $\{f^{-1}(a), f^{-1}(b)\} \in E(H)$.

**Definition 2.2.** Let $G$ be a tree such that $G$ is not a path. Denote three pairwise distinct leaves by $l, k, v \in V(G)$. Because $G$ is not a path then $G$ has at least 3 leaves, connect $l, k$ with a path $l = x_1, x_2, \ldots, x_m = k$. Connect $v, l$ with a path $l = v = y_1, y_2, \ldots, y_s = l$ and take the minimum of a set

$$i_m = \min \{i \mid \exists j \in \{1, \ldots, m\}, y_i = x_j\}.$$ 

Take $j_m$ such that $y_{i_m} = x_{j_m}$. Now we define $u = y_{i_m}, w = y_{i_m-1}, u^+ = x_{j_m-1}, u^- = x_{j_m+1}$.

**Example.**

![Diagram](image-url)

**Remark 2.3.** $l \neq u \neq k$.

**Definition 2.4.** Define a set $K(v, G) \subseteq V(G)$ as a set of vertices $z \in V(G)$ such the path between $z$ and $l$ uses the edge $\{w, u\}$.

**Remark 2.5.** $K(v, G)$ is the connected component of $(V(G), E(G) \setminus \{w, u\})$, $G$ without edge $\{w, u\}$, which contains $v$.

**Lemma 2.6.** (i) Paths between vertices from $K(v, G)$ don’t use the edge $\{w, u\}$.

(ii) Paths between vertices from $V(G) \setminus K(v, G)$ don’t use the edge $\{w, u\}$.

(iii) Paths joining a vertex from $V(G) \setminus K(v, G)$ to a vertex from $K(v, G)$ use the edge $\{w, u\}$.

**Proof.** Because $G$ is a tree, there is a unique path between each pair of vertices, then it is obvious by remark 2.5. \(\square\)
Definition 2.7. Define graphs
\[ \tilde{G} = (V(G), E(G) \setminus \{\{w, u\}\} \cup \{\{w, l\}\}) , \]
\[ \tilde{\tilde{G}} = (V(G), E(G) \setminus \{\{w, u\}\} \cup \{\{w, k\}\}) . \]

Lemma 2.8. \( \tilde{G} \) and \( \tilde{\tilde{G}} \) are trees.

Proof. At first we show connectivity, let \( a, b \in V(G) \), connect them with a path. If both are in \( K(v, G) \) or in \( V(G) \setminus K(v, G) \), then by Lemma 2.6 the path in \( G \) uses only edges which are also in \( \tilde{G} \), \( \tilde{\tilde{G}} \). Hence it is path also there.

Let \( a \in K(v, G) \) and \( b \in V(G) \setminus K(v, G) \). We can see \( w \in K(v, G) \), by Lemma 2.6 a path between \( a \) and \( w \), \( a = a_1, a_2, \ldots, a_p = w \), doesn’t use \( \{w, u\} \) and all vertices of this path are in \( K(v, G) \). If not, there is a path between vertices from \( K(v, G) \) and \( V(G) \setminus K(v, G) \) which doesn’t use \( \{w, u\} \), that is a contradiction with Lemma 2.6. Connect \( l \) and \( b \) with a path, \( l = b_1, b_2, \ldots, b_q = b \). It doesn’t use \( \{w, u\} \) and all vertices are in \( V(G) \setminus K(v, G) \). Then \( a = a_1, a_2, \ldots, a_p = w, \ l = b_1, b_2, \ldots, b_q = b \) is a path between \( a, b \) in the graph \( \tilde{G} \), analogously for \( \tilde{\tilde{G}} \).

Now we show that they don’t contain a cycle, for contradiction suppose that \( \tilde{G} \) contains a cycle \( C \subseteq G \). If \( C \) doesn’t use the edge \( \{w, l\} \), then \( C \subseteq G \), but \( G \) is a tree, this is a contradiction. If \( C \) uses \( \{w, l\} \), then there exists a path in \( G \) between \( w, l \), which doesn’t use the edge \( \{w, l\} \). Then there exists a path in \( G \) between \( w, l \), which doesn’t use the edge \( \{w, u\} \), but \( w \in K(v, G) \) and \( l \in V(G) \setminus K(v, G) \), that is contradiction with Lemma 2.6. Analogously for \( \tilde{G} \).

We want to show that
\[ s_H(G, f) \leq s_H(\tilde{G}, f) \]
or
\[ s_H(G, f) \leq s_H(\tilde{\tilde{G}}, f) . \]

Lemma 2.9.
\[ a, b \in K(v, G) , \] then \( \rho_G(a, b) = \rho_{\tilde{G}}(a, b) = \rho_{\tilde{\tilde{G}}}(a, b) . \]
\[ a, b \in V(G) \setminus K(v, G) , \] then \( \rho_G(a, b) = \rho_{\tilde{G}}(a, b) = \rho_{\tilde{\tilde{G}}}(a, b) . \)

Proof. A path in \( G \) between \( a, b \), by Lemma 2.6 doesn’t use \( \{u, w\} \), hence it is a path in \( \tilde{G} \) and \( \tilde{\tilde{G}} \) too, then the distance of \( a, b \) is the same in \( G, \tilde{G} \) and \( \tilde{\tilde{G}} \).

Definition 2.10. Define subsets
\[ F^+, F^-, F^0 \subseteq K(v, G) \times (V(G) \setminus K(v, G)) \]
such that \( (a, b) \in F^+ \) if a path between \( a, b \) uses the edge \( \{u, u^+\} \), \( (a, b) \in F^- \) if a path between \( a, b \) uses the edge \( \{u, u^-\} \) and \( (a, b) \in F^0 \) if a path between \( a, b \) doesn’t use either \( \{u, u^-\} \) nor \( \{u, u^+\} \).

Lemma 2.11. \( F^+, F^-, F^0 \) are pairwise disjoint and
\[ F^+ \cup F^- \cup F^0 = K(v, G) \times (V(G) \setminus K(v, G)) . \]
Moreover, both sides are equal, in the first inequality, if and only if vertex of degree 3.

Lemma 2.12. Let $l$ is a leaf, hence $z = l$. We get our inequality and we see that both are equal if and only if $y = l$ and, in the second inequality, if and only if $y = k$.

Proof. Let $z$ denote the first common vertex of paths $Q: l = y_1, y_2, \ldots, y_s = k$ and $P: y = x_1, x_2, \ldots, x_m = x$. Consider

$$i_m = \min \{ i \mid \exists j \in \{ 1, \ldots, m \}, y_i = x_j \}$$

and therefore $z = y_{i_m}$, let $T$ be the path from $z$ to $l$, we will show that $z$ is the only one common vertex of $T$ and $P$, vertices from $P$ split into the 4 subpaths, $P_1$ from $y$ to $z$, $P_2$ from $z$ to $u$, edge $\{ u, w \}$ and $P_3$ from $w$ to $x$. Vertices from $P_1$ are not in $Q$ (except for $z$) from the definition of $z$. Vertices from $P_2$ are not in $T$ (except for $z$) from the uniqueness of paths in trees and vertices from $P_3$ belong to $K(v, G)$ and every vertex of $T$ belongs to $V(G) \setminus K(v, G)$. By composition of paths $P_1, T, \{ l, w \}, P_3$, we get a path from $y$ to $x$ in the graph $G$.

Let $P$ denote the path from $\bar{y}$ to $\bar{x}$, analogously define $\bar{z}$ as the first common vertex of paths $\bar{P}$ and $Q$ (first in the direction from $\bar{y}$ to $\bar{x}$). We split $\bar{P}$ into the subpaths $\bar{P}_1$ from $\bar{y}$ to $\bar{z}$, $\bar{P}_2$ from $\bar{z}$ to $u$, edge $\{ u, w \}$ and $\bar{P}_3$ from $u$ to $\bar{x}$. Let $\bar{T}$ be the path from $u$ to $l$, analogously we get that $u$ is the only one common vertex of $\bar{P}$ and $\bar{T}$. Hence $\bar{P}_1, \bar{P}_2, \bar{T}, \{ l, w \}, \bar{P}_3$ is a path between $\bar{y}, \bar{x}$ in the graph $\bar{G}$.

And for paths from $u$ to $z$ and from $u$ to $\bar{z}$, $u$ is the only one common vertex, by uniqueness of path in trees.

Now we can calculate

$$\rho_G(x, y) = \rho_G(x, w) + 1 + \rho_G(u, z) + \rho_G(z, y),$$
$$\rho_G(\bar{x}, \bar{y}) = \rho_G(\bar{x}, w) + 1 + \rho_G(u, \bar{z}) + \rho_G(\bar{z}, \bar{y}),$$
$$\rho_G(x, y) = \rho_G(x, w) + 1 + \rho_G(l, z) + \rho_G(z, y),$$
$$\rho_G(\bar{x}, \bar{y}) = \rho_G(\bar{x}, w) + 1 + \rho_G(l, z) + \rho_G(z, u) + \rho_G(u, \bar{z}) + \rho_G(\bar{z}, \bar{y}),$$

hence

$$\rho_G(\bar{x}, \bar{y}) + \rho_G(x, y) = \rho_G(\bar{x}, \bar{y}) + \rho_G(x, y) + 2\rho_G(l, z).$$

Now we get our inequality and we see that both are equal if and only if $l = z$. But $l$ is a leaf, hence $z$ is a leaf, then $y = z = l$. For $\bar{G}$ analogously.

Example. Paths between $x, y$ and $\bar{x}, \bar{y}$ in graphs $G$ and $\bar{G}$. 

\[ \square \]
Lemma 2.13. Let \((x, y) \in F^0\) then

\[
\rho_{\bar{G}}(x, y) > \rho_G(x, y),
\]
\[
\rho_{\bar{G}}(x, y) > \rho_G(x, y).
\]

Proof. Let \(P\) be a path from \(x\) to \(y\) and \(Q\) be a path from \(l\) to \(k\) in \(G\), for \(P\) and \(Q\), \(u\) is the only one common vertex because \((x, y) \in F^0\). Hence \(x \rightarrow w \rightarrow l \rightarrow u \rightarrow y\) is a path in \(\bar{G}\), where paths of type \(a \rightarrow b\) are subpaths of \(P\) and \(Q\) and \(-\) denotes an edge. Now we can calculate the following

\[
\rho_{\bar{G}}(x, y) = \rho_G(x, u) + 1 + \rho_G(l, u) + \rho_G(u, y) = \rho_G(x, y) + \rho_G(l, u)
\]

and from \(l \neq u\) we have inequality. For \(\bar{G}\) analogously.

Lemma 2.14.

\[
\rho_{\bar{G}}(x, y) > \rho_G(x, y) \quad \text{for} \quad (x, y) \in F^-,
\]
\[
\rho_{\bar{G}}(x, y) > \rho_G(x, y) \quad \text{for} \quad (x, y) \in F^+.
\]

Proof. We will prove the first inequality. As well as in lemma 2.12 denote \(z\) the first common vertex of paths from \(y\) to \(x\) and from \(k\) to \(l\), formally we can define it as well as in lemma 2.12. Now we consider a path \(x \rightarrow w \rightarrow l \rightarrow u \rightarrow z \rightarrow y\). Hence

\[
\rho_{\bar{G}}(x, y) = \rho_G(x, w) + 1 + \rho_G(l, u) + \rho_G(u, z) + \rho_G(z, y)
\]

and from \(l \neq u\) we have inequality. For second inequality analogously.

Definition 2.15. Let \(G\) be a tree and \(H\) be a graph such that

\[
|V(G)| = |V(H)|
\]

and

\[
f : V(H) \rightarrow V(G)
\]
is a pseudoordering, we define a set
\[ L = \{ (x, y) \in K(v, G) \times (V(G) \setminus K(v, G)) \mid x \sim_{H,f} y \}, \]
where \( K(v, G) \) is the set from Definition 2.4.

Lemma 2.16. Let \( G \) be a tree and \( H \) be a graph such that, \( |V(G)| = |V(H)| \) and \( f : V(H) \to V(G) \) is a pseudoordering. Then
\[ s_H(f, \tilde{G}) \geq s_H(f, G) \]
or
\[ s_H(f, \tilde{G}) \geq s_H(f, G), \]
the first case occurs when
\[ |L \cap F^+| \leq |L \cap F^-|, \]
the second case occurs when
\[ |L \cap F^+| \geq |L \cap F^-|. \]

Proof. Denote \( n^+ = |L \cap F^+|, n^- = |L \cap F^-|, m = |L \cap F^0|, \)
\[ \tilde{m} = \left\{ (x, y) \in (K(v, G)^2) \cup ((V(G) \setminus K(v, G))^2) \mid x \sim_{H,f} y \right\}, \]
where square \( K(v, G)^2 \) means \( K(v, G) \times K(v, G) \). \( \tilde{m} \) is number of edges \( \{x, y\} \in E(H) \), which satisfy that \( f(x) \) and \( f(y) \) lie in the same component of \( (V(G), E(G) \setminus \{w, u\}) \).

Let \( n^+ \geq n^- \), the second case is analogous, we rearrange the sum \( s_H(f, G) \) in this way
\[ s_H(f, G) = \sum_{i=1}^{n^-} (\rho_G(x_i, y_i) + \rho_G(\bar{x}_i, \bar{y}_i)) + \sum_{i=n^-+1}^{n^+} \rho_G(x_i, y_i) \]
\[ + \sum_{i=1}^{m} \rho_G(a_i, b_i) + \sum_{i=1}^{\tilde{m}} \rho_G(c_i, d_i), \]
where
\[ (x_i, y_i) \in F^+, \ (\bar{x}_i, \bar{y}_i) \in F^-, \ (a_i, b_i) \in F^0, \]
\[ (c_i, d_i) \in \left\{ (x, y) \in (K(v, G)^2) \cup ((V(G) \setminus K(v, G))^2) \mid x \sim_{H,f} y \right\}. \]
Now, by Lemma 2.12
\[ \rho_G(x_i, y_i) + \rho_G(\bar{x}_i, \bar{y}_i) \leq \rho_{\tilde{G}}(x_i, y_i) + \rho_{\tilde{G}}(\bar{x}_i, \bar{y}_i), \]
by Lemma 2.14
\[ \rho_G(x_i, y_i) \leq \rho_{\tilde{G}}(x_i, y_i), \]
by Lemma 2.13
\[ \rho_G(a_i, b_i) \leq \rho_{\tilde{G}}(a_i, b_i) \]
and by Lemma 2.9
\[ \rho_G(c_i, d_i) = \rho_{\tilde{G}}(c_i, d_i). \]
Hence
\[ s_H(f, G) \leq \sum_{i=1}^{n-} (\rho_{\tilde{G}}(x_i, y_i) + \rho_G(x_i, y_i)) + \sum_{i=n-+1}^{n+} \rho_{\tilde{G}}(x_i, y_i) + \sum_{i=1}^{m} \rho_G(a_i, b_i) + \sum_{i=1}^{m} \rho_G(c_i, d_i) \]
\[ = s_H(f, \tilde{G}). \]
□

Lemma 2.17. Let \( G \) be a tree and \( H \) be a graph such that, \(|V(G)| = |V(H)|\) and \( f : V(H) \to V(G) \) is a pseudoordering. Then there exists a pseudoordering \( g : V(H) \to \{x_1, x_2, \ldots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \) such that
\[ s_H(f, G) \leq s_H(g, P_{|V(G)|-1}). \]

Proof. We denote
\[ \alpha(G) = \sum_{v \in V(G)} \deg_G v, \]
from the definition of \( u, l \) and \( k \) we know that \( \deg_G u \geq 3 \) and \( \deg_G l = \deg_G k = 1 \). From the construction of \( \overline{G} \) and \( \tilde{G} \) we have \( \deg_{\overline{G}} u = \deg_{\tilde{G}} u \leq \deg_G u, \deg_{\overline{G}} l = \deg_{\tilde{G}} k = 2 \) and all other vertices have the same degree as before. Hence
\[ \alpha(\tilde{G}) < \alpha(G), \]
\[ \alpha(\overline{G}) < \alpha(G). \]
Let \( S \) be a tree, which is not a path, we choose any three pairwise distinct leaves in \( V(S) \) and define \( S^* \) as one of graphs \( \tilde{S}, \bar{S} \), which satisfy \( s_H(f, S^*) \geq s_H(f, S) \). Denote \( G_0 = G \) and for \( i \geq 0 \) denote \( G_{i+1} = G_i^* \) if \( G_i \) is not a path, otherwise define \( G_{i+1} = G_i \). For contradiction we assume that the tree \( G_i \) is not a path for every \( i \in \mathbb{N}_0 \). We know \( \alpha(G_i) \in \mathbb{N}_0 \) for every \( i \) and
\[ \alpha(G_{i+1}) \leq \alpha(G_i) - 1, \]
hence
\[ \alpha(G_{\alpha(G_0)+1}) \leq \alpha(G_0) - \alpha(G_0) - 1 = -1 \]
and this is contradiction. Therefore there exists some \( j \) such that \( G_j \) is a path, from Lemma 2.16 we get
\[ s_H(f, G_{i+1}) \geq s_H(f, G_i) \]
and hence
\[ s_H(f, G_j) \geq s_H(f, G). \]

**Theorem 2.18.** Let \( G \) and \( H \) be graphs such that \( G \) is connected, \(|V(G)| = |V(H)|\) and \( f : V(H) \to V(G) \) is a pseudoordering, then there exists a pseudoordering
\[ g : V(H) \to \{x_1, x_2, \ldots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \]
such that
\[ s_H(f, G) \leq s_H(g, P_{|V(G)|-1}). \]

**Proof.** Let \( K \) be any spanning tree of \( G \), \( x, y \in V(G) \), we connect \( x \) and \( y \) with a path in graph \( K \), this path is also a path in \( G \). Hence
\[ \rho_G(x, y) \leq \rho_K(x, y) \]
for every \( x, y \), hence
\[ s_H(f, G) \leq s_H(f, K), \]
by Lemma 2.17 there exists a pseudoordering
\[ g : V(H) \to \{x_1, x_2, \ldots, x_{|V(G)|}\} = V(P_{|V(G)|-1}) \]
such that
\[ s_H(f, G) \leq s_H(f, K) \leq s_H(g, P_{|V(G)|-1}). \]

**Corollary 2.19.** Let \( G \) and \( H \) be graphs such that \( G \) is connected, \(|V(G)| = |V(H)|\), then
\[ h^+_{H}(G) \leq h^+_{H}(P_{|V(G)|-1}). \]

3. **Graphs with a maximal upper H-Hamiltonian number**

In this section we will prove that if in Corollary 2.19 the graph \( H \) is connected, then in the inequality in Corollary 2.19 both sides are equal.

**Remark 3.1.** For easier writing, we will denote vertices of \( H \) the same as vertices of \( G \), we will rename them in this way \( v \in H \mapsto f(v) \). We can naturally see it as graph with two sets of edges.

In inequalities in Lemma 2.16 both sides are equal under specific conditions, if \( L \setminus F^0 \neq \emptyset \), then in Lemma 2.13 there is a strict inequality and then also the same happens in Theorem 2.18.

If \( (L \setminus K(v, G) \times \{l\}) \cap F^+ \neq \emptyset \), then in Lemma 2.12 there is a strict inequality and then also the same happens in Theorem 2.18. Analogously if \( (L \setminus K(v, G) \times \{k\}) \cap F^- \neq \emptyset \).

Overall we get that the only nontrivial case is
\[ L \subseteq K(v, G) \times \{k, l\}. \]

**Remark 3.2.** Remark 3.1 holds for every triple of distinct leaves \( k, l, v \) in \( G \).
Lemma 3.3. Let $G$ be a tree, $H$ connected graph such that $|V(G)| = |V(H)|$ and $f: V(H) \to V(G)$ is a pseudoordering, which satisfy

$$s_H(f, G) = h^+_H(P_{|V(G)| - 1}),$$

then $G$ is path.

Proof. For contradiction suppose that $G$ is not a path, then there exist three pairwise distinct leaves $k, l, v$, we denote in the same way as before, vertex $u$ and set of vertices $K(v, G)$. Because graph $H$ is connected there exists a vertex $x$ such that $\{u, x\} \in E(H)$. Let $X \subseteq V(G)$ be a set of vertices of components of graph $G \setminus u$, containing $x$. $G \setminus u$ has, by definition of $u$, at least 3 components. Let now $\bar{v}$ be an arbitrary leaf (leaf in $G$) in $X$. Choose $\bar{k}, \bar{l}$ as arbitrary leaves in pairwise distinct components of $G \setminus u$ and different from $X$.

Now $(x, u) \in \bar{L}$, where $\bar{L}$ is alternative of $L$ for $\bar{k}, \bar{l}, \bar{v}$ and by Remark 3.1 for $\bar{k}, \bar{l}, \bar{v}$ and by $k \neq u \neq l$ we get contradiction.

□

Example. We show the idea of the last proof in the following picture.

![Graph Example]

Remark 3.4. Let $G$ be a graph with a maximal $H$-Hamiltonian number, then every spanning tree of $G$ has a maximal $H$-Hamiltonian number, therefore every spanning tree is a path. We will show that the only graphs with this property are cycles and paths.

Lemma 3.5. Let $G$ be a connected graph such that $|V(G)| \geq 2$, then there is a vertex, which is not an articulation point.

Proof. Consider a block-cut tree of $G$ and a block $B$, which is a leaf of the block-cut tree or if this tree has only one vertex, then $B = G$. $B$ is, by definition of a block, 2-connected. Because $B$ is leaf we get that in $B$ there is only one articulation and in $B$ there are at least 2 vertices. Hence in $B$ there is at least one vertex, which is not an articulation point. □

Lemma 3.6. Let $G$ be a finite connected graph such that $|V(G)| \geq 2$ and every spanning tree of $G$ is a path, then $G$ is a path or a cycle.
Theorem 3.7. Let $G$ and $H$ be connected finite graphs such that $|V(G)| = |V(H)|$, then

$$h_H^+(G) \leq h_H^+(P_{|V(G)|-1}),$$

moreover, both sides are equal if and only if $G$ is a path.

Proof. The first part follows from Theorem 2.18. Let $G$ be a graph, $f$ be a pseudoordering such that

$$s_H(f, G) = h_H^+(G) = h_H^+(P_{|V(G)|-1}).$$

From the proof of Theorem 2.18 we know that every spanning tree also satisfies the equation above. Hence, by Lemma 3.3, every spanning tree of $G$ is a path. By Lemma 3.6, $G$ is a path or a cycle, for contradiction we assume that it is a cycle. We denote $n = |V(G)|$, we will show that there are two vertices $v, u \in V(G)$ such that $v \sim_{H,f} u$ and $\rho_G(u, v) < \frac{n}{2}$.

Because $G$ is cycle, $|V(H)| = n \geq 3$ and $H$ is connected we see that there is a vertex of degree at least 2. Let $v$ be a vertex such that $\text{deg}_H(v) \geq 2$, there exists at least two vertices $u$ such that $v \sim_{H,f} u$. There exists at most one vertex such that $\rho_G(u, v) \geq \frac{n}{2}$, hence at least one of them satisfies $\rho_G(u, v) < \frac{n}{2}$.

Now we connect $v$ and $u$ with a shorter path in $G$. Let $e$ be some edge on this path, we define a graph $\tilde{G} = (V(G), E(G) \setminus e)$, it is a path, where every distance is greater or equal as in $G$. But $\rho_G(u, v) < \rho_{\tilde{G}}(u, v)$ and then

$$s_H(f, \tilde{G}) = s_H(f, G) > h_H^+(P_{|V(G)|-1}),$$

and this is contradiction with Theorem 2.18. □
4. Conclusion

When we use following equations which can be found for example in [2, Theorem 1.3] and [2, Corollary 2.2]

$$h^+(P_{|V(G)|-1}) = \left\lceil \frac{|V(G)|^2}{2} \right\rceil, \quad t^+(P_{|V(G)|-1}) = \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor - 1.$$  

This result is also calculated in [1] and when we use Theorem 3.7 for $H = P_{|V(G)|-1}$ and for $H = C_{|V(G)|}$ we get the following theorem.

**Theorem 4.1 ([2]).**

$$h^+(G) \leq \left\lceil \frac{|V(G)|^2}{2} \right\rceil, \quad t^+(G) \leq \left\lfloor \frac{|V(G)|^2}{2} \right\rfloor - 1.$$  

Moreover, both sides are equal if and only if $G$ is a path.

First part is [2, Corollary 2.2] and second part is [2, Theorem 4.2]. Now we can see, that Theorem 3.7 is generalization of Theorem 4.1 which is from article [2].

**References**


**Department of Mathematics and Statistics,**  
**Faculty of Science, Masaryk University,**  
**Kotlářská 2, 611 37 Brno, Czech Republic**  
**E-mail:** 451859@mail.muni.cz