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SOME PROPERTIES OF STATE FILTERS IN STATE RESIDUATED LATTICES

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Abstract. We consider properties of state filters of state residuated lattices and prove that for every state filter F of a state residuated lattice X:

(1) F is obstinate $\Leftrightarrow L/F \cong \{0,1\}$;
(2) F is primary $\Leftrightarrow L/F$ is a state local residuated lattice;

and that every g-state residuated lattice X is a subdirect product of $\{X/P_\lambda\}$, where P_λ is a prime state filter of X.

Moreover, we show that the quotient MTL-algebra X/P of a state residuated lattice X by a state prime filter P is not always totally ordered, although the quotient MTL-algebra by a prime filter is totally ordered.

Keywords: obstinate state filter; prime state filter; Boolean state filter; primary state filter; state filter; residuated lattice; local residuated lattice

MSC 2020: 03B47, 06B10

1. Introduction

The research on the theory of residuated lattices started in [24] and is progressing in many directions after finding the relation between fuzzy logics. For example, the class of MTL-algebras (BL-algebras, MV-algebras and so on), axiomatic extensions of residuated lattices, are proved to be an algebraic semantics for the monoidal t-norm logic (MTL) (the basic logic (BL), multiple valued logic (MV), respectively). From the result that every MTL-algebra is a subdirect product of totally ordered MTL-algebras, to show a formula A is provable in the MTL logic, it is enough to show that the formula A is valid on any totally ordered MTL-algebra. The situation

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corresponds to the fact that a formula in classical propositional logic is provable in CPL if it is valid in the Boolean algebra $\{0, 1\}$.

As another direction, a measure (called *state*) which corresponds to the measurement problem in the quantum logic is defined on MV-algebras. The notion of state, coming from the theory of quantum mechanics, was firstly applied to MV-algebras by Kôpka and Chovanec in [21] and then extended to non-commutative MV-algebras in [4], [22]. Since then, the theory of states on algebras has been applied to other algebras such as (pseudo) BL-algebras (see [10]), (non-commutative) Rℓ-monoids (see [6], [5]), (non-commutative) residuated lattices (see [17]) and now is becoming a hot research field in the theory of fuzzy logics and algebras. In [10], it is proved that the notion of Bosbach states is the same as that of Riečan states for good possibly bounded non-commutative Rℓ-monoids. On the other hand, it is proved in [1] that there is a Riečan state which is not a Bosbach state on a certain (non-commutative) residuated lattice.

A logic (called a quantum logic) which follows quantum mechanics has a mathematical model of the set $C(H)$ of all closed subspaces of a Hilbert space H. The set $C(H)$ is not a distributive lattice but an orthomodular lattice. On the other hand, since MV-algebras are distributive lattices, the notion of state does not fully reflect properties of states in the quantum logic. Therefore we need to extend the notion of state to non-distributive lattices.

Recently, another approach to MV-algebras, *state operators*, has been started by Flaminio and Montagna in [8]. A state operator is a mapping from MV-algebra X to itself satisfying some conditions representing properties of states on MV-algebras. They extend the language of MV-algebras by adding a new unary operator, a state operator, and consider the MV-algebra X with the state operator σ as a *state* MV-*algebra* (X, σ). They showed some fundamental results about state MV-algebras. After that, state operators are generalized and applied to more general algebras such as (pseudo) BL-algebras (see [3], [2]), (non-commutative) Rℓ-monoids (see [7]) and (non-commutative) residuated lattices (see [15]). In particular, state residuated lattices are defined and their basic properties are proved in [15], in which it is claimed that the class of all state residuated lattices does not form a variety. However, this is not true, as will be proved in this paper.

Owing to a shift from an external notion of state to an internal notion of state operators, logics with operators are considered as one kind of modal logics. Concretely speaking, state residuated lattices are an algebraic semantics of the following CRL-logic (commutative residuated lattices-based logic) (see [20]) with axioms corresponding to residuated lattices and

(1) $M(\perp) \to \perp$,
(2) $M(\varphi \to \psi) \to (M\varphi \to M\psi)$,

(3) $M(\varphi \to \psi) \leftrightarrow (M\varphi \to M(\varphi \wedge \psi))$,
(4) $M(\varphi \odot \psi) \leftrightarrow M\varphi \odot M(\varphi \to \varphi \odot \psi)$,
(5) $M(M\varphi \odot M\psi) \leftrightarrow (M\varphi \odot M\psi)$,
(6) $M(M\varphi \vee M\psi) \leftrightarrow (M\varphi \vee M\psi)$,
(7) $M(M\varphi \wedge M\psi) \leftrightarrow (M\varphi \wedge M\psi)$.

The rules of inference are Modus Ponens and Necessitation: from φ deduce $M\varphi$. Therefore, the logic above can be considered as the CRL-logic with a "modality" M (see [20]).

On the other hand, we have another important tool, filters, to develop residuated lattices. We define a state filter (simply called σ-filter) of a state residuated lattice (X, σ), where σ is a state operator, and also define some kind of σ-filters such as prime, primary, and obstinate.

We show the following results. For every state filter F of a state residuated lattice X:

(1) F is obstinate $\Leftrightarrow L/F \cong \{0, 1\}$;
(2) F is primary $\Leftrightarrow L/F$ is a state local residuated lattice;

and every state residuated lattice X is a subdirect product of $\{X/P_\lambda\}$, where P_λ is a prime state filter of X.

Moreover, we show that the quotient state MTL-algebra X/P of a state MTL-algebra (X, σ) by a state prime filter P is not always totally ordered, although the quotient MTL-algebra by a prime filter is totally ordered.

2. Residuated lattice and state operator

We recall a definition of bounded integral commutative residuated lattices (see [9]). An algebraic structure $(X, \wedge, \vee, \odot, \to, \mathbf{0}, \mathbf{1})$ is called a bounded integral commutative residuated lattice (simply called *residuated lattice* here) if

(1) $(X, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice;
(2) $(X, \odot, \mathbf{1})$ is a commutative monoid with unit element $\mathbf{1}$;
(3) For all $x, y, z \in X$, $x \odot y \leqslant z$ if and only if $x \leqslant y \to z$.

For all $x \in X$, by x', we mean $x' = x \to \mathbf{0}$, which is a negation in a sense. A residuated lattice X is called an Rℓ-*monoid* if it satisfies the *divisibility* condition

(div) $x \wedge y = (x \to y) \odot x$.

Moreover, if an Rℓ-monoid X satisfies the *pre-linearity* condition

(p-lin) $(x \to y) \vee (y \to x) = \mathbf{1}$,

then it is called a BL-*algebra*.

We have the following basic properties of residuated lattices (see [11], [12], [22], [24]).

Proposition 1. *Let X be a residuated lattice. For all $x, y, z \in X$, we have*

(1) $\mathbf{0}' = \mathbf{1}$, $\mathbf{1}' = \mathbf{0}$,

(2) $x \odot x' = \mathbf{0}$,

(3) $x \leqslant y \Leftrightarrow x \to y = \mathbf{1}$,

(4) $x \odot (x \to y) \leqslant y$,

(5) $x \leqslant y \Rightarrow x \odot z \leqslant y \odot z$, $z \to x \leqslant z \to y$, $y \to z \leqslant x \to z$,

(6) $\mathbf{1} \to x = x$,

(7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$,

(8) $(x \vee y)' = x' \wedge y'$,

(9) $(x \vee y)^{m+n} \leqslant x^m \vee y^n$ *for* $m, n \in \mathbb{N}$.

X is a residuated lattice in the rest of the paper. According to [15], [18], we define a state operator σ. A map $\sigma\colon X \to X$ is called a *state operator* of X if it satisfies the conditions:

(L1) $\sigma(0) = 0$,

(L2) $\sigma(x \to y) \leqslant \sigma(x) \to \sigma(y)$,

(L3) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$,

(L4) $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \to x \odot y)$,

(L5) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$,

(L6) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$,

(L7) $\sigma(\sigma(x) \vee \sigma(y)) = \sigma(x) \vee \sigma(y)$,

(L8) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

We note that, in [15], a state σ is defined by (L1), (L3)–(L8) and

(L2)* $x \leqslant y \Rightarrow \sigma(x) \leqslant \sigma(y)$.

It is easy to show that (L2) is equivalent to (L2)* under the condition (L3). Therefore our definition of state operators is the same as that defined in [15].

We have basic results about state operators.

Proposition 2. *Let (X, σ) be a state residuated lattice. Then we have*

(1) $\sigma(1) = 1$,

(2) $x \leqslant y \Rightarrow \sigma(x) \leqslant \sigma(y)$,

(3) $\sigma(x') = (\sigma(x))'$,

(4) $\sigma(\sigma(x)) = \sigma(x)$,

(5) $\sigma(X) = \mathrm{Fix}(\sigma) = \{x \in X\colon\ \sigma(x) = x\}$,

(6) $\sigma(X)$ *is a subalgebra of* X.

3. Filters and state filters

We define filters of residuated lattices according to [12], [13], [16], [22]. A nonempty subset $F \subseteq X$ is called a *filter* of X if

(F1) $x, y \in F \Rightarrow x \odot y \in F$;
(F2) $x \in F$ and $x \leqslant y \Rightarrow y \in F$.

It is proved (see [23]) that, for a nonempty subset F of X, F is a filter if and only if it satisfies the conditions $\mathbf{1} \in F$ and

(DS) $x \in F$ and $x \to y \in F \Rightarrow y \in F$.

Moreover a filter F is called a *state filter* (or simply *σ-filter*) if it satisfies the condition

$\triangleright$ $x \in F \Rightarrow \sigma(x) \in F$ for all $x \in X$.

By $\mathcal{F}(X)$ (or $\mathcal{F}_\sigma(X)$), we mean the set of all filters (or σ-filters) of (X, σ). For a nonempty subset $S \subseteq X$, by $[S)$ (or $[S)_\sigma$) we mean the filter (or σ-filter, respectively) generated by S. We have following results about the filter $[S)$ (see [9], [14], [12], [13]) and $[S)_\sigma$ (see [16]) generated by S.

Proposition 3. *For a nonempty subset $S \subseteq X$ and F a filter of X, we have*

(1) $[S) = \{x \colon \exists\, s_i \in S;\ s_{\mathbf{1}} \odot \ldots \odot s_n \leqslant x\}$;
(2) $a \in X \Rightarrow [F \cup \{a\}) = \{x \colon \exists\, u \in F, \exists\, n;\ u \odot a^n \leqslant x\}$;
(3) $[S)_\sigma = [S \cup \sigma S)$.

It is trivial that $\mathcal{F}(X)$ is a partially ordered set with respect to the set inclusion $\subseteq$. Moreover, it is easy to show the following result (see [9]).

Proposition 4. $(\mathcal{F}(X), \wedge, \vee, \to, \{\mathbf{1}\}, X)$ *is a complete Heyting algebra, where for all $F, G \in \mathcal{F}(X)$,*

$$F \wedge G = F \cap G, \quad F \vee G = [F \cup G), \quad F \to G = \{x \in X \colon F \cap [x) \subseteq G\}.$$

Hence we have $F \wedge \bigvee\limits_{\lambda} G_\lambda = \bigvee\limits_{\lambda} (F \wedge G_\lambda)$.

Proposition 5. *We have*

(1) $F_1, F_2 \in \mathcal{F}(X)$, $F_1 \vee F_2 = \{x \in X \colon \exists\, f_i \in F_i;\ f_1 \odot f_2 \leqslant x\}$;
(2) $[x) \vee [y) = [x \odot y) = [x \wedge y)$ and $[x) \wedge [y) = [x \vee y)$.

Moreover, a similar argument implies that the class of all state filters of (X, σ) is also a complete Heyting algebra.

Proposition 6 ([16]). *$(\mathcal{F}_\sigma(X), \wedge, \vee, \rightarrow, \{\mathbf{1}\}, X)$ is a complete Heyting algebra, but not subalgebra of $(\mathcal{F}(X), \wedge, \vee, \rightarrow, \{\mathbf{1}\}, X)$, where for all $F, G \in \mathcal{F}_\sigma(X)$,*

$$F \wedge G = F \cap G, \quad F \vee G = [F \cup G)_\sigma, \quad F \rightarrow G = \{x \in X \colon F \cap [x)_\sigma \subseteq G\}.$$

We define some types of state filters of a state residuated lattice (X, σ). For a state filter $F \in \mathcal{F}_\sigma(X)$, F is called

prime state filter: if $(x \odot \sigma(x)) \vee (y \odot \sigma(y)) \in F$ then $x \in F$ or $y \in F$;

primary state filte: if $(x \odot y)' \in F$ then there exists $n \in \mathbb{N}$ such that $((\sigma(x))^n)' \in F$ or $((\sigma(y))^n)' \in F$;

Boolean state filter: $(x \odot \sigma(x)) \vee (x' \odot \sigma(x')) \in F$;

obstinate state filter: if $x \notin F$ then there exists $n \in \mathbb{N}$ such that $(\sigma(x))'^n \in F$.

Moreover, a state filter is called *maximal* if there is no state filter containing it properly. We have a basic result about maximal state filters.

Proposition 7. *Let $F \in \mathcal{F}_\sigma(X)$. Then F is maximal if and only if for $x \notin F$ there exists $n \in \mathbb{N}$ such that $((\sigma(x))^n)' \in F$.*

Proof. We only show that if $x \notin F$ then there exists $n \in \mathbb{N}$ such that $((\sigma(x))^n)' \in F$ when F is maximal. Let F be a maximal state filter and $x \notin F$. Since $[F \cup \{x\})_\sigma = X$, there exist $f \in F$ and $m \in \mathbb{N}$ such that $f \odot (x \odot \sigma(x))^m = 0$ and thus $f \leqslant ((x \odot \sigma(x))^m)'$. This implies $((x \odot \sigma(x))^m)' \in F$. Moreover, since F is a state filter, we get $\sigma((x \odot \sigma(x))^m)' \in F$. From

$$\begin{aligned}\sigma((x \odot \sigma(x))^m)' &= (\sigma((x \odot \sigma(x)) \odot \ldots \odot (x \odot \sigma(x))))' \\ &\leqslant ((\sigma(x) \odot \sigma\sigma(x)) \odot \ldots \odot (\sigma(x) \odot \sigma\sigma(x)))' \\ &= ((\sigma(x) \odot \sigma(x)) \odot \ldots \odot (\sigma(x) \odot \sigma(x)))' \\ &= ((\sigma(x))^{2m})',\end{aligned}$$

we get that $((\sigma(x))^{2m})' \in F$, that is, there exists $n \in \mathbb{N}$ such that $((\sigma(x))^n)' \in F$. □

Corollary 1. *For every state filter F, if F is obstinate then it is maximal.*

Moreover, similar to the case of distributive lattices, we show that if F is a maximal state filter then it is a prime state filter.

Proposition 8. *Let $F \in \mathcal{F}_\sigma(X)$. If F is maximal then it is prime.*

P r o o f. Suppose that a maximal state filter F is not prime. We have $(a \odot \sigma a) \vee (b \odot \sigma b) \in F$ for some $a, b \notin F$. Since F is maximal, we get $((\sigma a)^m)', ((\sigma b)^n)' \in F$ and $((\sigma a)^m)' \odot ((\sigma b)^n)' \in F$, therefore $((\sigma a)^m)' \odot ((\sigma b)^n)' \leqslant ((\sigma a)^m)' \wedge ((\sigma b)^n)' = (((\sigma a)^m) \vee ((\sigma b)^n))' \in F$.

On the other hand, from $(a \odot \sigma a) \vee (b \odot \sigma b) \in F$, we get $\sigma((a \odot \sigma a) \vee (b \odot \sigma b)) \in F$. Since $\sigma((a \odot \sigma a) \vee (b \odot \sigma b)) \leqslant \sigma(\sigma a \vee \sigma b) = \sigma a \vee \sigma b$, we have $\sigma a \vee \sigma b \in F$ and thus $(\sigma a \vee \sigma b)^{m+n} \in F$. The fact that $(\sigma a \vee \sigma b)^{m+n} \leqslant (\sigma a)^m \vee (\sigma b)^n$ implies $(\sigma a)^m \vee (\sigma b)^n \in F$. However, this is a contradiction. $\square$

Theorem 1. *For every state filter F, F is obstinate if and only if it is prime and Boolean.*

P r o o f. Suppose that a state filter F is obstinate. It is enough to show that F is Boolean. Let $x \in X$. If $x \in F$, since $\sigma x \in F$, then we have $x \odot \sigma x \in F$ and hence $(x \odot \sigma x) \vee (x' \odot \sigma x') \in F$. If $x \notin F$, since F is obstinate, we get $(\sigma x)' = \sigma x' \in F$. If $x' \notin F$ then $(\sigma x')' \in F$ and $(\sigma x')' \odot (\sigma x') = 0 \in F$. This is a contradiction. This means that $x' \in F$. It follows from $x' \odot \sigma x' \in F$ that $(x \odot \sigma x) \vee (x' \odot \sigma x') \in F$. Namely, F is Boolean.

Conversely, we assume that F is a prime and Boolean state filter. For any $x \in X$, since F is Boolean, we have $(x \odot \sigma x) \vee (x' \odot \sigma x') \in F$. Moreover, F is prime, we get that $x \in F$ or $x' \in F$. This means that if $x \notin F$ then $x' \in F$ and hence $\sigma x' = (\sigma x)' \in F$ for any $x \in X$, namely, F is obstinate. $\square$

For every $F \in \mathcal{F}_\sigma(X)$, we define a relation θ_F on X as follows:

$$(x, y) \in \theta_F \quad \text{if and only if} \quad x \to y,\ y \to x \in F.$$

Then it is easy to show:

Proposition 9. *θ_F is a congruence on a state residuated lattice (X, σ) and thus the quotient structure $X/F = (X/F, \wedge, \vee, \odot, \to, \sigma_{X/F}, 0/F, 1/F)$ is a state residuated lattice, where $\sigma_{X/F}(x/F) = \sigma x/F$ for all $x/F \in X/F$.*

In the case of a state filter F being obstinate, the quotient structure has a simple structure.

Theorem 2. *Let F be a state filter of X. Then F is obstinate if and only if $X/F \cong \{0, 1\}$. Therefore, the following conditions are equivalent to each other:*

For ever state filter F,

(1) *F is obstinate;*

(2) *F is maximal and Boolean;*

(3) *F is prime and Boolean;*
(4) $X/F \cong \{0, 1\}$.

Next, we consider the case of F being Boolean.

Proposition 10. *Let F be a state filter of a state residuated lattice X. If F is Boolean, then X/F is a state Boolean algebra.*

Proof. For any $x \in X$, since $(x \odot \sigma x) \vee (x' \odot \sigma x') \in F$, $(x \odot \sigma x) \leqslant x$ and $x' \odot \sigma x' \leqslant x'$, we have $x \vee x' \in F$. This implies that $x/F \vee (x/F)' = 1/F$ in X/F and hence that each element x/F in X/F is a complemented element, that is, $x/F \in B(X/F)$ and thus $X/F = B(X/F)$. This means that X/F is a Boolean state algebra. $\square$

Now we have a natural question whether the converse holds, that is,

▷ Is F a Boolean state filter when X/F is a Boolean state algebra?

Unfortunately, this does not hold, as the following example shows. Let $= \{0, a, b, 1\}$ with $0 < a, b < 1$ and $\sigma a = \sigma 1 = 1$, $\sigma b = \sigma 0 = 0$. It is obvious that (X, σ) is a state Boolean algebra. Let $F = \{1\}$. It is easy to show that F is a state filter and that $X/F = X/\{1\} \cong X$ is the state Boolean algebra. But $(a \odot \sigma a) \vee (a' \odot \sigma a') = (a \odot 1) \vee (b \odot 0) = a \notin \{1\} = F$. Namely, the state filter F is not Boolean.

4. Generalized state operators

At first we recall a definition of a state operator on an Rℓ-monoid according to [7], where a state operator is defined on a pseudo Rℓ-monoid, that is, an operator $\odot$ is not commutative. However, for the sake of simplicity, we treat it on a commutative Rℓ-monoid. In [7], a state operator $\mu\colon A \to A$ on an Rℓ-monoid A is defined as follows:

For all $x, y \in A$, it satisfies

(m1) $\mu(\mathbf{0}) = \mathbf{0}$,
(m2) $\mu(x \to y) = \mu(x) \to \mu(x \wedge y)$,
(m3) $\mu(x \odot y) = \mu(x) \odot \mu(x \to x \odot y)$,
(m4) $\mu(\mu(x) \odot \mu(y)) = \mu(x) \odot \mu(y)$,
(m5) $\mu(\mu(x) \to \mu(y)) = \mu(x) \to \mu(y)$,
(m6) $\mu(\mu(x) \vee \mu(y)) = \mu(x) \vee \mu(y)$.

For the case of a state operator on a BL algebra A, since the join operation $\vee$ can be represented by two other operations $\wedge$ and $\to$ as $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$, the last condition (m6) above can be removed.

Proposition 11. *Let A be an $R\ell$-monoid and $\mu\colon A \to A$ be a map satisfying* (m2). *Then the condition* (m3): $\mu(x \odot y) = \mu(x) \odot \mu(x \to x \odot y)$ *is equivalent to the condition*

$$(*) \qquad \mu(x) \odot \mu(x \to y) \leqslant \mu(y).$$

P r o o f. ($\Rightarrow$) Suppose that μ satisfies the condition (m3). At first, we note that μ is order-preserving. Indeed, if $x \leqslant y$, then we have $\mu(x) = \mu(x \wedge y) = \mu(y \odot (y \to x)) = \mu(y) \odot \mu(y \to (y \odot (y \to x))) \leqslant \mu(y)$ and thus $\mu(x) \leqslant \mu(y)$. Now it follows from (m2) that $\mu(x) \odot \mu(x \to y) = \mu(x) \odot (\mu(x) \to \mu(x \wedge y)) \leqslant \mu(x \wedge y) \leqslant \mu(y)$.

($\Leftarrow$) Conversely, we assume the condition ($*$) for all $x, y \in A$. It is easy to show that $\mu(\mathbf{1}) = \mathbf{1}$ by (m2). We note that μ is also order-preserving in this case. Suppose that $x \leqslant y$. Since $\mu(\mathbf{1}) = \mathbf{1}$, we have $\mu(x) = \mu(x) \odot \mathbf{1} = \mu(x) \odot \mu(x \to y) \leqslant \mu(y)$. It follows from divisibility and (m2) that $\mu(x) \odot \mu(x \to x \odot y) = \mu(x) \odot (\mu(x) \to \mu(x \wedge (x \odot y))) = \mu(x) \odot (\mu(x) \to \mu(x \odot y)) = \mu(x) \wedge \mu(x \odot y) = \mu(x \odot y)$. $\square$

In [15], a state operator on a residuated lattice is defined as follows. A map $\tau\colon L \to L$ is called a *state operator* on a residuated lattice L if it satisfies the following conditions:

For any $x, y \in L$,

(L1) $\tau(\mathbf{0}) = \mathbf{0}$,
(L2) $x \to y = \mathbf{1}$ implies $\tau(x) \to \tau(y) = \mathbf{1}$,
(L3) $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$,
(L4) $\tau(x \odot y) = \tau(x) \odot \tau(x \to x \odot y)$,
(L5) $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$,
(L6) $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$,
(L7) $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$,
(L8) $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$.

We also have a similar result about (L2).

Proposition 12. *Let L be a residuated lattice and $\tau\colon L \to L$ be a map satisfying* (L3). *Then the condition* (L2) *is equivalent to the condition*

$$(**) \qquad \tau(x) \odot \tau(x \to y) \leqslant \tau(y).$$

P r o o f. ($\Rightarrow$) Suppose that τ satisfies (L2), that is, τ is order preserving. Then it follows from (L3) that $\tau(x) \odot \tau(x \to y) = \tau(x) \odot (\tau(x) \to \tau(x \wedge y)) \leqslant \tau(x \wedge y) \leqslant \tau(y)$.

($\Leftarrow$) We get $\tau(\mathbf{1}) = \tau(\mathbf{0} \to \mathbf{0}) = \tau(\mathbf{0}) \to \tau(\mathbf{0} \wedge \mathbf{0}) = \tau(\mathbf{0}) \to \tau(\mathbf{0}) = \mathbf{1}$. If $x \to y = \mathbf{1}$ then we have $\tau(x) = \tau(x) \odot \mathbf{1} = \tau(x) \odot \tau(\mathbf{1}) = \tau(x) \odot \tau(x \to y) \leqslant \tau(y)$, that is, $\tau(x) \to \tau(y) = \mathbf{1}$. $\square$

Moreover, we note that the condition (L6) can be proved by use of other conditions. Indeed, since $\tau(\tau(x)) = \tau(\tau(x)\odot\mathbf{1}) = \tau(\tau(x)\odot\tau(\mathbf{1})) = \tau(x)\odot\tau(\mathbf{1}) = \tau(x)\odot\mathbf{1} = \tau(x)$ by (L5), we have $\tau(\tau(x) \to \tau(y)) = \tau(\tau(x)) \to \tau(\tau(x) \wedge \tau(y)) = \tau(x) \to (\tau(x) \wedge \tau(y)) = (\tau(x) \to \tau(x)) \wedge (\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$. Hence the condition (L6) is redundant to define a state operator τ.

The results above mean that a state operator τ on a residuated lattice can be defined by the following conditions: (L1), $(**)$, (L3)–(L5), (L7) and (L8).

R e m a r k 1. It was described in [15] that the class of all state residuated lattices was only a quasivariety and not a variety. However, this is not true, because, as we proved before, state operators in [15] are defined by (L1), $(**)$, (L3)–(L5), (L7) and (L8). This implies that the class of all state residuated lattices forms a variety.

Taking into the results above, we define a generalized state operator on a residuated lattice X as follows. A map $\sigma\colon X \to X$ satisfying the following conditions:

(gs1) $\sigma(\mathbf{0}) = \mathbf{0}$,
(gs2) $\sigma(x \to y) \leqslant \sigma(x) \to \sigma(y)$,
(gs3) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$,
(gs4) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$,
(gs5) $\sigma(\sigma(x) \vee \sigma(y)) = \sigma(x) \vee \sigma(y)$,
(gs6) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$

is called a *generalized state operator* (or simply g-*state operator*) and (X, σ) is called a g-*state residuated lattice*. It is trivial that the class of all g-state residuated lattices (X, σ) forms a variety.

It is easy to show the following results.

Proposition 13. *Let σ be a g-state operator on a residuated lattice X. Then we have*

(1) $\sigma(\mathbf{1}) = \mathbf{1}$;
(2) $\sigma(\sigma(x)) = \sigma(x)$;
(3) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$;
(4) $\sigma(x') = (\sigma(x))'$;
(5) $\sigma(x) \odot \sigma(y) \leqslant \sigma(x \odot y)$;
(6) $\ker(\sigma)$ *is a filter, where* $\ker(\sigma) = \{x \in X\colon\ \sigma(x) = \mathbf{1}\}$;
(7) $\sigma(X)$ *is a* $\{\wedge, \vee, \odot, \to\}$*-reduct subalgebra of* X. *Hence* $\sigma(X)$ *is a residuated lattice;*
(8) $a \in \sigma(A)$ *if and only if* $\sigma(a) = a$;
(9) *If σ is faithful, that is,* $\ker\sigma = \{\mathbf{1}\}$, *then $x < y$ implies $\sigma(x) < \sigma(y)$.*

We also show that a condition $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \to x \odot y)$ in the definition of state operators on Rℓ-monoids (see [7]) and on BL-algebras (see [2]) is redundant, because it holds in any residuated lattice with divisibility $x \wedge y = x \odot (x \to y)$.

Proposition 14. *Let σ be a g-state operator on a residuated lattice X with divisibility $x \wedge y = x \odot (x \to y)$ for all $x, y \in X$. Then we have $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \to x \odot y)$ for all $x, y \in X$.*

P r o o f. Since $x \odot y \leqslant x$, it follows from (gs3) that $\sigma(x) \odot \sigma(x \to x \odot y) = \sigma(x) \odot (\sigma(x) \to \sigma(x \wedge (x \odot y))) = \sigma(x) \odot (\sigma(x) \to \sigma(x \odot y)) = \sigma(x) \wedge \sigma(x \odot y) = \sigma(x \odot y)$. □

The above implies that:

Corollary 2. *All state MV-algebras (state BL-algebras, state Rℓ-monoids, state residuated lattices) are g-state residuated lattices.*

Let (X, σ) be a g-state residuated lattice. A nonempty set F of X is called a *σ-filter* if F is a filter of X and $x \in F$ implies $\sigma(x) \in F$. We denote $\mathcal{F}_\sigma(X)$ the class of all σ-filters of (X, σ). We give a characterization theorem of σ-filters below.

Proposition 15 (Characterization of σ-filters)**.** *For a nonempty subset $S \subseteq X$ of a g-state residuated lattice (X, σ), the σ-filter $[S)_\sigma$ generated by S is*

$$[S)_\sigma = [S) \vee [\sigma(S)).$$

P r o o f. Let $\Gamma = [S) \vee [\sigma(S))$. It is sufficient to show that Γ is the least σ-filter including S. It is obvious that $S \subseteq \Gamma$ and Γ is a filter of X. Suppose that $x \in \Gamma$. There exist $s_i, t_i \in S$ such that $(s_1 \odot \sigma(t_1)) \odot \ldots \odot (s_k \odot \sigma(t_k)) \leqslant x$ and hence that $\sigma((s_1 \odot \sigma(t_1)) \odot \ldots \odot (s_k \odot \sigma(t_k))) \leqslant \sigma(x)$. Since

$$\begin{aligned}
&\sigma((s_1 \odot \sigma(t_1)) \odot \ldots \odot (s_k \odot \sigma(t_k))) \\
&\qquad \geqslant \sigma(s_1 \odot \sigma(t_1)) \odot \ldots \odot \sigma(s_k \odot \sigma(t_k)) \\
&\qquad \geqslant (\sigma(s_1) \odot \sigma(\sigma(t_1))) \odot \ldots \odot (\sigma(s_k) \odot \sigma(\sigma(t_k))) \\
&\qquad = (\sigma(s_1) \odot \sigma(t_1)) \odot \ldots \odot (\sigma(s_k) \odot \sigma(t_k)) \\
&\qquad \geqslant ((s_1 \odot \sigma(s_1)) \odot (t_1 \odot \sigma(t_1))) \odot \ldots \odot ((s_k \odot \sigma(s_k)) \odot (t_k \odot \sigma(t_k))),
\end{aligned}$$

we have $\sigma(x) \in \Gamma$, that is, Γ is a σ-filter. For any σ-filter G including S, it is clear that $\Gamma \subseteq G$. This means that Γ is the least σ-filter including S, namely, $[S)_\sigma = [S) \vee [\sigma(S))$. □

Corollary 3. *We have*

(1) $F \in \mathcal{F}(X) \;\Rightarrow\; [F)_\sigma = F \vee [\sigma(F))$;
(2) $[x)_\sigma = [x) \vee [\sigma(x)) = [x \odot \sigma(x)) = [x \wedge \sigma(x))$;
(3) $F \in \mathcal{F}_\sigma(X) \;\Rightarrow\; [F \cup \{x\})_\sigma = F \vee [x)_\sigma = F \vee [x \odot \sigma(x))$.

The next result is proved for BL-algebras in [2]; however, it can be proved without difficulty in all residuated lattices.

Proposition 16. *Let P be a proper σ-filter of (X, σ). Then the following conditions are equivalent to each other.*

(1) *If $P_1, P_2 \in \mathcal{F}_\sigma(X)$ and $P_1 \cap P_2 = P$, then $P = P_1$ or $P = P_2$;*
(2) *If $P_1, P_2 \in \mathcal{F}_\sigma(X)$ and $P_1 \cap P_2 \subseteq P$, then $P_1 \subseteq P$ or $P_2 \subseteq P$;*
(3) *If $x, y \in X$ such that $(x \odot \sigma(x)) \vee (y \odot \sigma(y)) \in P$ then $x \in P$ or $y \in P$.*

A proper σ-filter P of (X, σ) is called a *prime σ-filter* if it satisfies one of the conditions above. We denote by $\mathrm{Spec}_\sigma(X)$ the class of all prime σ-filters of (X, σ).

For $F, G \in \mathcal{F}_\sigma(X)$, we define $F \bigvee\limits_\sigma G = \sup\limits_{\mathcal{F}_\sigma(X)} \{F, G\}$ and $F \bigwedge\limits_\sigma G = \inf\limits_{\mathcal{F}_\sigma(X)} \{F, G\}$. Then we have

Proposition 17. *For $F, G \in \mathcal{F}_\sigma(X)$,*

(1) $F \bigvee\limits_\sigma G = F \vee G$,
(2) $F \bigwedge\limits_\sigma G = F \wedge G$.

Hence $\mathcal{F}_\sigma(X)$ is a sublattice of $\mathcal{F}(X)$.

Proof. It is sufficient to show that both $F \vee G$ and $F \wedge G$ are σ-filters for all $F, G \in \mathcal{F}_\sigma(X)$. Suppose $x \in F \vee G$. There exist $f \in F$ and $g \in G$ such that $f \odot g \leqslant x$. Since $\sigma(x) \geqslant \sigma(f \odot g) \geqslant \sigma(f) \odot \sigma(g)$ and $F, G \in \mathcal{F}_\sigma(X)$, we have $\sigma(f) \in F$ and $\sigma(g) \in G$. This implies $\sigma(x) \in F \vee G$, that is, $F \vee G$ is also a σ-filter and thus $\sup\limits_{\mathcal{F}_\sigma(X)} \{F, G\} = F \vee G$. The other case can be proved similarly. $\square$

Remark 2. The result above means that the class $\mathcal{F}_\sigma(X)$ of all σ-filters is a sublattice of $\mathcal{F}(X)$ and thus $\mathcal{F}_\sigma(X)$ is a distributive lattice. In the following, we show that $\mathcal{F}_\sigma(X)$ is also a Heyting algebra but that it is not a subalgebra of $\mathcal{F}(X)$.

We prove the following result without difficulty, so we omit its proof.

Proposition 18. *For every σ-filter F, a σ-filter F is maximal if and only if $a \notin F$ implies $((\sigma(a))^n)' \in F$ for some $n \geqslant 1$.*

Proposition 19. *For all elements $x, y \in X$, we have*

(1) $[x)_\sigma \wedge [y)_\sigma = [x \odot \sigma(x)) \wedge [y \odot \sigma(y)) = [(x \odot \sigma(x)) \vee (y \odot \sigma(y)))$,

(2) $[x)_\sigma \vee [y)_\sigma = [x \wedge y)_\sigma = [x \odot y)_\sigma$.

P r o o f. We only show the case of (2). The other case can be directly proved from (2) in Corollary 3. Since $x, y \geqslant x \wedge y$, we have $x, y \in [x \wedge y)_\sigma$ and $[x)_\sigma, [y)_\sigma \subseteq [x \wedge y)_\sigma$. For any σ-filter F such that $[x)_\sigma, [y)_\sigma \subseteq F$, it follows from $x, y \in F$ that $x \wedge y \in F$ and hence that $[x \wedge y)_\sigma \subseteq F$. This means that $[x)_\sigma \vee [y)_\sigma = [x \wedge y)_\sigma$. Similarly, we have $[x)_\sigma \vee [y)_\sigma = [x \odot y)_\sigma$. $\square$

Now we have a natural question whether the equation $[x)_\sigma \wedge [y)_\sigma = [x \vee y)_\sigma$ holds or not. It follows from the example below that $[x)_\sigma \wedge [y)_\sigma = [x \vee y)_\sigma$ does not hold in general. This example is provided as an Example 3.22 in [15].

E x a m p l e 1. Let $X = \{\mathbf{0}, a, b, c, d, \mathbf{1}\}$ be a set with $\mathbf{0} < a < c < \mathbf{1}$, $\mathbf{0} < b < c$, $d < \mathbf{1}$ and

$\odot$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	a	$\mathbf{0}$	a	$\mathbf{0}$	a
b	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	b	b
c	$\mathbf{0}$	a	$\mathbf{0}$	a	b	c
d	$\mathbf{0}$	$\mathbf{0}$	b	b	d	d
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$

$\to$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a	d	$\mathbf{1}$	d	$\mathbf{1}$	d	$\mathbf{1}$
b	c	c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
c	b	c	d	$\mathbf{1}$	d	$\mathbf{1}$
d	a	a	c	c	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$

We define a map σ on X as follows:

$$\sigma(x) = \begin{cases} \mathbf{0}, & x = \mathbf{0}, b, d, \\ \mathbf{1}, & x = a, c, \mathbf{1}. \end{cases}$$

Then we have $[a)_\sigma = [a)$, $[b)_\sigma = [\mathbf{0}) = X$ and thus $[a)_\sigma \wedge [b)_\sigma = [a)$ but $[a \vee b)_\sigma = [c)_\sigma = [c) \neq [a)$, therefore, $[a)_\sigma \wedge [b)_\sigma \neq [a \vee b)_\sigma$.

We define an operation $\to_\sigma$ in $\mathcal{F}_\sigma(X)$ by

$$F \to_\sigma G = \{x \in X \colon F \cap [x)_\sigma \subseteq G\}, \quad F, G \in \mathcal{F}_\sigma(X).$$

Then we have

Proposition 20. *For all $F, G, H \in \mathcal{F}_\sigma(X)$, we have*

(1) $F \to_\sigma G \in \mathcal{F}_\sigma(X)$;

(2) $F \cap G \subseteq H$ *if and only if* $F \subseteq G \to_\sigma H$. *Thus,* $(\mathcal{F}_\sigma(X), \wedge, \vee, \to_\sigma, \{\mathbf{1}\}, X)$ *is a complete Heyting algebra.*

We show the prime filter theorem for σ-filters. A nonempty set $I \subseteq X$ is called a $\vee$-*closed system* if $x \vee y \in I$ for $x, y \in I$.

Lemma 1 (Prime filter theorem, see [19]). *Let $F \in \mathcal{F}_\sigma(X)$ and I be a $\vee$-closed system. If $F \cap I = \emptyset$ then there is a prime σ-filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Let $\Gamma = \{G\colon\ G \in \mathcal{F}_\sigma(X),\ F \subseteq G,\ G \cap I = \emptyset\}$. It is easy to prove that there is a maximal element P in Γ by Zorn's lemma. We only show that P is prime. Otherwise, there exist $a, b \in X$ such that $(a \odot \sigma(a)) \vee (b \odot \sigma(b)) \in P$ but $a, b \notin P$. By maximality of P, we have $(P \vee [a)_\sigma) \cap I \neq \emptyset$ and $(P \vee [b)_\sigma) \cap I \neq \emptyset$. There are elements $x, y \in I$ such that $x \in P \vee [a)_\sigma$ and $y \in P \vee [b)_\sigma$. It follows that $x \vee y \in I$ because I is $\vee$-closed. On the other hand, $x, y \leqslant x \vee y$ implies $x \vee y \in P \vee [a)_\sigma$ and $x \vee y \in P \vee [b)_\sigma$ and hence $x \vee y \in (P \vee [a)_\sigma) \cap (P \vee [b)_\sigma) = P \vee ([a)_\sigma \wedge [b)_\sigma) = P \vee [(a \odot \sigma(a)) \vee (b \odot \sigma(b))) = P$. Hence $x \vee y \in P \cap I$, but this is a contradiction. This means that P is a prime σ-filter. $\square$

Corollary 4. *If $a \notin F$ for a σ-filter F, then there exists a prime σ-filter P such that $F \subseteq P$ but $a \notin P$. Therefore, every σ-filter F is an intersection of prime σ-filters P such that $F \subseteq P$, that is,*

$$F = \bigcap \{P \in \mathrm{Spec}_\sigma\colon\ F \subseteq P\}.$$

Corollary 5. *If $a \neq \mathbf{1}$, then there exists a prime σ-filter P such that $a \notin P$.*

Corollary 6. $\bigcap \mathrm{Spec}_\sigma(X) = \{\mathbf{1}\}$.

This implies that:

Theorem 3 ([19]). *Every g-state residuated lattice X is a subdirect product of $\{X/P_\lambda\}$, where P_λ is a prime state filter of X.*

We may ask whether $[P)_\sigma$ is a prime σ-filter if P is a prime filter. The answer to that question is no. In fact, if we consider the residuated lattice X in the example above, then $[d) = \{d, \mathbf{1}\}$ is a prime filter but $[d)_\sigma = X$ is not a proper filter. Thus $[d)_\sigma$ is not a prime σ-filter even if $[d)$ is a prime filter.

Now we consider pre-linearity in a g-state residuated lattice (X, σ). As is well known, for a residuated lattice X satisfying the pre-linearity condition $(x \to y) \vee (y \to x) = \mathbf{1}$ for all $x, y \in X$, if P is a prime filter then the quotient algebra X/P is a linearly ordered residuated lattice. Now we have the following natural questions:

(Q1) Is X/P linearly ordered if X satisfies the pre-linearity condition and P is a prime σ-filter?

(Q2) Under what condition in a g-state residuated lattice is the quotient structure X/P by prime σ-filter P linearly ordered?

For the first question Q1, we have a negative answer, as the following example in [3] shows.

Example 2. Let $X = \{\mathbf{0}, a, b, c, d, \mathbf{1}\}$ with the order $\mathbf{0} < a < b$, $d < \mathbf{1}$ and $\mathbf{0} < c < d < \mathbf{1}$. We define $\odot$ and $\rightarrow$ by the tables

$\odot$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	$\mathbf{0}$	a	$\mathbf{0}$	$\mathbf{0}$	a
b	$\mathbf{0}$	a	b	$\mathbf{0}$	a	b
c	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	c	c	c
d	$\mathbf{0}$	$\mathbf{0}$	a	c	c	d
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$

$\rightarrow$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
a	d	$\mathbf{1}$	$\mathbf{1}$	d	$\mathbf{1}$	$\mathbf{1}$
b	c	d	$\mathbf{1}$	c	d	$\mathbf{1}$
c	b	b	b	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
d	a	b	b	d	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	$\mathbf{1}$

It is easy to show that X is a BL-algebra. We define a map σ on X as follows:

$$\sigma(x) = \begin{cases} \mathbf{0}, & x = \mathbf{0}, a, b, \\ \mathbf{1}, & x = c, d, \mathbf{1}. \end{cases}$$

Then it is clear that (X, σ) is a g-state residuated lattice. In this example, $\{\mathbf{1}\}$ is not a prime filter, because $b \vee d = \mathbf{1} \in \{\mathbf{1}\}$ and $b, d \neq \mathbf{1}$. But it is obvious that $\{\mathbf{1}\}$ is a prime σ-filter. However, the quotient algebra $X/\{\mathbf{1}\} \cong X$ is not a linearly ordered residuated lattice.

This example proves the following.

Theorem 4. *The quotient residuated lattice X/P by a prime σ-filter P is not necessary linearly ordered even if X satisfies the pre-linearity condition $(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$ for all $x, y \in X$.*

Therefore, the quotient algebras X/P of state $R\ell$-monoids (state BL-algebras) X by prime σ-filter are not necessary linearly ordered.

As to the second question Q2, we define a new condition in the case of g-state residuated lattices (called a *σ-pre-linearity* here), which corresponds to the pre-linearity condition in the case of residuated lattices

σ-pre-linearity: $((x \rightarrow y) \odot \sigma(x \rightarrow y)) \vee ((y \rightarrow x) \odot \sigma(y \rightarrow x)) = \mathbf{1}$.

Then it is easy to show

Theorem 5. *Let (X, σ) be a g-residuated lattice with the σ-pre-linearity condition and P be a prime σ-filter. Then the quotient structure X/P is linearly ordered.*

It is obvious that if X satisfies σ-pre-linearity then it satisfies pre-linearity. However, the converse does not hold in general. In fact, we have a counter-example in a Boolean algebra: Let $X = \{\mathbf{0}, a, b, \mathbf{1}\}$ with $\mathbf{0} < a, b < \mathbf{1}$. We define σ by $\sigma(a) = \sigma(\mathbf{1}) = \mathbf{1}$ and $\sigma(b) = \sigma(\mathbf{0}) = \mathbf{0}$. Then we have $(x \to y) \vee (y \to x) = \mathbf{1}$ for all $x, y \in X$, that is, the pre-linearity condition holds. But $((a \to b) \odot \sigma(a \to b)) \vee ((b \to a) \odot (b \to a)) = (b \odot \sigma(b)) \vee (a \odot \sigma(a)) = \mathbf{0} \vee a = a \neq \mathbf{1}$. Consequently, the σ-pre-linearity condition does not hold.

5. Extended filters

Extended filters in Rℓ-monoids were introduced in [14]. After that, extended filters are considered in pseudo-BL algebras in [1], [3]. For a subset $B \subseteq X$ and a filter $F \in \mathcal{F}(X)$,

$$E_F(B) = \{x \in X \colon\ x \vee b \in F \text{ for all } b \in B\}$$

is called an *extended filter* associated with B.

In [16], extended filters in $\mathcal{F}(X)$ are characterized in the case of a residuated lattice X. The notion of extended filter is generalized to the case of pseudo state BL-algebras in [2] and to the case of state residuated lattices in [15], where extended state filters are called *co-annihilators*. According to the definitions of these papers [1], [15], we define an extended σ-filter as follows.

Let (X, σ) be a g-state residuated lattice and B a subset $B \subseteq X$ and F a σ-filter of (X, σ). We put

$$E_F^{\sigma}(B) = \{x \in X \colon\ (x \odot \sigma(x)) \vee (b \odot \sigma(b)) \in F \text{ for all } b \in B\},$$

which is called an *extended σ-filter* associated with B.

Here we give a simple characterization theorem of extended σ-filters for a g-state residuated lattice (X, σ).

Theorem 6 (Characterization of extended σ-filters)**.** *Let F be a σ-filter and B a subset of X. Then we have*

$$E_F^{\sigma}(B) = [B)_{\sigma} \to_{\sigma} F$$

in the Heyting algebra $(\mathcal{F}_{\sigma}(X), \wedge, \vee, \to_{\sigma}, \{\mathbf{1}\}, X)$.

P r o o f. Let $x \in E_F^{\sigma}(B)$. For all $u \in [B)_{\sigma} \cap [x)_{\sigma}$, there are $b_i \in B$ and $n \geqslant 1$ such that $(b_1 \odot \sigma(b_1)) \odot \ldots \odot (b_k \odot \sigma(b_k)) \leqslant u$ and $(x \odot \sigma(x))^n \leqslant u$. Since $x \in E_F^{\sigma}(B)$ and $b_i \in B$, we have $(x \odot \sigma(x)) \vee (b_i \odot \sigma(b_i)) \in F$. This implies $((x \odot \sigma(x)) \vee$

$(b_1\odot\sigma(b_1)))\odot\ldots\odot((x\odot\sigma(x))\vee(b_k\odot\sigma(b_k)))\in F$ and $((x\odot\sigma(x))\vee(b_1\odot\sigma(b_1)))\odot\ldots\odot((x\odot\sigma(x))\vee(b_k\odot\sigma(b_k)))\leqslant(x\odot\sigma(x))\vee((b_1\odot\sigma(b_1))\odot\ldots\odot(b_k\odot\sigma(b_k)))\leqslant(x\odot\sigma(x))\vee u$, thus $(x\odot\sigma(x))\vee u\in F$. We note that $((x\odot\sigma(x))\vee u)^{n+1}\in F$ and $((x\odot\sigma(x))\vee u)^{n+1}\leqslant(x\odot\sigma(x))^n\vee u=u$. This means that if $u\in[B)_\sigma\cap[x)_\sigma$ then $u\in F$, namely, $[B)_\sigma\cap[x)_\sigma\subseteq F$. Hence we have $x\in[B)_\sigma\to_\sigma F$ and $E_F^\sigma(B)\subseteq[B)_\sigma\to_\sigma F$.

Conversely, suppose that $x\in[B)_\sigma\to_\sigma F$. For every $b\in B$, since $b\odot\sigma(b)\leqslant(x\odot\sigma(x))\vee(b\odot\sigma(b))$ and $x\odot\sigma(x)\leqslant(x\odot\sigma(x))\vee(b\odot\sigma(b))$, we get $(x\odot\sigma(x))\vee(b\odot\sigma(b))\in[B)_\sigma$, $(x\odot\sigma(x))\vee(b\odot\sigma(b))\in[x)_\sigma$ and hence $(x\odot\sigma(x))\vee(b\odot\sigma(b))\in[B)_\sigma\cap[x)_\sigma\subseteq F$. This means that $(x\odot\sigma(x))\vee(b\odot\sigma(b))\in F$ for all $b\in B$. Thus we have $x\in E_F^\sigma(B)$ and $[B)_\sigma\to_\sigma F\subseteq E_F^\sigma(B)$.

Therefore, we get that $E_F^\sigma(B)=[B)_\sigma\to_\sigma F$ in the Heyting algebra $\mathcal{F}_\sigma(X)$ for the g-state residuated lattice X. □

We recall a definition of state on a residuated lattice X. A map $s\colon X\to[0,1]$ is called a *Bosbach state* on a residuated lattice X if it satisfies

(BS1) $s(x)+s(x\to y)=s(y)+s(y\to x)$,
(BS2) $s(\mathbf{0})=0$ and $s(\mathbf{1})=1$, where "+" is an usual sum in [0,1].

Taking x by $x\wedge y$ and y by x, we have $s(x)+s(x\to y)=1+s(x\wedge y)$.

We also give a definition of another type of state, Riečan state. A map $s\colon X\to[0,1]$ is called a *Riečan state* on X if it satisfies

(RS1) If $x\perp y$ then $s(x\oplus y)=s(x)+s(y)$, where $x\oplus y=(x'\odot y')'$ and $x\perp y$ is defined by $x''\leqslant y'$, that is, $x\odot y=\mathbf{0}$;
(RS2) $s(\mathbf{0})=0$ and $s(\mathbf{1})=1$.

As the name of g-state operators shows, g-state operators induce new states on residuated lattices.

Proposition 21. *Let σ be a g-state operator and $s\colon X\to[0,1]$ be a (Bosbach, Riečan) state on $\sigma(X)$. Then $s_\sigma\colon\sigma(X)\to[0,1]$, defined by $s_\sigma(x)=s(\sigma(x))$ for all $x\in X$, is a (Bosbach, Riečan) state on X.*

Proof. At first, we consider the case of Riečan state. Let s be a Riečan state on $\sigma(X)$. It is obvious that $s_\sigma(\mathbf{1})=s(\sigma(\mathbf{1}))=s(\mathbf{1})=1$. For all $x,y\in X$ such that $x\perp y$, since $x\odot y=\mathbf{0}$, we have $\mathbf{0}=\sigma(\mathbf{0})=\sigma(x\odot y)\geqslant\sigma(x)\odot\sigma(y)$. This means that $\sigma(x)\perp\sigma(y)$ in $\sigma(X)$. Since s is the Riečan state on $\sigma(X)$, we get $s_\sigma(x)+s_\sigma(y)=s(\sigma(x))+s(\sigma(y))=s(\sigma(x)\oplus\sigma(y))$. On the other hand, $\sigma(x\oplus y)=\sigma((x'\odot y')')=\sigma(x'\to y'')=\sigma(x')\to\sigma(x'\wedge y'')$. It follows from $x''\leqslant y'$ that $y''\leqslant x'''=x'$ and $x'\wedge y''=y''$. Hence we have $\sigma(x\oplus y)=\sigma(x')\to\sigma(y'')=(\sigma(x))'\to(\sigma(y))''=((\sigma(x))'\odot(\sigma(y))')'=\sigma(x)\oplus\sigma(y)$ and $s(\sigma(x\oplus y))=s(\sigma(x)\oplus\sigma(y))$. This implies $s_\sigma(x\oplus y)=s_\sigma(x)+s_\sigma(y)$ and thus s_σ is a Riečan state.

For the case of Bosbach state s, s_σ is proved to be a Bosbach state as follows:

$$\begin{aligned}
s_\sigma(x \to y) + s_\sigma(x) &= s(\sigma(x \to y)) + s(\sigma(x)) \\
&= s(\sigma(x) \to \sigma(x \wedge y)) + s(\sigma(x)) \\
&= s(\sigma(x \wedge y)) + s(\sigma(x \wedge y) \to \sigma(x)) \\
&= 1 + s(\sigma(x \wedge y)) \\
&= 1 + s_\sigma(x \wedge y).
\end{aligned}$$

Thus s_σ is also a Bosbach state. $\square$

Remark 3. In [15], the result above is proved as Theorem 3.10 under the additional condition that $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ for all $x, y \in X$. However, this condition is redundant as our proof shows.

We consider a relation between $\mathcal{F}_\sigma(X)$ and $\mathcal{F}(\sigma(X))$.

Proposition 22. *For every $F \in \mathcal{F}_\sigma(X)$ and $G \in \mathcal{F}(\sigma(X))$, we have*

(1) $F \in \mathcal{F}_\sigma(X) \Rightarrow \sigma(F) \in \mathcal{F}(\sigma(X))$,
(2) $G \in \mathcal{F}(\sigma(X)) \Rightarrow \sigma^{-1}(G) \in \mathcal{F}_\sigma(X)$.

Proof. (1) Let $F \in \mathcal{F}_\sigma(X)$. It is clear that $\mathbf{1} = \sigma(\mathbf{1}) \in \sigma(F)$. If $\sigma(x), \sigma(y) \in \sigma(F)$ $(x, y \in F)$, since $\sigma(F) \subseteq F$, then $\sigma(x), \sigma(y) \in F$ and $\sigma(x) \odot \sigma(y) \in F$. This implies $\sigma(x) \odot \sigma(y) = \sigma(\sigma(x) \odot \sigma(y)) \in \sigma(F)$. At last, if $\sigma(x) \in \sigma(F)$ $(x \in F)$ and $\sigma(x) \leqslant \sigma(y)$, then we have $\sigma(x) \in F$ and $\sigma(y) \in F$. Thus $\sigma(y) = \sigma(\sigma(y)) \in \sigma(F)$. Hence $\sigma(F) \in \mathcal{F}(\sigma(X))$.

(2) It is trivial that $\mathbf{1} \in \sigma^{-1}(G)$. Suppose that $x, y \in \sigma^{-1}(G)$. It follows from $\sigma(x), \sigma(y) \in G$ that $\sigma(x) \odot \sigma(y) \in G$ and $\sigma(x) \odot \sigma(y) \leqslant \sigma(x \odot y)$, hence $\sigma(x \odot y) \in G$. This means that $x \odot y \in \sigma^{-1}(G)$. If $x \in \sigma^{-1}(G)$ and $x \leqslant y$, then we get $\sigma(x) \leqslant \sigma(y)$ and $\sigma(y) \in G$, that is, $y \in \sigma^{-1}(G)$. Moreover, for $x \in \sigma^{-1}(G)$, since $\sigma(\sigma(x)) = \sigma(x) \in G$, we have $\sigma(x) \in \sigma^{-1}(G)$ and hence $\sigma^{-1}(G) \in \mathcal{F}_\sigma(X)$. $\square$

We also have similar results about maximal filters. We denote by $\mathrm{Max}_\sigma(X)$ (or $\mathrm{Max}(\sigma(X))$) the class of all maximal σ-filters of (X, σ) (or maximal filter of $\sigma(X)$, respectively).

Corollary 7. *For every $M \in \mathcal{F}_\sigma(X)$ and $N \in \mathcal{F}(\sigma(X))$, we have*

(1) $M \in \mathrm{Max}_\sigma(X) \Rightarrow \sigma(M) \in \mathrm{Max}(\sigma(X))$,
(2) $N \in \mathrm{Max}(\sigma(X)) \Rightarrow \sigma^{-1}(N) \in \mathrm{Max}_\sigma(X)$.

Proof. (1) Suppose that $\sigma(x) \notin \sigma(M)$. Since $x \notin M$ and M is a maximal σ-filter, there is $n \geqslant 1$ such that $((\sigma(x))^n)' \in M$ and $\sigma(((\sigma(x))^n)') \in \sigma(M)$. Now, it follows from $\sigma((\sigma(x))^n) = \sigma(\sigma(x) \odot \ldots \odot \sigma(x)) \geqslant \sigma(\sigma(x)) \odot \ldots \odot \sigma(\sigma(x)) = \sigma(x) \odot \ldots \odot \sigma(x) = (\sigma(x))^n$ that $\sigma(((\sigma(x))^n)') = (\sigma((\sigma(x))^n))' \leqslant ((\sigma(x))^n)'$ and thus $((\sigma(x))^n)' \in M$. This implies that $\sigma(M)$ is a maximal filter of $\sigma(X)$, that is, $\sigma(M) \in \operatorname{Max}(\sigma(X))$.

(2) If $x \notin \sigma^{-1}(N)$, since N is a maximal filter and $\sigma(x) \notin N$, then there exists $n \geqslant 1$ such that $((\sigma(x))^n)' \in N \subseteq \sigma(X)$. This implies $\sigma(((\sigma(x))^n)') = ((\sigma(x))^n)' \in N$ and $((\sigma(x))^n)' \in \sigma^{-1}(N)$. Namely, $\sigma^{-1}(N)$ is the maximal σ-filter of (X, σ) and thus $\sigma^{-1}(N) \in \operatorname{Max}_\sigma(X)$. $\square$

6. Quotient structures of g-state residuated lattices

Let (X, σ) be a g-state residuated lattice. For a σ-filter F, we define a relation $\equiv_F$ by

$$x \equiv_F y \Leftrightarrow x \rightarrow y, \quad y \rightarrow x \in F.$$

It is easy to prove that $\equiv_F$ is a congruence relation on X, and thus we consider a quotient structure X/F by the congruence relation $\equiv_F$. We denote the congruence structure $X/F = \{x/F \colon x \in X\}$ and $x/F = \{y \in X \colon x \equiv_F y\}$. We define an operator $\sigma/F \colon X/F \rightarrow X/F$ on the residuated lattice X/F by $(\sigma/F)(x/F) = \sigma(x)/F$. It is obvious that the quotient structure X/F is a residuated lattice. We note that σ/F is well-defined. Indeed, if $x/F = y/F$, since $x \rightarrow y, y \rightarrow x \in F$ and F is the σ-filter, then we have $\sigma(x \rightarrow y), \sigma(y \rightarrow x) \in F$ and $\sigma(x \rightarrow y) \leqslant \sigma(x) \rightarrow \sigma(y)$, $\sigma(y \rightarrow x) \leqslant \sigma(y) \rightarrow \sigma(x)$. This yields that $\sigma(x) \rightarrow \sigma(y), \sigma(y) \rightarrow \sigma(x) \in F$ and hence $\sigma(x)/F = \sigma(y)/F$. Moreover, it is clear that σ/F is a g-state operator on X/F. Hence we get

Theorem 7. *For a σ-filter F, the structure $(X/F, \sigma/F)$ is a g-state residuated lattice.*

Corollary 8. *$(\sigma/F)(X/F)$ is a subalgebra of X/F.*

Lemma 2. *For a σ-filter F, we have $F \cap \sigma(X) = \sigma(F)$.*

Proof. It is clear from $\sigma(F) \subseteq F$ that $\sigma(F) \subseteq F \cap \sigma(X)$. Conversely, suppose that $a \in F \cap \sigma(X)$. There exists $x \in X$ such that $a = \sigma(x) \in F$. It follows that $a = \sigma(x) = \sigma(\sigma(x)) = \sigma(a) \in \sigma(F)$ and hence $F \cap \sigma(X) = \sigma(F)$. $\square$

Since we know that $\sigma(X)$ is a $\{\wedge, \vee, \odot, \rightarrow\}$-reduct subalgebra of X and $\sigma(F)$ is a filter of $\sigma(X)$, a quotient structure $\sigma(X)/\sigma(F)$ is a residuated lattice. Therefore, we ask about a relation between $(\sigma/F)(X/F)$ and $\sigma(X)/\sigma(F)$.

Theorem 8. *Let (X, σ) be a g-state residuated lattice and F be a σ-filter of X. Then*

$$(\sigma/F)(X/F) \cong \sigma(X)/\sigma(F).$$

Proof. Let $\xi\colon \sigma(X) \rightarrow (\sigma/F)(X/F)$ defined by $\xi(\sigma(x)) = (\sigma/F)(x/F) = \sigma(x)/F$. It is clear that ξ is well-defined and surjective. Moreover, it is easy to show that ξ is a homomorphism between residuated lattices $\sigma(X)$ and $(\sigma/F)(X/F)$. It follows from the homomorphism theorem that $\sigma(X)/\ker(\xi) \cong (\sigma/F)(X/F)$. For the kernel $\ker(\xi)$ of ξ, we get

$$\begin{aligned}
\sigma(x) \in \ker(\xi) &\Leftrightarrow \xi(\sigma(x)) = \mathbf{1}/F \text{ and } \sigma(x) \in \sigma(X)\\
&\Leftrightarrow \sigma(x)/F = \mathbf{1}/F \text{ and } \sigma(x) \in \sigma(X)\\
&\Leftrightarrow \sigma(x) \in F \text{ and } \sigma(x) \in \sigma(X)\\
&\Leftrightarrow \sigma(x) \in F \cap \sigma(X)\\
&\Leftrightarrow \sigma(x) \in \sigma(F).
\end{aligned}$$

Hence we have

$$(\sigma/F)(X/F) \cong \sigma(X)/\sigma(F).$$

□

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