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REPRESENTATION AND CONSTRUCTION OF HOMOGENEOUS AND QUASI-HOMOGENEOUS N-ARY AGGREGATION FUNCTIONS

Yong Su and Radko Mesiar

Homogeneity, as one type of invariantness, means that an aggregation function is invariant with respect to multiplication by a constant, and quasi-homogeneity, as a relaxed version, reflects the original output as well as the constant. In this paper, we characterize all homogeneous/quasi-homogeneous n-ary aggregation functions and present several methods to generate new homogeneous/quasi-homogeneous n-ary aggregation functions by aggregation of given ones.

Keywords: aggregation functions, invariantness, homogeneity, quasi-homogeneity

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1. INTRODUCTION

Aggregation functions, as tools for aggregation (fusion) process that combine numerical values into a single representative value, have proved to be useful in computer and engineering science, economic and finance, social science as well as many other applied fields of physics and natural science [6]. There exist a great number of aggregation functions owing to the diversified demands of different areas, and their selection generally depends on the context to which they are going to be applied.

The scale types of variables being aggregated as an essential factor when choosing an appropriate aggregation function should be taken into consideration. The *scale type* of a variable is defined by the class of *admissible transformations*, such as that from grams to pounds, that change the scale into an alternative acceptable scale. Many decisions are achieved by aggregation functions, which are "meaningful", in the sense that they do not depend upon the particular scales of measurement chosen for the variables, but only upon their scale types. A function invariant under appropriate admissible transformation is the one not depending on a given scale. According to the type of scales we can speak of several types of invariantnesses of aggregation functions.

Invariantness with respect to any scale which is rather restrictive has been studied in [12]. If we fix the beginning of our scale (i. e., "zero" is fixed) but letting free the unit (recall, e. g., the mass measurement in kilograms and in pounds), homogeneous

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aggregation functions are needed [13]. The homogeneity of order k has been studied for t-norms/t-conorms [2], for copulas [9] and for general binary aggregation functions in [15]. For general aggregation functions, only the homogeneity of order 1 was discussed in [13], see also [6]. The homogeneity is a rather restrictive property, and we need a more relaxed version, one that reflects the original output as well as the concentration constant. A relaxed homogeneity, called quasi-homogeneity, was introduced in [4]. The quasi-homogeneity has been investigated for t-norms [4], for copulas [10] and for general binary aggregation functions [15]. In this paper, homogeneity/quasi-homogeneity n-ary aggregation functions are characterized and several methods to construct homogeneous/quasi-homogeneous n-ary aggregation functions by aggregation of given ones are presented.

The paper is organized as follows. In Section 2 we present the preliminary notions and results that are necessary in the remainder of this paper. In Section 3 the characterizations of (quasi-)homogeneous n-ary aggregation functions are presented and Section 4 is devoted to presenting several methods to generate homogeneous/quasi-homogeneous n-ary aggregation functions by aggregation of given ones. Finally, Section 5 includes some conclusions and future work.

2. PRELIMINARIES

First, let us review essential prerequisites. Let n be any non-zero natural integer and set $[n] := \{1, \ldots, n\}$. We will often use bold symbols to denote n-tuples. For instance, $(x_1, \ldots, x_n)$ will often be written as $\mathbf{x}$. In particular, $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$. We will frequently use vector inequality $\mathbf{x} \leq \mathbf{y}$, which means that $x_i \leq y_i$ for any $i \in [n]$. For any n-tuples $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \leq \mathbf{y}$, we denote by $\frac{\mathbf{x}}{\mathbf{y}}$ the n-tuple $(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n})$ with convention $\frac{0}{0} = 1$. For any non-zero natural integer k and any $x \in [0,1]$, we set $k \cdot x := x, \ldots, x$ (k times). For instance, $\mathbf{F}(3 \cdot x, 2 \cdot y) = \mathbf{F}(x, x, x, y, y)$.

An n-ary *aggregation function* on $[0,1]^n$ is merely a function $\mathbf{A}^{(n)} : [0,1]^n \to [0,1]$ that

(i) is nondecreasing (in each variable), i.e., $\mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{A}^{(n)}(\mathbf{x}) \leq \mathbf{A}^{(n)}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$;

(ii) fulfills the boundary conditions $\mathbf{A}^{(n)}(\mathbf{0}) = 0$ and $\mathbf{A}^{(n)}(\mathbf{1}) = 1$.

When no confusion can arise, the aggregation functions will simply be written as $\mathbf{A}$ instead of $\mathbf{A}^{(n)}$.

An n-ary function $\mathbf{F} : [0,1]^n \to [0,1]$ is

- *homogeneous of order* $k > 0$ if, for any $\lambda \in [0,1]$ and any $\mathbf{x} \in [0,1]^n$,

$$\mathbf{F}(\lambda \mathbf{x}) = \lambda^k \mathbf{F}(\mathbf{x}); \tag{1}$$

- *meaningful on a single ratio scale* [6] if, for any $\lambda \in [0,1]$, there exists $R(\lambda) > 0$ such that

$$\mathbf{F}(\lambda \mathbf{x}) = R(\lambda) \mathbf{F}(\mathbf{x}) \tag{2}$$

for all $\mathbf{x} \in [0,1]^n$;

- *quasi-homogeneous* [4] if there exists some continuous, strictly monotonic function $\varphi : [0,1] \to \mathbb{R}$ and some function $f : [0,1] \to [0,1]$ such that, for any $\lambda \in [0,1]$ and any $\mathbf{x} \in [0,1]^n$,

$$\mathbf{F}(\lambda \mathbf{x}) = \varphi^{-1}(f(\lambda)\varphi(\mathbf{F}(\mathbf{x}))). \tag{3}$$

  In this case, we also say that $\mathbf{F}$ is (φ, f)-quasi-homogeneous.

Note that if an n-ary function $\mathbf{F}$ satisfies Eq. (1) with $k = 1$, then we say that $\mathbf{F}$ is *ratio scale invariant* or *positively homogeneous* (see [6]).

By means of any n-ary aggregation function $\mathbf{A}$, a positively homogeneous n-ary function $\mathbf{H^A}$ can be constructed by $\mathbf{H^A}(\mathbf{0}) = 0$ and

$$\mathbf{H^A}(\mathbf{x}) = \mathbf{Max}(\mathbf{x})\mathbf{A}\left(\frac{\mathbf{x}}{\mathbf{Max}(\mathbf{x})}\right) \quad \text{if } \mathbf{x} \neq \mathbf{0},$$

where $\mathbf{Max}(\mathbf{x}) := \max(x_1, \ldots, x_n)$. Observe that $\mathbf{H^A}$ is positively homogeneous, however, it needs not be monotone. A necessary and sufficient condition for a function $\mathbf{H^A}$ to be monotone was presented in [13].

Theorem 2.1. Let $\mathbf{A}$ be an n-ary aggregation function. The function $\mathbf{H^A} : [0,1]^n \to [0,1]$ is an n-ary aggregation function if and only if

$$\frac{\mathbf{A}(\mathbf{x})}{\mathbf{A}(\mathbf{y})} \geq \mathbf{Min}\left(\frac{\mathbf{x}}{\mathbf{y}}\right) \tag{4}$$

with convention $\frac{0}{0} = 1$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ such that $\mathbf{x} \leq \mathbf{y}$ and $x_i = y_i = 1$ for some $i \in [n]$.

By using the logarithmic transformation of an n-ary aggregation function $\mathbf{B}$ into $\mathbf{L} : [0,\infty]^n \to [0,\infty]$, namely,

$$\mathbf{L}(\mathbf{x}) := -\log(\mathbf{B}(\exp(-x_1), \ldots, \exp(-x_n))),$$

property (4) can be rewritten as

$$\mid \mathbf{L}(\mathbf{x}) - \mathbf{L}(\mathbf{y}) \mid \leq \|\mathbf{x} - \mathbf{y}\|_\infty \tag{5}$$

where $\|\cdot\|_\infty$ is the standard Chebyshev norm, i.e., $\|\mathbf{x}\|_\infty = \max_i |x_i|$. The property (5) will be called *kernel property.* From the monotonicity of $\mathbf{L}$, it suffices to deal with $\mathbf{x}, \mathbf{y} \in [0,\infty]^n$ such that $x_i = y_i = 0$ for some $i \in [n]$. Note that aggregation functions with property (5) for all $\mathbf{x}, \mathbf{y} \in [0,\infty]^n$ are equivalently characterized by

$$\mathbf{L}(\mathbf{x} + s\mathbf{1}) \leq \mathbf{L}(\mathbf{x}) + s$$

for all $\mathbf{x} \in]0,\infty[^n$ and $s \in [0,\infty]$, and as a prominent example we recall the Choquet integral-based aggregation functions (see pp. 279–280 in [6] for details).

Example 2.2. For any weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n) \in [0,1]^n$ such that $\sum_{i=1}^n w_i = 1$, in the class of quasi-arithmetic means (weighted quasi-arithmetic means), positively homogeneous aggregation functions form a subclass $(\mathbf{A}_p)_{p\in]-\infty,\infty[}$ of so called power-root operators [3],

$$\mathbf{A}_p(x_1, \dots, x_n) = \left(\sum_{i=1}^n w_i x_i^p\right)^{\frac{1}{p}}$$

for $p \neq 0$ and $\mathbf{A}_0(x_1, \dots, x_n) = \prod_{i=1}^n x_i^{w_i}$ is weighted geometric mean.

Finally, we review a general composition construction method that yields a (new) aggregation function from given ones: For an m-ary outer aggregation function $\mathbf{A}$, any system $(E_1, \dots, E_m)$ of non-empty subsets of $[n]$ with a fixed position of its members (i. e., $E_i = \{p_{i,1}, \dots, p_{i,n_i}\}$ a subset of $[n]$ with cardinality n_i), and aggregation functions B_i related to dimension $n_i = card\, E_i$, for $\mathbf{x} \in [0,1]^n$ define n_i-ary input vector $\mathbf{x}_{E_i} = (x_{p_{i,1}}, \dots, x_{p_{i,n_i}})$, then the composite function

$$\mathbf{C}(\mathbf{x}) = \mathbf{A}(\mathbf{B}_1(\mathbf{x}_{E_1}), \dots, \mathbf{B}_m(\mathbf{x}_{E_m})) \tag{6}$$

is an n-ary aggregation function.

Observe that the general composition construction method above includes those two composition construction methods in Chapter 6.3 of [6], that is,

- *Composition based on partition*: an n-ary outer aggregation function $\mathbf{A}$ and m inner aggregation functions $\mathbf{B}_1, \dots, \mathbf{B}_m$ defined on n_1-, ..., n_m-dimensional input vectors, i. e., for $n = \sum_{i=1}^m n_i$ we obtain an aggregation function $\mathbf{D}_{\mathbf{A};\mathbf{B}_1,\dots,\mathbf{B}_m} : [0,1]^n \to [0,1]$ given by

$$\mathbf{D}_{\mathbf{A};\mathbf{B}_1,\dots,\mathbf{B}_m}(\mathbf{x}_1, \dots, \mathbf{x}_m) = \mathbf{A}(\mathbf{B}_1(\mathbf{x}_1), \cdots, \mathbf{B}_m(\mathbf{x}_m)) \tag{7}$$

  where $\mathbf{x}_i \in [0,1]^{n_i}$, $i = 1, \dots, m$.

- *Composition of functions*: Let $m, n \in \mathbb{N}\backslash\{1\}$. Let $\mathbf{A} : [0,1]^m \to [0,1]$ and $\mathbf{B}_1, \dots, \mathbf{B}_m : [0,1]^n \to [0,1]$ be aggregation functions. Then the function $\mathbf{C} = \mathbf{A}(\mathbf{B}_1, \dots, \mathbf{B}_m) : [0,1]^n \to [0,1]$ given by

$$\mathbf{C}(\mathbf{x}) = \mathbf{A}(\mathbf{B}_1(\mathbf{x}), \mathbf{B}_2(\mathbf{x}), \dots, \mathbf{B}_m(\mathbf{x})) \tag{8}$$

  is an n-ary aggregation function.

3. CHARACTERIZATIONS OF (QUASI-)HOMOGENEOUS N-ARY AGGREGATION FUNCTIONS

Positively homogeneous aggregation functions were characterized in [13] (e. g., Theorem 2.1). In this section, we will characterize (quasi-)homogeneous n-ary aggregation functions in terms of positively homogeneous aggregation functions. First, we show that for an n-ary aggregation function, Eqs. (1) and (2) are equivalent.

Proposition 3.1. An n-ary aggregation function $\mathbf{A}$ is meaningful on a single ratio scale if and only if it is homogeneous of order k for some $k > 0$.

Proof. The sufficiency is obvious.

Suppose that $\mathbf{A}$ is meaningful on a single ratio scale. Then $R(x) = R(x)\mathbf{A}(\mathbf{1}) = \mathbf{A}(n \cdot x)$ for any $x \in [0,1]$ and hence, R is increasing. Further

$$R(xy) = \mathbf{A}(n \cdot xy) = R(x)\mathbf{A}(n \cdot y) = R(x)R(y),$$

and hence, R solves Cauchy functional equation $R(xy) = R(x)R(y)$. Thus, $R(x) = x^k$ for some $k > 0$. In this case, $\mathbf{A}$ is homogeneous of order k. □

3.1. A characterization of homogeneous n-ary aggregation functions

By means of any n-ary aggregation function $\mathbf{A}$, for any $k > 0$, a new n-ary aggregation function $\mathbf{A}^k$ can be constructed by $\mathbf{A}^k(\mathbf{x}) = (\mathbf{A}(\mathbf{x}))^k$ for any $\mathbf{x} \in [0,1]^n$. Before characterizing homogeneous n-ary aggregation functions, we present a method to construct homogeneous aggregation functions from given ones and by this result, we can reduce the study of aggregation functions that homogeneous of order k to that of positively homogeneous aggregation functions.

The following corollary follows from the definition of homogeneous functions:

Corollary 3.2. An n-ary aggregation function $\mathbf{A}$ is homogeneous of order k if and only if $\mathbf{A}^{\frac{1}{k}}$ is positively homogeneous.

Finally, we present a characterization of n-ary aggregation functions that are homogeneous of order k in terms of positively homogeneous aggregation functions.

Theorem 3.3. An n-ary aggregation function $\mathbf{A}$ is homogeneous of order k if and only if there exist a positively homogeneous n-ary aggregation function $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^k$.

Proof. Suppose that $\mathbf{A}$ is homogeneous of order k. Consider $\mathbf{B} = \mathbf{A}^{\frac{1}{k}}$. Clearly, $\mathbf{A} = \mathbf{B}^k$. From Corollary 3.2, it follows that $\mathbf{B}$ is positively homogeneous.

Conversely, suppose that there exists an n-ary aggregation function $\mathbf{B}$ such that $\mathbf{B}$ is positively homogeneous and $\mathbf{A} = \mathbf{B}^k$. Then $\mathbf{A}^{\frac{1}{k}} = \mathbf{B}$ is positively homogeneous, which, together with Corollary 3.2, yields that $\mathbf{A}$ is homogeneous of order k. □

3.2. A characterization of quasi-homogeneous n-ary aggregation functions

This section is devoted to characterizing quasi-homogeneous n-ary aggregation functions. If an n-ary aggregation function $\mathbf{A}$ is (φ, f)-quasi-homogeneous, then $\varphi(1) \neq 0$, otherwise

$$\mathbf{A}(n \cdot \lambda) = \varphi^{-1}(f(\lambda)\varphi(\mathbf{A}(\mathbf{1}))) = \varphi^{-1}(0) = 1 \quad \text{for each } \lambda \in [0,1],$$

it is impossible. Let $\widetilde{\varphi}(x) = \varphi(x)/\varphi(1)$. Direct checking verifies that $\widetilde{\varphi}(1) = 1$ and that $\mathbf{A}$ is (φ, f)-quasi-homogeneous if and only if $\mathbf{A}$ is $(\widetilde{\varphi}, f)$-quasi-homogeneous. In the following, we presuppose that $\varphi(1) = 1$.

Proposition 3.4. If an n-ary aggregation function $\mathbf{A}$ is (φ, f)-quasi-homogeneous, then

(i) its diagonal section $\delta(x) = \mathbf{A}(n \cdot x)$ is an increasing bijection.

(ii) $f(x) = x^c$ and $\varphi(x) = (\delta^{-1}(x))^c$ for some arbitrarily chosen $c > 0$.

(iii) $\delta^{-1} \circ \mathbf{A}$ is positively homogeneous.

Proof. Set $x_i = 1$ ($i \in [n]$) in (3) to get $\varphi(\delta(\lambda)) = f(\lambda)$ for every $\lambda \in [0,1]$ and further that f is monotonic. Putting $x_i = x$ ($i \in [n]$) in (3), we have $\delta(\lambda x) = \varphi^{-1}(f(\lambda)\varphi(\delta(x)))$, or equivalently, $f(\lambda x) = f(\lambda)f(x)$ for any $\lambda, x \in [0,1]$. Thus, f satisfies the multiplicative Cauchy equation, and hence $f(\lambda) = \lambda^c$ for every $\lambda \in [0,1]$ with $c > 0$ (see [1]). As a consequence, $\varphi(\delta(\lambda)) = \lambda^c$, whence δ must be an increasing bijection and further $\varphi(x) = (\delta^{-1}(x))^c$. From $f(\lambda) = \lambda^c$ and $\varphi(x) = (\delta^{-1}(x))^c$, (3) can be rewritten as $\delta^{-1} \circ \mathbf{A}(\lambda \mathbf{x}) = \lambda\delta^{-1} \circ \mathbf{A}(\mathbf{x})$, i.e., $\delta^{-1} \circ \mathbf{A}$ is positively homogeneous. □

For brevity, in the following take δ to mean the diagonal of the n-ary aggregation function $\mathbf{A}$, i.e., $\delta(x) = \mathbf{A}(n \cdot x)$ for each $x \in [0,1]$. For any idempotent aggregation function $\mathbf{A}$, $\delta = id$, whence the following corollary follows:

Corollary 3.5. An n-ary idempotent aggregation function is quasi-homogeneous if and only if it is positively homogeneous.

Theorem 3.6. An n-ary aggregation function $\mathbf{A}$ is quasi-homogeneous if and only if its diagonal section δ is an increasing bijection and $\delta^{-1} \circ \mathbf{A}$ is positively homogeneous. In this case, $\mathbf{A}$ is (f, φ)-quasi-homogeneous with $f(x) = x^c$ and $\varphi(x) = (\delta^{-1}(x))^c$ for some arbitrarily chosen $c > 0$.

Proof. The necessity is guaranteed from Proposition 3.4.

Sufficiency: Suppose that δ is an increasing bijection and $\delta^{-1} \circ \mathbf{A}$ is positively homogeneous. Consider $f(x) = x^c$ and $\varphi(x) = (\delta^{-1}(x))^c$ (or equivalently, $\delta(x) = \varphi^{-1}(x^c)$) for some arbitrarily chosen $c > 0$. Since $\delta^{-1} \circ \mathbf{A}$ is positively homogeneous, we then have that for any $\lambda \in [0,1]$ and any $\mathbf{x} \in [0,1]^n$, $\delta^{-1} \circ \mathbf{A}(\lambda \mathbf{x}) = \lambda\delta^{-1} \circ \mathbf{A}(\mathbf{x})$. Thus,

$$\mathbf{A}(\lambda \mathbf{x}) = \delta(\lambda\delta^{-1} \circ \mathbf{A}(\mathbf{x})) = \varphi^{-1}(\lambda^c\varphi(\mathbf{A}(\mathbf{x}))) = \varphi^{-1}(f(\lambda)\varphi(\mathbf{A}(\mathbf{x})))$$

i.e., $\mathbf{A}$ is (f, φ)-quasi-homogeneous. □

By virtue of Theorem 3.6, we can see that the study of quasi-homogeneous n-ary aggregation functions can reduce to that of positively homogeneous aggregation functions. By Theorems 3.3 and 3.6, the following theorem follows.

Theorem 3.7. An n-ary aggregation function $\mathbf{A}$ is quasi-homogeneous if and only if its diagonal section δ is an increasing bijection and there exists a positively homogeneous n-ary aggregation function $\mathbf{B}$ such that $\mathbf{A} = \delta \circ \mathbf{B}$.

Example 3.8. Let $\delta : [0,1] \to [0,1]$ be the increasing bijection given by $\delta(x) = \max(x/3, 3x-2)$. By applying Theorem 3.7 to δ and $(\mathbf{A}_p)_{p\in]-\infty,\infty[}$ in Example 2.2, we obtain a family of one-parametric quasi-homogeneous aggregation functions $(\mathbf{B}_p)_{p\in]-\infty,\infty[}$ given by

$$\mathbf{B}_p(x_1,\ldots,x_n) = \max\left(\frac{1}{3}\left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}}, 3\left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} - 2\right)$$

for $p \neq 0$ and

$$\mathbf{B}_0(x_1,\ldots,x_n) = \max\left(\frac{1}{3}\prod_{i=1}^{n} x_i^{w_i}, 3\prod_{i=1}^{n} x_i^{w_i} - 2\right).$$

Moreover, the diagonal section of $(\mathbf{B}_p)_{p\in]-\infty,\infty[}$ is δ for any $p \in]-\infty,\infty[$.

Example 3.9. Fix $k \in [0,1[$ and α such that $\max(0, 2k-1) \leq \alpha \leq k$. Let $\delta_{k,\alpha} : [0,1] \to [0,1]$ be the increasing bijection given by

$$\delta_{k,\alpha}(x) = \begin{cases} \frac{\alpha x}{k} & \text{if } x \leq k, \\ \frac{\alpha-1}{k-1}x + \frac{k-\alpha}{k-1} & \text{otherwise.} \end{cases}$$

By applying Theorem 3.7 to $\delta_{k,\alpha}$ and $(\mathbf{A}_p)_{p\in]-\infty,\infty[}$ in Example 2.2, we obtain a family of three-parametric quasi-homogeneous aggregation functions $\mathbf{A}_{p,k,\alpha}$ given by

$$\mathbf{A}_{p,k,\alpha}(x_1,\ldots,x_n) = \begin{cases} \frac{\alpha}{k}\left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} & \text{if } \left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} \leq k, \\ \frac{\alpha-1}{k-1}\left(\sum_{i=1}^{n} w_i x_i^p\right)^{\frac{1}{p}} + \frac{k-\alpha}{k-1} & \text{otherwise} \end{cases}$$

for $p \neq 0$ and

$$\mathbf{A}_{0,k,\alpha}(x_1,\ldots,x_n) = \begin{cases} \frac{\alpha}{k}\prod_{i=1}^{n} x_i^{w_i} & \text{if } \prod_{i=1}^{n} x_i^{w_i} \leq k, \\ \frac{\alpha-1}{k-1}\prod_{i=1}^{n} x_i^{w_i} + \frac{k-\alpha}{k-1} & \text{otherwise.} \end{cases}$$

4. CONSTRUCTION METHODS

In this section, we introduce several composition construction methods to generate (quasi-)homogeneous aggregation functions and illustrate these construction methods by several examples.

Proposition 4.1. Let the m-ary aggregation function $\mathbf{A}$ be homogeneous of order k, let the system $(E_1,\ldots,E_m)$ be non-empty subsets of $[n]$ with a fixed position of its members and let aggregation functions $\mathbf{B}_i$ related to dimension $n_i = card\, E_i$ be homogeneous of order r. Then the aggregation function $\mathbf{C} : [0,1]^n \to [0,1]$ given by (6) is homogeneous of order kr.

Proof. For any $\lambda \in [0,1]$ and any $\mathbf{x} \in [0,1]^n$, it holds

$$\begin{aligned}\mathbf{C}(\lambda\mathbf{x}) &= \mathbf{A}(\mathbf{B}_1(\lambda\mathbf{x}_{E_1}),\ldots,\mathbf{B}_m(\lambda\mathbf{x}_{E_m}))\\ &= \mathbf{A}(\lambda^r\mathbf{B}_1(\mathbf{x}_{E_1}),\ldots,\lambda^r\mathbf{B}_m(\mathbf{x}_{E_m}))\\ &= \lambda^{kr}\mathbf{A}(\mathbf{B}_1(\mathbf{x}_{E_1}),\ldots,\mathbf{B}_m(\mathbf{x}_{E_m}))\\ &= \lambda^{kr}\mathbf{C}(\mathbf{x}),\end{aligned}$$

i. e., $\mathbf{C}$ is homogeneous of order kr. □

From Proposition 4.1, the next two corollaries follows.

Corollary 4.2. Let the n-ary aggregation function $\mathbf{A}$ be homogeneous of order k and let n_i-ary aggregation functions $\mathbf{B}_i$ be positively homogeneous, $i = 1,..,m$. If $\sum_{i=1}^m n_i = n$, then the aggregation function $\mathbf{C} : [0,1]^n \to [0,1]$ given by (7) is homogeneous of order k.

Corollary 4.3. Let $m,n \in \mathbb{N}\backslash\{1\}$. Let $\mathbf{A} : [0,1]^m \to [0,1]$ and $\mathbf{B}_1,\ldots,\mathbf{B}_m : [0,1]^n \to [0,1]$ be aggregation functions. If $\mathbf{A}$ be homogeneous of order k, and $\mathbf{B}_i$ is positively homogeneous, $i = 1,..,m$, then the n-ary aggregation function $\mathbf{C}$ given by (8) is homogeneous of order k.

Example 4.4. Let $1 \le k \le 2$ and $\theta = 2-k$. The Cuadras-Augé copula

$$\mathbf{C}_\theta(x_1,x_2) = (x_1 \wedge x_2)^\theta (x_1x_2)^{1-\theta}$$

is homogeneous of order k (see Theorem 3.4.2 in [9] for details). Clearly, the binary geometric mean function $\mathbf{GM} : [0,1]^2 \to [0,1]$ given by $\mathbf{GM}(x_1,x_2) = \sqrt{x_1x_2}$ and the binary arithmetic mean function $\mathbf{AM} : [0,1]^2 \to [0,1]$ given by $\mathbf{AM}(x_1,x_2) = \frac{x_1+x_2}{2}$ are positively homogeneous aggregation functions.

(i) Applying Corollary 4.2 to $\mathbf{C}_\theta$, $\mathbf{GM}$ and $\mathbf{AM}$, we know that the aggregation function $\mathbf{A}_1 : [0,1]^4 \to [0,1]$ given by

$$\begin{aligned}\mathbf{A}_1(x_1,x_2,x_3,x_4) &= \mathbf{C}_\theta(\mathbf{GM}(x_1,x_2),\mathbf{AM}(x_3,x_4))\\ &= \left(\sqrt{x_1x_2} \bigwedge \frac{x_3+x_4}{2}\right)^\theta \left(\frac{\sqrt{x_1x_2}(x_3+x_4)}{2}\right)^{1-\theta}\end{aligned}$$

is homogeneous of order k.

(ii) Applying Corollary 4.2 to $\mathbf{C}_\theta$, $\mathbf{GM}$ and $\mathbf{AM}$, we know that the aggregation function $\mathbf{A}_2 : [0,1]^2 \to [0,1]$ given by

$$\begin{aligned}\mathbf{A}_2(x_1,x_2) &= \mathbf{C}_\theta(\mathbf{GM}(x_1,x_2),\mathbf{AM}(x_1,x_2))\\ &= \sqrt{x_1x_2}\left(\frac{x_1+x_2}{2}\right)^{1-\theta}\end{aligned}$$

is homogeneous of order k.

Example 4.5. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ be any weight vector such that $\sum_{i=1}^n w_i = 1$. Let $p_i \in]-\infty, \infty[\setminus\{0\}$, $i = 1, 2, \ldots, k$. Clearly, the product

$$\sqcap(\mathbf{x}) = \prod_{i=1}^{k} x_i$$

is homogeneous of order k. Consider power-root operators $(\mathbf{A}_p)_{p\in]-\infty,\infty[}$ in Example 2.2. From Corollary 4.3, we know that the n-ary aggregation function $\mathbf{C}$ given by

$$\mathbf{C}(\mathbf{x}) \quad = \quad \prod_{i=1}^{k} \mathbf{A}_{p_i}(\mathbf{x}) = \prod_{j=1}^{k} \left(\sum_{i=1}^{n} w_i x_i^{p_j} \right)^{\frac{1}{p_j}}$$

is homogeneous of order k.

Finally, we present a method to generate quasi-homogeneous aggregation functions.

Proposition 4.6. Let $\delta : [0,1] \to [0,1]$ be an increasing bijection. Let the m-ary aggregation function $\mathbf{A}$ be positively homogeneous, let the system $(E_1, \ldots, E_m)$ be non-empty subsets of $[n]$ with a fixed position of its members and let aggregation functions B_i related to dimension $n_i = card\, E_i$ be quasi-homogeneous aggregation functions. If δ_i is the diagonal section of $\mathbf{B}_i$, $i = 1, \ldots, m$, then the aggregation function $\mathbf{C} : [0,1]^n \to [0,1]$ given by

$$\mathbf{C}(\mathbf{x}) = \delta \circ \mathbf{A}(\delta_1^{-1} \circ \mathbf{B}_1(\mathbf{x}_{E_1}), \cdots, \delta_m^{-1} \circ \mathbf{B}_m(\mathbf{x}_{E_1}))$$

is a quasi-homogeneous aggregation function.

Proof. Clearly, $\mathbf{C}$ is an n-ary aggregation function. Since the n_i-ary aggregation function $\mathbf{B}_i$ is quasi-homogeneous and its diagonal section is δ_i, $i = 1, \ldots, m$, we the have that $\delta_i^{-1} \circ \mathbf{B}_i$ is positively homogeneous, which, together with Proposition 4.1, gives that the aggregation function $\widetilde{\mathbf{C}} : [0,1]^n \to [0,1]$ given by

$$\widetilde{\mathbf{C}}(\mathbf{x}) = \mathbf{A}(\delta_1^{-1} \circ \mathbf{B}_1(\mathbf{x}_{E_1}), \cdots, \delta_m^{-1} \circ \mathbf{B}_m(\mathbf{x}_{E_1}))$$

is positively homogeneous. This proposition follows from Theorems 2.1 and 3.7. □

By the proposition above, we have the following corollaries:

Corollary 4.7. Let $\delta : [0,1] \to [0,1]$ be an increasing bijection. Let the aggregation function $\mathbf{A} : [0,1]^m \to [0,1]$ be positively homogeneous and let n_i-ary aggregation functions $\mathbf{B}_i$ be quasi-homogeneous aggregation functions, $i = 1, .., m$. If $\sum_{i=1}^m n_i = n$ and δ_i is the diagonal section of $\mathbf{B}_i$, $i = 1, \ldots, m$, then the aggregation function $\mathbf{C} : [0,1]^n \to [0,1]$ given by

$$\mathbf{C}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \delta \circ \mathbf{A}(\delta_1^{-1} \circ \mathbf{B}_1(\mathbf{x}_1), \cdots, \delta_m^{-1} \circ \mathbf{B}_m(\mathbf{x}_m))$$

with $\mathbf{x}_i \in [0,1]^{n_i}$, $i = 1, \ldots, m$, is a quasi-homogeneous aggregation function.

Corollary 4.8. Let $m, n \in \mathbb{N}\backslash\{1\}$ and let $\delta : [0,1] \to [0,1]$ be an increasing bijection. Let the aggregation function $\mathbf{A} : [0,1]^m \to [0,1]$ be positively homogeneous, and let $\mathbf{B}_1, \ldots, \mathbf{B}_m : [0,1]^n \to [0,1]$ be quasi-homogeneous aggregation functions. If δ_i is the diagonal section of $\mathbf{B}_i$, $i = 1, \ldots, m$, then the function $\mathbf{C} : [0,1]^n \to [0,1]$ defined by

$$\mathbf{C}(\mathbf{x}) = \delta \circ \mathbf{A}(\delta_1^{-1} \circ \mathbf{B}_1(\mathbf{x}), \cdots, \delta_m^{-1} \circ \mathbf{B}_m(\mathbf{x}))$$

is a quasi-homogeneous aggregation function.

Example 4.9. Fix $p \in]1, \infty[$ and $\beta \in] - \infty, 0[$. Let $\delta : [0,1] \to [0,1]$ be the increasing bijection defined by

$$\delta(x) = \frac{px}{1 + (p-1)x}.$$

The t-norm $\mathbf{T} : [0,1]^2 \to [0,1]$ given by

$$\mathbf{T}(x_1, x_2) = (x_1^\beta + x_2^\beta - 1)^{\frac{1}{\beta}}$$

is a quasi-homogeneous 2-ary aggregation function, of which the diagonal section is $\delta_1(x) = (2x^\beta - 1)^{\frac{1}{\beta}}$ (see Theorem 2.3 in [4]) for details; we also mention that the considered t-norm $\mathbf{T}$ is a strict Schweizer-Sklar t-norm, also called strict Clayton copula (see Remark 4.4 in [7]) and the product $\sqcap : [0,1]^n \to [0,1]$ is a quasi-homogeneous n-ary aggregation function (further an aggregation function that is homogeneous of order n). Applying corollary 4.7 to the binary geometric mean function $\mathbf{GM}$, $\mathbf{T}$ and $\sqcap$, we obtain a family of two-parametric quasi-homogeneous $n+2$-ary aggregation functions $\mathbf{C}_{p,\beta}$ given by

$$\mathbf{C}_{p,\beta}(x_1, x_2, x_3, \ldots, x_{n+2}) = \frac{p \prod\limits_{i=3}^{n+2} x_i^{\frac{1}{2n}} \left(\frac{(x_1^\beta + x_2^\beta - 1)^{\frac{1}{\beta}} + 1}{2} \right)^{\frac{1}{2\beta}}}{1 + (p-1) \prod\limits_{i=3}^{n+2} x_i^{\frac{1}{2n}} \left(\frac{(x_1^\beta + x_2^\beta - 1)^{\frac{1}{\beta}} + 1}{2} \right)^{\frac{1}{2\beta}}}.$$

Example 4.10. Fix $c \in]0, \infty[$, $p \in]1, \infty[$ and $\beta \in] - \infty, 0[$. Let $\delta : [0,1] \to [0,1]$ be the increasing bijection defined by $\delta(x) = x^c$.

The aggregation function $\mathbf{A} : [0,1]^2 \to [0,1]$ given by

$$\mathbf{A}(x_1, x_2) = \max(x_1, x_2) \exp\left(1 - \sqrt{1 + \log\left|\frac{x_2}{x_1}\right|}\right)$$

is positively homogeneous. Applying corollary 4.8 to $\mathbf{A}$, the binary geometric mean function $\mathbf{GM}$ and the binary arithmetic mean function $\mathbf{AM}$, we obtain a quasi-homogeneous 2-ary aggregation functions $\mathbf{C}$ given by

$$\mathbf{C}(x_1, x_2) = \left(\frac{x_1 + x_2}{2}\right)^c \exp\left(c - c\sqrt{1 + \log\left|\frac{x_1 + x_2}{2\sqrt{x_1 x_2}}\right|}\right).$$

5. CONCLUDING REMARKS AND FUTURE WORK

In this paper, all (quasi-)homogeneous n-ary aggregation functions were characterized in terms of positively homogeneous n-ary aggregation functions (see Theorems 3.3 and 3.7) and several methods to construct (quasi-)homogeneous n-ary aggregation functions were presented. We have discussed homogeneity problems from a general point of view, not focusing on some particular class of aggregation functions. Observe, for example, that in the class of integrals, all decomposition integrals [5] or finite super-decomposition integrals [11] are positively homogeneous. In the literature, there exists an extended quasi-homogeneity, called pseudo-homogeneity. A function $O : [0,1]^2 \to [0,1]$ is *pseudo-homogeneous* [14] if there exists some $G : [0,1]^2 \to [0,1]$ such that, for any $x, y, \lambda \in [0,1]$,

$$O(\lambda x, \lambda y) = G(\lambda, O(x, y)). \tag{9}$$

Clearly, the quasi-homogeneity implies the pseudo-homogeneity. Further, for any aggregation function O such that its diagonal section δ is a bijection, the quasi-homogeneity and pseudo-homogeneity are equivalent. Indeed, putting $x = y$ in (9), we have $G(\lambda, x) = \delta(\lambda\delta^{-1}(x))$, and further O is quasi-homogeneous with $f = \mathbf{id}$ and $\varphi = \delta^{-1}$. However, when δ is not a bijection, pseudo-homogeneity is much more complex. Another direction of the future research in this domain should be devoted to the study of n-ary aggregation functions which are pseudo-homogeneous.

In decision making, homogeneous aggregation functions are needed whenever one fixes the beginning of our scale (i. e., "zero" is fixed) but letting free the unit [13]; and in image processing, for many techniques in image segmentation, e. g. thresholding, edge detection, or enhancement, the homogeneity plays a key role (for more details see [8] and the references therein). Therefore, this work will be beneficial to decision making and image processing.

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