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On the completeness of total spaces of horizontally conformal submersions

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Abstract. In this paper, we address the completeness problem of certain classes of Riemannian metrics on vector bundles. We first establish a general result on the completeness of the total space of a vector bundle when the projection is a horizontally conformal submersion with a bound condition on the dilation function, and in particular when it is a Riemannian submersion. This allows us to give completeness results for spherically symmetric metrics on vector bundle manifolds and eventually for the class of Cheeger-Gromoll and generalized Cheeger-Gromoll metrics on vector bundle manifolds. Moreover, we study the completeness of a subclass of g-natural metrics on tangent bundles and we extend the results to the case of unit tangent sphere bundles. Our proofs are mainly based on techniques of metric topology and on the Hopf-Rinow theorem.

Introduction and main results

Horizontally conformal maps arise naturally from the theory of harmonic morphisms. Indeed, harmonic morphisms have been characterized as harmonic maps which are horizontally conformal (cf. [10], [14], [11], [12]). Further, horizontally conformal submersions arise as generalizations of Riemannian submersions (cf. [8], [9]). Besides Riemannian submersions, there are two geometrically interesting classes of horizontally conformal submersions:

- The projection maps of tangent bundles endowed with special Riemannian g-natural metrics: the geometry of tangent bundles had been deeply studied from many point of views (cf. [1], [2], [3], [4], [17]), but the introduction

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of the large class of g-natural metrics (cf. [17]) had given a second wind to research in this topic during the last decades. When some restrictions are taken on these metrics, the projection map of the tangent bundle becomes a horizontally conformal submersion;

- The projection maps of vector bundles equipped with *spherically symmetric metrics* (cf. [5]), or with the *Cheeger-Gromoll metric* or more generally (p,q)-*-metrics* (cf. [6]). Spherically symmetric metrics on vector bundle manifolds are two-weights Riemannian metrics naturally constructed from a Riemannian metric on the base and a fiber metric on the vector bundle together with a compatible connection. When we restrict ourselves to the case of a tangent bundle, spherically symmetric metrics are examples of g-natural metrics.

In this paper, we are interested in the study of the completeness problem of the total space of a vector bundle whose projection is a horizontally conformal submersion, with a special focus on the cases of g-natural tangent bundles and spherically symmetric vector bundles. We shall refer to this as the completeness problem (**CP**). To the best of our knowledge, this problem had not been solved in its full generality, and only one partial result had been established in the context of Riemannian submersions. Indeed, R. Hermann proved that if the total space of a Riemannian submersion is complete then so is the base (cf. [13]). On the other hand, R. Albuquerque conjectured that the total space, when endowed with a spherically symmetric metric, is complete if and only if the base manifold and fibers are complete (cf. [5]).

We shall prove that the conjecture of Albuquerque is true even with weaker hypotheses. More precisely, we have the following:

Main Theorem 1. *Let G be a Riemannian metric on a vector bundle (E,π,M) for which the projection $\pi:(E,G)\longrightarrow(M,g)$ is a horizontally conformal submersion such that the dilation function λ satisfies $a\leq\lambda^2\leq b$, for some real numbers $a,b>0$. If the base (M,g) is complete, then so is the total space (E,G).*

The converse holds for Riemannian metrics on a vector bundle E for which the projection is a Riemannian submersion (cf. [13]). Actually, it holds also for the case described in the following corollary:

Corollary 1. *Let G be a Riemannian metric on a vector bundle (E,π,M) for which the projection $\pi:(E,G)\longrightarrow(M,g)$ is a horizontally homothetic submersion with constant dilation function. Then, (E,G) is complete if and only if (M,g) is complete.*

Since the projections of vector bundles endowed with spherically symmetric metrics are horizontally conformal submersions, Main Theorem 1 achieves an answer in affirmative to the problem **CP** not only for spherically symmetric metrics but generally when the projection is a horizontally conformal submersion with a bounded dilation function but without assuming completeness of fibers.

In particular, Main Theorem 1 answers the question of completeness for some well known classes of Riemannian metrics on vector bundles for which the projection is a Riemannian submersion. For example, the Cheeger-Gromoll and the generalized Cheeger-Gromoll metrics (cf. [6], [7], [20]). It also establishes the completeness of the class of Kaluza-Klein metrics (cf. [1], [2], [4]) on tangent bundles and tangent sphere bundles. Those applications will be explored in detail.

The proof of our main results are based on topological techniques and the Hopf-Rinow theorem. It relies on a comparison of the distances induced from the metrics of the base manifold and total space.

This paper will be organized as follows. The first section is devoted to recall the definitions of horizontally conformal submersions and spherically symmetric metrics as well as a concise account of g-natural metrics on tangent bundles and unit tangent sphere bundles. The second section contains the proofs of claims. The third section concerns the application of Main Theorem 1 to solve the **CP** in the different contexts as described above.

All manifolds are assumed to be smooth (by smooth we mean differentiable of class C^∞) and connected. All geometric objects (functions, vector fields, ...etc.) are smooth.

1 Preliminaries

This section serves to recall the main concepts we will be using in the forthcoming sections.

Generally speaking, given a map $\phi : (N,h) \longrightarrow (M,g)$, then for any point $x \in N$, set $\mathcal{V}_x(\phi) = \ker(d_x\phi)$, this space is called the *vertical subspace* at x associated with ϕ, and we denote $\mathcal{H}_x(\phi) = \mathcal{V}_x(\phi)^\perp$ and we call it the *horizontal subspace* at x associated with ϕ.

Definition 1. Let $\phi : (N,h) \longrightarrow (M,g)$ be a non-constant smooth map. If $C_\phi := \{x \in N : \ d_x\phi \equiv 0\}$ and $\tilde{N} = N \setminus C_\phi$, then ϕ is said to be *horizontally conformal* if there exists $\lambda : \tilde{N} \longrightarrow \mathbb{R}^+$ such that

$$g_{\phi(x)}(d_x\phi(X), d_x\phi(Y)) = \lambda(x)^2 h_x(X,Y),$$

for all $X, Y \in \mathcal{H}_x(\phi)$ and $x \in \tilde{N}$. The function λ is then extended to the whole of N by putting $\lambda_{|C_\phi} = 0$. The extended function $\lambda : N \longrightarrow \mathbb{R}^+$ is called the *dilation* of ϕ.

It turns out that the dilation function of a horizontally conformal map is smooth (see [12]). The gradient vector field $grad(\lambda^2)$ can be decomposed, with respect to the decomposition $T_xM = \mathcal{V}_x(\phi) \oplus \mathcal{H}_x(\phi)$, as

$$grad(\lambda^2) = grad_{\mathcal{V}}(\lambda^2) + grad_{\mathcal{H}}(\lambda^2).$$

Definition 2. A non-constant map $\phi : (N,h) \longrightarrow (M,g)$ is said to be *horizontally homothetic* if it is horizontally conformal with dilation function λ such that $grad_{\mathcal{H}}(\lambda^2) = 0$ on $\tilde{N}$.

Horizontally conformal submersions are direct generalizations of Riemannian submersions, it suffices to take the dilation function $\lambda \equiv 1$. Riemannian submersions have been extensively studied, for instance B. O'Neill introduced two fundamental tensors which relate the geometry of the base and total space in exactly the same way the second fundamental form do (cf. [22]). Further, he gave the equations, analogous to Guass and Codazzi, which relate the curvature tensors of the two manifolds (cf. [22]). Moreover, geodesics of Riemannian submersions were studied by many authors. For instance, B. O'Neill studied geodesics of the total space of a Riemannian submersion in their relation to those of the base, he also related the respective Jacobi fields and index forms (cf. [23]).

One of the geometrically interesting examples of horizontally conformal submersions are the projection of certain vector bundles when endowed with certain classes of Riemannian metrics. In [5], R. Albuquerque introduced the class of *spherically symmetric metrics* on vector bundle manifolds.

More precisely, let (E, π, M) be a vector bundle with a Riemannian base (M, g). We assume that E is endowed with a fiber metric h and a compatible connection D (i.e. $Dh = 0$). Denote by $\mathcal{V}E$ (resp. $\mathcal{H}E$) the *vertical* (resp. *horizontal*) subbundle of TE. Let K denote the connection map (the connector) associated with the connection D. Vectors and vector fields which lie in $\mathcal{H}E$ (resp. $\mathcal{V}E$) are said to be horizontal (resp. vertical).

First, we recall some basic facts from the theory of connections on fiber bundles and in particular connection theory on vector bundles. A curve in E is said to be *horizontal* if its tangent vector is a horizontal vector at each point. Horizontal curves are ultimately related to horizontal lifts of vector fields. More precisely, the integral curves of the horizontal lift X^h of a vector field X are the horizontal lifts of integral curves of X.

Definition 3. Let γ be a curve in M, a *horizontal lift* of γ is a horizontal curve γ^* in E such that $\pi(\gamma^*(t)) = \gamma(t)$, for all t.

The following result guarantees the existence of horizontal lifts of curves in the base, for a detailed proof in the general context of fiber bundles, we refer the reader to [15].

Proposition 1. *Let $\gamma : [0,1] \longrightarrow M$ be a curve starting at x and let $e \in E$ be a point in E such that $\pi(e) = x$, then there exists a unique horizontal lift γ^* of γ starting at e.*

We also recall the following useful lemma (cf. [5]).

Lemma 1. *Let f be a smooth real scalar function on E depending on $r = h(e,e)$. Then, for any horizontal (resp. vertical) vector X^H (resp. Y^V) on E, we have*

i) $X^H(f(r)) = 0$;

ii) $Y^V(f(r)) = 2f'(r)\xi^\flat(Y^V)$.

Consider the metric defined by

$$G_e(X,Y) = e^{2\varphi_1(r)} g_{\pi(e)}((d\pi)_e(X), (d\pi)_e(Y)) + e^{2\varphi_2(r)} h_{\pi(e)}(K_e(X), K_e(Y)), \quad (1)$$

for all $e \in E$ and $X, Y \in T_eE$, with $r = h(e,e)$ and φ_1, φ_2 are smooth scalar functions on E depending only on the norm $r = h(e,e)$ and smooth at $r = 0$

on the right. R. Albuquerque called those metrics *spherically symmetric metrics* (**SS-metrics**).

There are other classes of Riemannian metrics G on vector bundle manifolds among which we cite:

Cheeger-Gromoll metric: this metric is a generalization of the classical Cheeger-Gromoll metric on tangent bundles [1], [2], [20], given by

$$G_e^{cg}(X,Y) = g(d\pi(X), d\pi(Y)) + \frac{1}{1+r}\Big(h(K_eX, K_eY) + h(K_eX, e)h(K_eY, e)\Big), \tag{2}$$

for $e \in E$ and $X, Y \in T_eE$, with $r = h(e,e)$.

Generalized Cheeger-Gromoll metrics (cf. [7]): they are generalization of the previous Cheeger-Gromoll metric and are given by:

$$G_e^{p,q}(X,Y) = g(d\pi(X), d\pi(Y)) + \big(\frac{1}{1+r}\big)^p \big(h(K_e(X), K_e(Y)) + qh(K_eX, e)h(K_eY, e)\big), \tag{3}$$

for $e \in E$ and $X, Y \in T_eE$, with $r = h(e,e)$. Those metrics were called (p,q)--metrics (cf. [7] for more details). For, $p = q = 1$, one recovers the Cheeger-Gromoll metric G^{cg}.

It's clear that the projection of generalized Cheeger-Gromoll metrics, and hence the classical Cheeger-Gromoll metric, are Riemannnian submersions. Now, we shall prove that the projections of vector bundles endowed with SS-metrics are horizontally conformal submersions.

Proposition 2. *Let G be an SS-metric on E of the form (1), then the projection $\pi : (E, G) \longrightarrow (M, g)$ is a horizontally homothetic submersion with dilation function $\lambda : E \longrightarrow \mathbb{R}$ with $\lambda(e) = e^{-\varphi_1(r)}$.*

Proof. Since π is a submersion, $C_\pi = \emptyset$, thus $\tilde{E} = E \setminus C_\pi = E$. Furthermore, $V_e(\pi) = \mathcal{V}_eE$ and since the horizontal and vertical distributions are orthogonal with respect to G, we have $\mathcal{H}_e(\pi) = \mathcal{H}_eE = \ker(K_e)$. As a matter of fact, for all $e \in E$ and $Y_1 = X_1^h, Y_2 = X_2^h \in \mathcal{H}_eE$, we have

$$\begin{aligned} g_{\pi(e)}(d\pi(Y_1), d\pi(Y_2)) &= g_{\pi(e)}(X_1, X_2) \\ &= e^{-2\varphi_1} G_e(Y_1, Y_2), \end{aligned}$$

whence π is horizontally conformal with dilation function $\lambda : e \mapsto e^{-\varphi_1(\|e\|^2)}$.

Let $\{e_i\}$ be a local linear orthonormal frame of M and $\{\sigma_p\}$ a local orthonormal frame of E, denote by

$$E_i = e^{-\varphi_1} e_i^h \quad \text{and} \quad E_{n+p} = e^{-\varphi_2} \sigma_p^v,$$

for $i = 1, \dots, n$ and $p = 1, \dots, k$. Hence $\{E_I; I = 1, \dots, n+k\}$ is a local linear orthonormal frame of E.

Since, by virtue of Lemma 1, we have $E_i.\lambda = e^{-\varphi_1} e_i^h.\lambda = 0$, for all $i = 1, \dots, m$. Hence $grad_{\mathcal{H}}(\lambda) = 0$, thus π is horizontally homothetic. ☐

Remark 1. If we choose an SS-metric with weights φ_1 and φ_2 such that $\varphi_1 = 0$, then the projection is a Riemannian submersion.

Though tangent bundles are examples of vector bundles, they constitute a particular exception. Indeed, they can be endowed with a far more large family of naturally constructed metrics for many good reasons, firstly a Riemannian metric on the base gives at the same time a linear connection on the tangent bundle and inner products on fibers, secondly local frames of the tangent bundle are nothing but the coordinate frames, that is local trivializations of the tangent bundle arise naturally from the smooth structure of the base. For all of these considerations, it becomes quite natural to have a huge class of metrics on the tangent bundle to a Riemannian manifold, namely g-natural metrics which are defined as follows.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ its Levi-Civita connection. At each point (x, u) of its tangent bundle TM, the tangent space to TM splits into the horizontal and vertical subspaces with respect to ∇ as follows:

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)}TM \oplus \mathcal{V}_{(x,u)}TM.$$

For any vector $X \in T_xM$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}TM$, called the *horizontal lift* of X to $(x, u) \in TM$, such that $\pi_* X^h = X$, where $\pi : TM \to M$ is the natural projection. The *vertical lift* of a vector $X \in T_xM$ to $(x, u) \in TM$ is a vector $X^v \in \mathcal{V}_{(x,u)}TM$ such that $X^v(df) = Xf$, for all functions f on M(we consider 1-forms df on M as functions on TM via the equation $df(x, u) = uf$). The map $X \mapsto X^h$ is an isomorphism between the vector spaces T_xM and $\mathcal{H}_{(x,u)}TM$. Similarly, the map $X \mapsto X^v$ is an isomorphism between T_xM and $\mathcal{V}_{(x,u)}TM$. Each tangent vector $\tilde{Z} \in T_{(x,u)}TM$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in T_xM$ are uniquely determined tangent vectors. Horizontal and vertical lifts of vector fields on M can be defined in an obvious way and are uniquely defined vector fields on TM.

Classically, the total space of the tangent bundle was endowed with the Sasaki metric, but it turns out that the Sasaki metric presents a strong rigidity in the sense of [16] and [20]. In order to overcome such a rigidity, many generalizations of the Sasaki metric were introduced (e.g. the Cheeger-Gromoll metric, Oproiu metrics etc). We refer the reader to the paper [1] for a detailed survey of the geometry of the tangent bundle. The introduction of g-natural metrics comes from the description of all second order natural transformations of Riemannian metrics on manifolds into metrics on their tangent bundles [17], for more details about the concept of naturality and related notions we refer the reader to [17], [18] and [19]. g-natural metrics have been characterized as follows:

Proposition 3. *(see [4]) Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are six smooth functions α_i, $\beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that for every u, X, $Y \in T_xM$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r) g_x(X, Y) + (\beta_1 + \beta_3)(r) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) &= \alpha_2(r) g_x(X, Y) + \beta_2(r) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) &= \alpha_2(r) g_x(X, Y) + \beta_2(r) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) &= \alpha_1(r) g_x(X, Y) + \beta_1(r) g_x(X, u) g_x(Y, u), \end{cases}$$

where $r = g_x(u,u)$. *For* $n = 1$, *the same holds with* $\beta_i = 0$, *for* $i = 1, 2, 3$.

In the sequel, we shall use the following notations:

(i) $\phi_i = \alpha_i(t) + t\beta_i(t)$,

(ii) $\alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(\text{t})$,

(iii) $\phi(t) = \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t)$,

for all $t \in \mathbb{R}^+$.

Riemannian g-natural metrics are characterized as follows:

Proposition 4. *(see [4]) The necessary and sufficient conditions for a* g*-natural metric* G *on the tangent bundle of a Riemannian manifold* (M, g) *to be Riemannian are that the functions of Proposition 3, defining* G*, satisfy the inequalities*

$$\begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases} \tag{4}$$

for all $t \in \mathbb{R}^+$.

For $n = 1$ *the system reduces to* $\alpha_1(t) > 0$ *and* $\alpha(t) > 0$, *for all* $t \in \mathbb{R}^+$.

The *unit tangent sphere bundle* over a Riemannian manifold (M, g) is the hypersurface $T_1M = \{(x, u) \in TM | g_x(u, u) = 1\}$ in TM. The tangent space to T_1M, at a point $(x, u) \in T_1M$, is given by

$$T_{(x,u)}T_1M = \{X^h + Y^v / X \in T_xM, Y \in \{u\}^\perp \subset T_xM\}.$$

By a (pseudo)-Riemannian g-natural metric $\tilde{G}$ on T_1M we mean a (pseudo)--Riemannian metric induced by a g-natural metric G on TM. It's well known that a g-natural metric on T_1M is completely determined by the values of four real constants, namely:

$$a = \alpha_1(1), \quad b = \alpha_2(1), \quad c = \alpha_3(1), \quad d = (\beta_1 + \beta_3)(1),$$

where α_i and β_i are the weight functions of a g-natural metric as given in Proposition 3. In the same way, a g-natural metric $\tilde{G}$ on T_1M is Riemannian if and only if

$$a > 0, \quad \alpha = a(a + c) - b^2 > 0, \quad \phi = a(a + c + d) - b^2 > 0.$$

If $\tilde{G}$ is a g-natural metric on T_1M, then at each point $(x, u) \in T_1M$, the metric $\tilde{G}$ is completely determined by

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a + c)g_x(X, Y) + dg_x(X, u)g_x(Y, u) \\ \tilde{G}_{(x,u)}(X^h, Y^v) = \tilde{G}_{(x,u)}(Y^v, X^h) = bg_x(X, Y) \\ \tilde{G}_{(x,u)}(X^v, Y^v) = ag_x(X, Y), \end{cases}$$

It is worth mentioning that the tangent sphere bundle is not a vector bundle, it is a fiber bundle so our results do not work directly in this case, but we shall extend them without troubles nor extra assumptions.

Let G be a Riemannian g-natural metric with weight functions α_i and β_i, $i = 1, 2, 3$, such that,

$$\alpha_2 = 0, \beta_2 = 0, \beta_1 + \beta_3 = 0.$$

This way, we obtain a class of metrics known as Kaluza-Klein metrics (**KK-**-metrics). Let G be a KK-metric, then using similar techniques to those used in the proof of Proposition 2, one can prove that the projection $\pi : (TM, G) \longrightarrow (M, g)$ (resp. $\pi' : (T_1M, \tilde{G}) \longrightarrow (M, g)$) is a horizontally homothetic submersion with dilation function $\lambda = \sqrt{(\alpha_1 + \alpha_3)}$ (resp. horizontally homothetic with constant dilation function $\lambda = \sqrt{(a + c)}$).

2 Proofs of the claims

This section is devoted to prove the claims. The proof of the main theorem is based on a comparison of the distances induced from the metric tensors of the total space and the base.

Lemma 2. *Let $\pi : (N, h) \longrightarrow (M, g)$ be a horizontally conformal submersion with dilation function λ. Assume that M is complete and the square of the dilation function is bounded from above by $\mu > 0$ i.e. $\lambda^2 \leq \mu$. Then, for all $e_1, e_2 \in E$, we have*

$$d_g(\pi(e_1), \pi(e_2) \leq \sqrt{\mu} d_h(e_1, e_2),$$

where d_g and d_h are the (geodesic) distances induced by the metric tensors g and h, respectively.

Proof. Let $e_1, e_2 \in E$ with $x_i = \pi(e_i)$, $i = 1, 2$. For any curve $\delta : [a, b] \longrightarrow E$ relating e_1 to e_2, $\pi \circ \delta : [a, b] \longrightarrow M$ is a curve relating x_1 to x_2. Since (M, g) is complete, there exists a geodesic $c : I \longrightarrow M$ relating x_1 to x_2, such that $l(c) = d_g(x_1, x_2)$, where $l(c)$ denotes the length of c, with respect to g.

For a curve $\delta : I \longrightarrow E$, we have

$$\begin{aligned} h(\dot{\delta}(t), \dot{\delta}(t)) &= h(\mathcal{H}(\dot{\delta}(t)), \mathcal{H}(\dot{\delta}(t))) + h(\mathcal{V}(\dot{\delta}(t)), \mathcal{V}(\dot{\delta}(t))) \\ &= \lambda(\delta(t))^{-2} g\big(\pi_* \dot{\delta}(t), \pi_* \dot{\delta}(t)\big) + h(\mathcal{V}(\dot{\delta}(t)), \mathcal{V}(\dot{\delta}(t))), \end{aligned}$$

where $\mathcal{H}$ (resp. $\mathcal{V}$) denotes the horizontal (resp. vertical) part of a vector. If $L(\delta)$ denotes the length of δ, with respect to h, then

$$\begin{aligned} L(\delta) &= \int_a^b \big(h(\dot{\delta}(t), \dot{\delta}(t)\big)^{\frac{1}{2}} dt \\ &= \int_a^b \big(\lambda(\delta(t))^{-2} g(\dot{\overbrace{\pi \circ \delta}}(t), \dot{\overbrace{\pi \circ \delta}}(t)) + h(\mathcal{V}(\dot{\delta}(t)), \mathcal{V}(\dot{\delta}(t)))\big)^{\frac{1}{2}} dt \\ &\geq \frac{1}{\sqrt{\mu}} \int_a^b \big(g(\dot{\overbrace{\pi \circ \delta}}(t), \dot{\overbrace{\pi \circ \delta}}(t))\big)^{\frac{1}{2}} dt \\ &\geq \frac{l(\pi \circ \delta)}{\sqrt{\mu}} \end{aligned}$$

Hence, for any curve (potentially piecewise smooth) $\delta : [a,b] \longrightarrow E$, we have

$$L(\delta) \geq \frac{l(\pi\circ\delta)}{\sqrt{\mu}} \geq \frac{l(c)}{\sqrt{\mu}} = \frac{1}{\sqrt{\mu}} d_g(x_1, x_2),$$

whence the result. □

Proof of Main Theorem 1. By virtue of the Hopf-Rinow theorem, it suffices to show that (E, d_G) is a complete metric space. Let $(e_n)_n$ be a Cauchy sequence of the metric space (E, d_G). By virtue of Lemma 2, we have $d_g(\pi(e_{n+p}), \pi(e_n) \leq \sqrt{b} d_G(e_{n+p}, e_n)$, for all $n, p \in \mathbb{N}$, we conclude that the sequence (x_n), with $x_n = \pi(e_n)$, is a Cauchy sequence of the metric space (M, d_g). By virtue of the Hopf-Rinow theorem, the sequence (x_n) converges to a limit $x_0 \in M$.

Let $\epsilon > 0$. Since the sequence $(x_n)_n$ converges to x_0, there exists $N_\epsilon \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ for all $n \geq N_\epsilon$. So, for $n \geq N_\epsilon$, let $c : [0,1] \longrightarrow M$ be a geodesic relating x_n to x_0 and let c^* be the horizontal lift of c starting at e_n as given in Proposition 1.

We get

$$\|\dot{\widehat{c^*}}(t)\|_G^2 = G(\dot{\widehat{c^*}}(t), \dot{\widehat{c^*}}(t)) = \lambda(c^*(t))^{-2} g(\dot{c}(t), \dot{c}(t)),$$

thus

$$L(c^*) \leq \frac{l(c)}{\sqrt{a}} = \frac{1}{\sqrt{a}} d_g(x_n, x_0).$$

In particular, we have $d_G(e_n, c^*(1)) \leq \dfrac{1}{\sqrt{a}} d_g(x_n, x_0) \leq \dfrac{\epsilon}{\sqrt{a}}$. Thus, the sequence $(e_n)_n$ converges to the limit point $e = c^*(1)$. Once again, the Hopf-Rinow theorem gives the result. □

Remark 2. In Lemma 2 and the proof of Main Theorem 1, we have proved that if (M, g) is complete and G is a Riemannian metric on E such that the projection $\pi : (E, G) \longrightarrow (M, g)$ is a horizontally conformal map with dilation function λ satisfying $a \leq \lambda^2 \leq b$, for some real numbers $a, b > 0$, then we have

$$\sqrt{a} d_G(e_1, e_2) \leq d_g(\pi(e_1), \pi(e_2) \leq \sqrt{b} d_G(e_1, e_2),$$

where $e_1, e_2 \in E$ are the ends of the lift of a minimizing geodesic relating x_1 and x_2, with $x_i = \pi(e_i)$ for $i = 1, 2$.

3 Applications

This section is devoted to the application of Main Theorem 1 to settle the completeness problem of spherically symmetric metrics and eventually the completeness of the Cheeger-Gromoll and (p, q)-metrics, we will also study the completeness of g-natural metrics on tangent bundles and tangent sphere bundles.

It is well known that if the total space of a Riemannian submersion is complete then the base is also complete (cf. [13], [21]). Further, if G is a complete Riemannian metric on E, then the fibers E_x are complete submanifolds because the fibers are closed submanifolds of (E, G).

Lemma 3. *Let G be a Riemannian metric on E such that the projection $\pi : (E,G) \longrightarrow (M,g)$ is a horizontally homothetic submersion with constant dilation function λ. If (E,G) is complete, then so is (M,g).*

Proof. Using the fundamental tensors of a submersion introduced in [22] and their application in the case of horizontally conformal submersions studied in [11] and [12], one can prove that

$$\begin{aligned}(\nabla^G)_{\dot{\gamma}^*}\dot{\gamma}^* &= (\pi^*\nabla^g_{\dot{\gamma}})\dot{\gamma} + \frac{1}{2}\big(\mathcal{V}[\dot{\gamma}^*,\dot{\gamma}^*] + \lambda^2 h(\dot{\gamma}^*,\dot{\gamma}^*)grad(\frac{1}{\lambda^2})\big)\\ &= (\pi^*\nabla^g_{\dot{\gamma}^*})\dot{\gamma}^*,\end{aligned}$$

where ∇^G and ∇^g are the Levi-Civita connections of (E,G) and (M,g), respectively, and $\pi^*\nabla^g$ is the pullback connection of ∇^g by π.

Thus, a curve in M is a geodesic if and only if its horizontal lift is a geodesic in E. Hence the result. □

As an application we present a partial answer to the **CP** and some completeness results for Cheeger-Gromoll and generalized Cheeger-Gromoll metrics:

Corollary 2. *Assume that E is endowed with a metric G belonging to one of the following classes:*

(a) SS-metrics with φ_1 is constant,

(c) (p,q)-metrics $G^{p,q}$.

Then, the Riemannian manifold (E,G) is complete if and only if (M,g) is complete.

Proof. It follows from Lemma 3 and Main Theorem 1. □

For spherically symmetric metrics on vector bundle manifolds for which the dilation function is not constant, the zero section is a global section that allows one to embed the base manifold as a submanifold of the total space, but this can not ensure the completeness of (M,g) from that of (E,G).

Corollary 3. *Let G be an SS-metric on E with weight functions φ_1 and φ_2. Assume that φ_1 is bounded and (M,g) is complete, then (E,G) is complete.*

Proof. The projection π is horizontally conformal with dilation function $\lambda = e^{-\varphi_1}$. Since φ_1 is bounded, then we can find $a,b > 0$ such that $a \leq \lambda^2 \leq b$. Hence the result follows from Main Theorem 1. □

Example 1. Take $\varphi_1(r) = cos(r)$, and φ_2 an arbitrary weight function as in section 1. If (M,g) is complete, then (E,G) is complete, where G is the spherically symmetric metric with weights φ_1 and φ_2.

Now, we focus on tangent bundles and tangent sphere bundles with g-natural metrics.

Proposition 5. *Let (M, g) be a Riemannian manifold and let TM be its tangent bundle endowed with a KK-metric G with weight functions α_i and β_i, for $i = 1, 2, 3$. Assume that $\alpha_1 + \alpha_3$ satisfies $a \leq \alpha_1 + \alpha_3 \leq b$, for some real numbers $a, b > 0$, and (M, g) is complete, then (TM, G) is complete. Furthermore, the equivalence holds if $\alpha_1 + \alpha_3$ is constant.*

Proof. It follows from Lemma 2 and Main Theorem 1. □

Proposition 6. *Let (M, g) be a Riemannian manifold and let T_1M be its unit tangent sphere bundle endowed with $\tilde{G}$ induced from a KK-metric G with weight functions α_i and β_i, for $i = 1, 2, 3$. Then $(T_1M, \tilde{G})$ is complete if and only if (M, g) is complete.*

Proof. We give only a sketch of the proof since it stems from the proof of Main Theorem 1. Since the projection $\pi : (T_1M, G) \longrightarrow (M, g)$ is a horizontally conformal submersion with constant dilation function $\lambda = \sqrt{(a + c)}$ with $a = \alpha_1(1)$ and $c = \alpha_3(1)$, Lemma 2 implies that the projection of a Cauchy sequence of the total space is a Cauchy sequence of the base M. Moreover, the horizontal lift of a geodesic, with respect to the Levi-Civita connection of $\tilde{G}$, exists by virtue of a more general version of Proposition 1 in the case of fiber bundles (cf. [15]). Hence, one obtains a limit point for the considered sequence, which implies at once that (E, G) is complete. The converse follows from Lemma 3. □

Remark 3. The class of *Kaluza-Klein metrics* on (unit) tangent bundles to Riemannian manifolds includes the Sasaki metric [24], the Cheeger-Gromoll metric and the Cheeger-Gromoll type metrics $G^{p,q}$ defined in [6].

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