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PACKING FOUR COPIES OF A TREE INTO A COMPLETE BIPARTITE GRAPH

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Abstract. In considering packing three copies of a tree into a complete bipartite graph, H. Wang (2009) gives a conjecture: For each tree T of order n and each integer $k \geqslant 2$, there is a k-packing of T in a complete bipartite graph B_{n+k-1} whose order is $n+k-1$. We prove the conjecture is true for $k=4$.

Keywords: packing; bipartite packing; embedding

MSC 2020: 05C05, 05C70

1. Introduction

We discuss only finite simple graphs and use standard terminology and notation from [6] except as indicated. For any graph G we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. A forest is a graph without cycles. A tree is a connected forest. We use B_n (or $K_{t,n-t}$) to represent a complete bipartite graph of order n. A bipartite graph G admits (a,b)-bipartition if G has a bipartition (X,Y) such that $|X|=a$ and $|Y|=b$. Note that up to isomorphism, B_n ($K_{t,n-t}$) is not uniquely defined for $n \geqslant 4$ and $t \geqslant 1$.

An isomorphism from a simple graph G to a simple graph H is a bijection $f\colon V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say that G is isomorphic to H, written as $G \cong H$. By an embedding σ of a bipartite graph G in B_n, we mean that σ is an injection from $V(G)$ into $V(B_n)$ such that $\sigma(V_0) \subseteq X_0$ and $\sigma(V_1) \subseteq X_1$, where (V_0,V_1) and (X_0,X_1) are the given bipartitions of G and B_n, respectively. A k-packing of T in the graph G is a partition of edges of subgraph of the

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graph G such that each element of the partition induces a subgraph isomorphic to T, where k is the number of the elements in the partition. (Later, denote the subgraph of G by G'. Let σ_i be a bijection satisfying $\sigma_i(G') \simeq T$ for $1 \leqslant i \leqslant k$.) There have been some results found on k-packing of T in G for various k, T and G. When T is a path and G is a complete bipartite graph, some results can be found in [7], [8]. Hobbs, Bourgeois and Kasiraj in [2] proved that any two trees of order m and n with $m < n$ can be packed into a complete bipartite graph $K_{n-1,\lceil\frac{1}{2}n\rceil}$. It is proved in [1], [5] that for any disconnected forest F of order n, there is a 2-packing of F in a complete bipartite graph B_n. Wang in [3] showed that any two forests of order n admitting the same (a, b)-bipartition can be packed into a complete bipartite graph of order at most $n+1$. Wang in [4] also proved that for any tree T of order n a 3-packing of T in some B_{n+2} can be found. Wang gives a conjecture in paper (see [4]): For any tree T of order n and each integer $k \geqslant 2$, there is a k-packing of T in some B_{n+k-1}. The conjecture is true for $k = 2$ and $k = 3$ by the results in [3], [4], [5]. We will show it is true for $k = 4$.

Theorem 1.1. *For each tree T of order n, there is a 4-packing of T in some B_{n+3}.*

Its proof can be found in Section 3 while in Section 2, some lemmas, which are important for the proof of the main theorem, are given.

2. Preliminary

We first give some terminology and notation. Given a bipartite graph G, we say that two vertices of G are *strongly independent* if they are not adjacent and they do not have any common neighbor either. A *node* of G is a vertex of G that is adjacent to an endvertex of G. A *supernode* of G is a vertex x of G such that, with one exception, every neighbor of x is an endvertex of G. If G is a tree but not a star, we readily see that G has at least two distinct supernodes by observing a longest path of G. If (X, Y) is the given bipartition of G, then any subgraph H of G has $(X \cap V(H), Y \cap V(H))$ as its given bipartition. For a 4-packing (b, g, r, s) of G in B_n, we say that a vertex x is 4-*placed* if $b(x)$, $g(x)$, $r(x)$ and $s(x)$ are distinct. A linear forest is a forest such that each of its components is a path. By adopting the method in [4], we give Lemmas 2.1 and 2.2, which are important for the proof of the main theorem. Let $P = x_i x_{i+1} \ldots x_{i+l}$ denote a path of length l with vertex set $V(P) = \{x_{i+t}\colon\ 0 \leqslant t \leqslant l\}$ and edge set $E(P) = \{x_{i+t-1}x_{i+t}\colon\ 1 \leqslant t \leqslant l\}$. Let $K_{s,t}(V_s, V_t)$ denote a bipartite graph with vertex set $V(K_{s,t}) = V_s \cup V_t$ and edge set $E(K_{s,t}) = \{ab\colon\ a \in V_s,\ b \in V_t\}$.

Lemma 2.1. *Let x, y, z and p be four strongly independent endvertices in the same partite of a tree T. If there is a 4-packing of $T - x - y - z - p$ in B_n, then there is a 4-packing of T in B_{n+4}.*

P r o o f. Let $\{u, v, w, q\} \subseteq V(T)$ be such that $\{xu, yv, zw, pq\} \subseteq E(T)$. Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be a 4-packing of $T - x - y - z - p$ in B_n. For $i \in \{1, 2, 3, 4\}$, let $A_i = \{\sigma_i(u), \sigma_i(v), \sigma_i(w), \sigma_i(q)\}$. Obviously $|A_i| = 4$. Note that $\bigcup_{i=1}^{4} A_i$ is contained in one partite of B_n. Let $V(B_{n+4}) = V(B_n) \cup \{x, y, z, p\}$ such that $\{x, y, z, p\}$ is in the partite that does not contain $\bigcup_{i=1}^{4} A_i$. For each $i \in \{1, 2, 3, 4\}$ we add a set E_i consisting of four independent edges between $\{x, y, z, p\}$ and A_i to $\sigma_i(T - x - y - z - p)$ to obtain a copy of T in B_{n+4}. Note that $\left|\bigcup_{i=1}^{4} E_i\right| = 16$ and $4 \leqslant \left|\bigcup_{i=1}^{4} A_i\right| \leqslant 20$. The edges in $\bigcup_{i=1}^{4} E_i$ comes from the complete bipartite graph M with partite sets $\{x, y, z, p\}$ and $\bigcup_{i=1}^{4} A_i$. Obviously $|E(M)| \geqslant 16$. It is easy to choose E_i $(1 \leqslant i \leqslant 4)$ satisfying $E_i \cap E_j = \varphi$ for $1 \leqslant i < j \leqslant 4$. Thus, we extend each σ_i to an embedding of T in B_{n+4} such that $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ becomes a 4-packing of T in B_{n+4}. □

Lemma 2.2. *Let H be a subgraph of a tree T such that each vertex of $T - V(H)$ is an endvertex of T. If there is a 4-packing of H in B_n such that each vertex x of H with $xy \in E(T)$ for some $y \in V(T) - V(H)$ is 4-placed, then there is a 4-packing of T in B_{n+m}, where $m = |V(T)| - |V(H)|$.*

P r o o f. Let $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ be a 4-packing of H in B_n so that if $xy \in E(T)$ with $y \in V(T) - V(H)$, then x is 4-placed. Note that $\sigma_1(x)$, $\sigma_2(x)$, $\sigma_3(x)$ and $\sigma_4(x)$ are in the same partite for all $x \in V(H)$. We obtain B_{n+m} by adding each $y \in V(T) - V(H)$ to B_n so that if $xy \in E(T)$, then y and $\sigma_1(x)$ are in the opposite partites. Then for each $i \in \{1, 2, 3, 4\}$ we extend σ_i to an embedding of T in B_{n+m} so that $\sigma_i(y) = y$ for each $y \in V(T) - V(H)$. Then $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ is a 4-packing of T in B_{n+m}. □

We also need the following lemmas in order to prove our main theorem.

Lemma 2.3. *The following two statements hold:*

(1) *If P is a linear forest of order $2k$ with $k \geqslant 8$, then there is a 4-packing of P in $K_{k,k}$ such that each vertex of P is 4-placed.*

(2) *If P is a path of order $2k$ with $k \in \{5, 6, 7\}$, then there is a 4-packing of P in $K_{k+1,k+2}$ such that each vertex of P is 4-placed.*

P r o o f. To prove (1), without loss of generality, suppose $P = x_1y_1x_2y_2\ldots x_ky_k$ is a path with $(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_k\})$ as its bipartition. The subscript modulo k is in $\{1, 2, \ldots, k\}$. Define a 4-packing (b, g, r, s) of P in $K_{k,k}(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_k\})$ as follows. For all $z \in V(P)$, let $b(z) = z$; for $i = \{1, 2, \ldots, k\}$, let $g(x_i) = x_{i+1}$ and $g(y_i) = y_{i+3}$; $r(x_i) = x_{i+2}$ and $r(y_i) = y_{i+6}$; $s(x_i) = x_{i+3}$ and $s(y_i) = y_{i+1}$.

To prove (2), let $(\{x_1,\dots,x_k,w_1\},\{y_1,\dots,y_k,w_2,w_3\})$ be the bipartition of $K_{k+1,k+2}$. Say $P=x_1y_1x_2y_2\dots x_ky_k$. In $K_{\{x_1,\dots,x_k,w_1\},\{y_1,\dots,y_k,w_2,w_3\}}$, i.e., $K_{k+1,k+2}$, we define four embeddings (b,g,r,s) of P with b being identity embedding as follows.

If $k=5$, define g, r and s such that $g(P)=x_2y_3x_5w_3w_1w_2x_1y_5x_3y_1$, $r(P)=x_3y_4w_1y_3x_1w_3x_2w_2x_4y_2$, $s(P)=x_4w_3x_3w_2x_5y_1w_1y_2x_1y_4$, with $g(x_1)=x_2$, $r(x_1)=x_3$ and $s(x_1)=x_4$.

If $k=6$, define g, r and s such that $g(P)=x_2y_4x_6w_2w_1y_6x_5w_3x_4y_2x_1y_5$, $r(P)=x_3y_5x_4y_6x_2w_3x_6y_1w_1y_3x_5w_2$, $s(P)=x_6y_2x_5y_1x_3w_2x_2y_3x_1y_4w_2w_3$, with $g(x_1)=x_2$, $r(x_1)=x_3$ and $s(x_1)=x_6$.

If $k=7$, define g, r and s such that $g(P)=x_2y_3x_5y_6w_1w_3x_1y_5x_7w_2x_3y_1x_4y_2$, $r(P)=x_3y_4x_7y_1x_5w_2x_2y_6x_1y_7w_1y_2x_6w_3$, $s(P)=x_5y_7x_3y_5x_2y_4x_6w_2w_1w_3x_7y_3x_1y_6$, with $g(x_1)=x_2$, $r(x_1)=x_3$ and $s(x_1)=x_5$. □

Lemma 2.4. *Let P be a path of order n from x to y. The following three statements hold:*

(1) *If $n\in\{4,6,8\}$, there is a 4-packing (b,g,r,s) of P in B_{n+3} such that z is 4-placed for each $z\in V(P)-\{y\}$.*

(2) *If $n=5$, there is a 4-packing (b,g,r,s) of P in B_{n+3} such that z is 4-placed for each $z\in V(P)-\{x,y\}$. Furthermore,*

$$\{b(x),g(x),r(x),s(x)\}\cap\{b(y),g(y),r(y),s(y)\}=\emptyset.$$

(3) *If $n\in\{7,9\}$, there is a 4-packing (b,g,r,s) of P in B_{n+3} such that z is 4-placed for each $z\in V(P)$.*

P r o o f. To prove (1), when $n=4$, let $P=x_1x_2x_3x_4$. Set $V_0=\{x_1,x_3\}$ and $V_1=\{x_2,x_4\}$. Let (V_0,V_1) be the partition of P and $(V_0\cup\{x_5,x_7\},V_1\cup\{x_6\})$ be the bipartition of B_7. Define the required 4-packing (b,g,r,s) of P in B_7 with b being identity embedding as follows: $g(P)=x_3x_6x_1x_4$, $r(P)=x_7x_4x_5x_6$ and $s(P)=x_5x_2x_7x_6$ with $g(x_1)=x_3$, $r(x_1)=x_7$ and $s(x_1)=x_5$.

When $n=6$, let $P=x_1x_2x_3x_4x_5x_6$. Set $V_0=\{x_1,x_3,x_5\}$ and $V_1=\{x_2,x_4,x_6\}$. Let (V_0,V_1) be the partition of P and $(V_0\cup\{x_7\},V_1\cup\{x_8,x_9\})$ be the bipartition of B_9. Define the required 4-packing (b,g,r,s) of P in B_9 with b being identity embedding as follows: $g(P)=x_7x_4x_1x_8x_3x_6$, $r(P)=x_3x_9x_5x_2x_7x_6$ and $s(P)=x_5x_8x_7x_9x_1x_6$ with $g(x_1)=x_7$, $r(x_1)=x_3$ and $s(x_1)=x_5$.

When $n=8$, let $P=x_1x_2x_3x_4x_5x_6x_7x_8$, $V_0=\{x_1,x_3,x_5,x_7\}$ and $V_1=\{x_2,x_4,x_6,x_8\}$. Let (V_0,V_1) be the partition of P and $(V_0\cup\{x_9\},V_1\cup\{x_{10},x_{11}\})$ be the bipartition of B_{11}. Define the required 4-packing (b,g,r,s) of P in B_{11} with b being identity embedding as follows: $g(P)=x_1x_{11}x_5x_{10}x_7x_2x_9x_4$, $r(P)=x_1x_{10}x_9x_6x_3x_8x_5x_2$ and $s(P)=x_7x_4x_1x_8x_9x_{11}x_3x_{10}$ with $g(x_1)=x_1$, $r(x_1)=x_1$ and $s(x_1)=x_7$.

To prove (2), let $P = x_1x_2x_3x_4x_5$. Set $V_0 = \{x_1, x_3, x_5\}$ and $V_1 = \{x_2, x_4\}$. Let (V_0, V_1) be the partition of P and $(V_0 \cup \{x_7\}, V_1 \cup \{x_6, x_8\})$ be the bipartition of B_8. Define the required 4-packing (b, g, r, s) of P in B_8 with b being identity embedding as follows: $g(P) = x_7x_6x_5x_8x_3$, $r(P) = x_1x_4x_7x_2x_5$ and $s(P) = x_7x_8x_1x_6x_3$ with $g(x_1) = x_7$, $r(x_1) = x_1$ and $s(x_1) = x_7$.

To prove (3), when $n = 7$, let $P = x_1x_2x_3x_4x_5x_6x_7$. Set $V_0 = \{x_1, x_3, x_5, x_7\}$ and $V_1 = \{x_2, x_4, x_6\}$. Let (V_0, V_1) be the partition of P and $(V_0 \cup \{x_8, x_9\}, V_1 \cup \{x_{10}\})$ be the bipartition of B_{10}. Define the required 4-packing (b, g, r, s) of P in B_{10} with b being identity embedding as follows: $g(P) = x_3x_6x_8x_{10}x_7x_2x_9$, $r(P) = x_7x_4x_9x_6x_1x_{10}x_3$ and $s(P) = x_9x_{10}x_5x_2x_8x_4x_1$ with $g(x_1) = x_3$, $r(x_1) = x_7$ and $s(x_1) = x_9$.

When $n = 9$, let $P = x_1x_2x_3x_4x_5x_6x_7x_8x_9$. Set $V_0 = \{x_1, x_3, x_5, x_7, x_9\}$ and $V_1 = \{x_2, x_4, x_6, x_8\}$. Let (V_0, V_1) be the partition of P and $(V_0 \cup \{x_{11}\}, V_1 \cup \{x_{10}, x_{12}\})$ be the bipartition of B_{12}. Define the required 4-packing (b, g, r, s) of P in B_{12} with b being identity embedding as follows: $g(P) = x_3x_8x_{11}x_6x_9x_{10}x_1x_4x_7$, $r(P) = x_5x_{10}x_7x_2x_{11}x_4x_9x_{12}x_1$ and $s(P) = x_7x_{12}x_5x_8x_1x_6x_3x_{10}x_{11}$ with $g(x_1) = x_3$, $r(x_1) = x_5$ and $s(x_1) = x_7$. $\square$

To state Lemma 2.5, we define graphs G_i ($1 \leqslant i \leqslant 18$ and $i \neq 8, 13$ or 17) to be the subgraphs of $K_{8,8}$ (V_0, V_1), where $V_0 = \{x_1, x_3, x_5, \ldots, x_{15}\}$ and $V_1 = \{x_2, x_4, x_6, \ldots, x_{16}\}$. Let G_8 be the graph $K_{4,6}$ (U_0, U_1), where $U_0 = \{x_1, x_3, x_5, x_7\} \subset V_0$ and $U_1 = \{x_2, x_4, x_6, x_8, x_9, x_{10}\} \subset V_1$. Let G_{13} be the graph $K_{6,7}$ (U_0, U_1), where $U_0 = \{x_1, x_3, x_5, x_7, x_9, x_{11}\} \subset V_0$ and $U_1 = \{x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{13}\} \subset V_1$, and G_{17} be the graph $K_{7,8}$ (U_0, U_1), where $U_0 = \{x_1, x_3, x_5, x_7, x_9, x_{11}, x_{13}\} \subset V_0$ and $U_1 = \{x_2, x_4, x_6, x_8, x_{10}, x_{12}, x_{14}, x_{15}\} \subset V_1$. Let

$$G_1 = x_1x_2x_3x_4x_5x_6 \cup x_3x_8x_7,$$

where $x_1x_2x_3x_4x_5x_6$ is a path of length 5 with edges x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6.

$$\begin{aligned}
G_2 &= x_1x_2x_3x_4x_5x_6x_7 \cup x_3x_8x_9,\\
G_3 &= x_1x_2x_3x_4x_5 \cup x_3x_6x_7x_8x_9x_{10}x_{11},\\
G_4 &= x_1x_2x_3x_4x_5x_6x_7 \cup x_3x_8x_9x_{10}x_{11},\\
G_5 &= x_1x_2x_3x_4x_5 \cup x_3x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13},\\
G_6 &= x_1x_2x_3x_4x_5x_6x_7 \cup x_3x_8x_9x_{10}x_{11}x_{12}x_{13},\\
G_7 &= x_1x_2x_3x_4x_5x_6x_7x_8x_9 \cup x_5x_{10}x_{11}x_{12}x_{13},\\
G_8 &= x_1x_2x_3x_4x_5 \cup x_3x_6x_7,\\
G_9 &= x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11}x_{12}x_{13} \cup x_3x_{14}x_{15},
\end{aligned}$$

$$G_{10} = x_1x_2x_3x_4x_5x_6x_7x_8x_9x_{10}x_{11} \cup x_3x_{12}x_{13}x_{14}x_{15},$$
$$G_{11} = x_1x_2x_3x_4x_5x_6x_7x_8x_9 \cup x_5x_{10}x_{11}x_{12}x_{13}x_{14}x_{15},$$
$$G_{12} = x_1x_2x_3x_4x_5x_6x_7x_8x_9 \cup x_3x_{10}x_{11}x_{12}x_{13}x_{14}x_{15},$$
$$G_{13} = x_1x_2x_3x_4x_5x_6 \cup x_9x_{10}x_3x_8x_7,$$
$$G_{14} = x_1x_2x_3x_4x_5x_6x_7 \cup x_{11}x_{10}x_3x_8x_9,$$
$$G_{15} = x_1x_2x_3x_4x_5x_6x_7 \cup x_{13}x_{12}x_3x_8x_9x_{10}x_{11},$$
$$G_{16} = x_1x_2x_3x_4x_5x_6x_7x_8x_9 \cup x_{13}x_{12}x_3x_{10}x_{11},$$
$$G_{17} = x_1x_2x_3x_4x_5x_6 \cup x_7x_8x_3x_{10}x_9 \cup x_3x_{12}x_{11},$$
$$G_{18} = x_1x_2x_3x_4x_5x_6x_7 \cup x_9x_8x_3x_{10}x_{11} \cup x_3x_{12}x_{13}.$$

Lemma 2.5. *The following statements hold:*

(1) *There is a 4-packing of G_1 in B_{11} such that, except x_1, x_6 and x_7, every vertex of G_1 is 4-placed.*

(2) *There is a 4-packing of G_2 in B_{12} such that, except x_7 and x_9, every vertex of G_2 is 4-placed.*

(3) *There is a 4-packing of G_3 in B_{14} such that every vertex of G_3 is 4-placed.*

(4) *There is a 4-packing of G_4 in B_{14} such that every vertex of G_4 is 4-placed.*

(5) *There is a 4-packing of G_5 in B_{16} such that every vertex of G_5 is 4-placed.*

(6) *There is a 4-packing of G_6 in B_{16} such that every vertex of G_6 is 4-placed.*

(7) *There is a 4-packing of G_7 in B_{16} such that every vertex of G_7 is 4-placed.*

(8) *There is a 4-packing of G_8 in B_{10} such that, except x_1, x_5 and x_7, every vertex of G_8 is 4-placed.*

(9) *There is a 4-packing of G_9 in B_{18} such that every vertex of G_9 is 4-placed.*

(10) *There is a 4-packing of G_{10} in B_{18} such that every vertex of G_{10} is 4-placed.*

(11) *There is a 4-packing of G_{11} in B_{18} such that every vertex of G_{11} is 4-placed.*

(12) *There is a 4-packing of G_{12} in B_{18} such that every vertex of G_{12} is 4-placed.*

(13) *There is a 4-packing of G_{13} in B_{13} such that, except x_1, x_6 and x_9, every vertex of G_{13} is 4-placed.*

(14) *There is a 4-packing of G_{14} in B_{14} such that every vertex of G_{14} is 4-placed.*

(15) *There is a 4-packing of G_{15} in B_{16} such that every vertex of G_{15} is 4-placed.*

(16) *There is a 4-packing of G_{16} in B_{16} such that every vertex of G_{16} is 4-placed.*

(17) *There is a 4-packing of G_{17} in B_{15} such that every vertex of G_{17} is 4-placed.*

(18) *There is a 4-packing of G_{18} in B_{16} such that every vertex of G_{18} is 4-placed.*

The proof can be found in Appendix (I).

To state Lemma 2.6, we define graphs F_i $(1 \leqslant i \leqslant 8)$ to be the subgraphs of $K_{10,9}$ (V_0, V_1), where $V_0 = \{x_1, x_3, x_5, \ldots, x_{19}\}$ and $V_1 = \{x_2, x_4, x_6, \ldots, x_{18}\}$. Let

$$F_1 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_8,$$
$$F_2 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_{12}x_{11}x_8,$$
$$F_3 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_{14}x_{13}x_{12}x_{11}x_8,$$
$$F_4 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_{16}x_{15}x_{14} \cup x_8x_{11}x_{12}x_{13},$$
$$F_5 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_8x_{11}x_{12},$$
$$F_6 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_{14}x_{13}x_8x_{11}x_{12},$$
$$F_7 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_3x_{16}x_{15}x_{14}x_{13}x_8x_{11}x_{12},$$
$$F_8 = x_1x_2x_3x_4x_5 \cup x_6x_7x_8x_9x_{10} \cup x_{15}x_{14}x_{13}x_8x_{11}x_{12} \cup x_3x_{18}x_{17}x_{16}.$$

Lemma 2.6. *The following statements hold:*

(1) *There is a* 4*-packing of* F_1 *in* B_{13} *such that, except* x_{10}*, every vertex of* F_1 *is* 4*-placed.*
(2) *There is a* 4*-packing of* F_2 *in* B_{15} *such that every vertex of* F_2 *is* 4*-placed.*
(3) *There is a* 4*-packing of* F_3 *in* B_{17} *such that every vertex of* F_3 *is* 4*-placed.*
(4) *There is a* 4*-packing of* F_4 *in* B_{16} *such that, except* x_1 *and* x_5*, every vertex of* F_4 *is* 4*-placed.*
(5) *There is a* 4*-packing of* F_5 *in* B_{15} *such that every vertex of* F_5 *is* 4*-placed.*
(6) *There is a* 4*-packing of* F_6 *in* B_{17} *such that, except* x_6 *and* x_{12}*, every vertex of* F_6 *is* 4*-placed.*
(7) *There is a* 4*-packing of* F_7 *in* B_{19} *such that every vertex of* F_7 *is* 4*-placed.*
(8) *There is a* 4*-packing of* F_8 *in* B_{18} *such that every vertex of* F_8 *is* 4*-placed.*

The proof can be found in Appendix (II).

3. Proof of the main theorem

Now we are in the position to prove our main result Theorem 3.1.

Theorem 3.1. *For each tree* T *of order* n*, there is a* 4*-packing of* T *in some* B_{n+3}*.*

P r o o f. To avoid considering many classes of non-isomorphic trees with the same order n, the theorem is proved by contradiction. Let T be a tree with the smallest order such that the theorem fails for T. Say $|V(T)| = n$. By Lemma 2.1, T does not contain four strongly independent endvertices in the same partite. Thus, T contains at most six supernodes. Clearly, $n \geqslant 4$ and T is not a star. By observing a longest path, we see that T has at least two supernodes. We need to consider only the trees of order n with t supernodes ($2 \leqslant t \leqslant 6$). We divide the proof into several cases by the numbers of supernodes of T. In every case, we manage to define a subgraph H of T. Then from the 4-packing of H in B_n, we shall obtain a 4-packing of T in B_{n+3}.

Case 1: T has exactly two supernodes.

In this case, let $P = x_1x_2\dots x_t$ be a longest path. Then every vertex of $T - V(P)$ is an endvertex of T. If $t = 2k$ and $k \notin \{3,4\}$, then by Lemma 2.3 (1) and (2), there is a 4-packing of P in B_{2k+3} such that each vertex of P is 4-placed, and thus the theorem holds by Lemma 2.2. If $k \in \{3,4\}$, we apply Lemma 2.4 (1) to P and Lemma 2.2 to T, and see that the theorem holds. If $t = 2k+1$, let $P' = P - x_{2k+1}$. For the same reason, if $k \notin \{3,4\}$, then the theorem holds. If $k \in \{3,4\}$, we apply Lemma 2.4 (3) to P and Lemma 2.2 to T, and see that the theorem holds.

Case 2: T has at least three but at most six supernodes.

In this case, T has a vertex-cut U with $|U| \leqslant 3$ such that no component of $T - U$ contains two distinct supernodes of T. We choose such a vertex-cut U with $|U|$ minimal. Let w_1, w_2 and w_3 be three distinct vertices not in T. In the following, we shall define a subgraph H of T. Then from a 4-packing of H we shall obtain a 4-packing of T in B_{n+3} with $V(B_{n+3}) = V(T) \cup \{w_1, w_2, w_3\}$. We divide this case into the following three subcases.

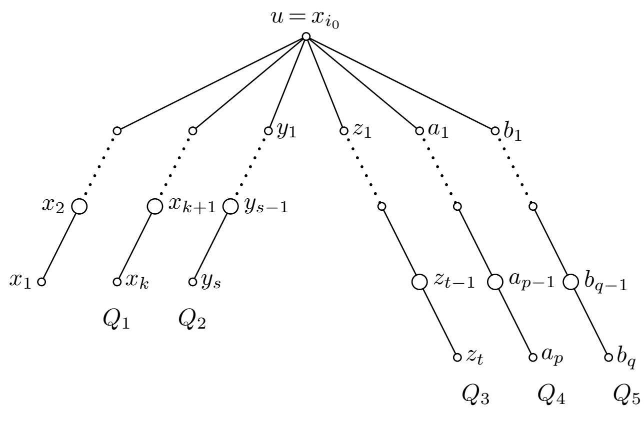

Figure 1. $|U| = 1$. (The larger dots are supernodes.)

Subcase 2.1: $|U| = 1$. Say $U = \{u\}$. As T has at least three supernodes, there exists a path $Q_1 = x_1x_2\dots x_k$ in T such that x_1 and x_k are two endvertices while x_2 and x_{k-1} are two distinct supernodes. Furthermore, $u = x_{i_0}$ for some $i_0 \in \{3,4,\dots,k-2\}$. Let x_1 and x_k be two endvertices in the opposite partites when T has at least four supernodes. In this situation, k is even and $k \geqslant 6$. Let $Q_2 = y_1y_2\dots y_s$ be a path vertex-disjoint from Q_1 such that $x_{i_0}y_1 \in E(T)$ and y_{s-1}

is a supernode of T. Thus, y_s is an endvertex of T. If T has four supernodes, let $Q_3 = z_1z_2 \dots z_t$ be the path vertex-disjoint from $Q_1 \cup Q_2$ such that $x_{i_0}z_1 \in E(T)$ and z_{t-1} is a supernode of T. If T has five supernodes, let $Q_4 = a_1a_2 \dots a_p$ be the path vertex-disjoint from $Q_1 \cup Q_2 \cup Q_3$ such that $x_{i_0}a_1 \in E(T)$ and a_{p-1} is a supernode of T. If T has six supernodes, let $Q_5 = b_1b_2 \dots b_q$ be the path vertex-disjoint from $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ such that $x_{i_0}b_1 \in E(T)$ and b_{q-1} is a supernode of T, see Figure 1.

Subcase 2.1.1: We suppose that T has exactly three distinct supernodes. In this situation, let $H = Q_1 \cup Q_2$. If x_1, x_k and y_s are in the opposite partites, we may assume that $\{x_1, x_{i_0}, y_s\} \subseteq V_0$, $x_k \in V_1$. Then $|V(H)| = 2h$ for some $h \geqslant 4$. It is easy to see that each vertex of $T - V(H)$ is an endvertex of T, for otherwise T would have four distinct supernodes. Since H does not contain the edge $x_{i_0}y_1$, each of Q_1 and Q_2 is a component of H, i.e., H is a linear forest. Assume for the moment $h \geqslant 8$. By Lemma 2.3 (1), there is a 4-packing (b, g, r, s) of H in B_{2h} such that each vertex of H is 4-placed. We may assume that b is the identity embedding. We extend the embeddings b, g, r, s to $H + x_{i_0}y_1$ in B_{2h+3} by adding w_1, w_2 and w_3 and defining $b(x_{i_0}) = x_{i_0}$, $g(x_{i_0}) = w_1$, $r(x_{i_0}) = w_2$ and $s(x_{i_0}) = w_3$. By Lemma 2.2 there is a 4-packing of T in B_{n+3}. Therefore, $h = 4$, 5, 6 or 7. If $h = 4$, then $T[V(H)] \cong G_1$. If $h = 5$, then $k = 6$ or $k = 8$. Furthermore, we see that if $k = 6$, then $T[V(H)] - x_6 \cong G_2$, and if $k = 8$, then $T[V(H)] - x_8 \cong G_2$. If $h = 6$, then $k = 6$, 8, or 10. Furthermore, we see that if $k = 6$, then $T[V(H)] - x_6 \cong G_3$, if $k = 8$, then $T[V(H)] - x_8 \cong G_4$, and if $k = 10$, then $T[V(H)] - x_{10} \cong G_3$ or G_4. If $h = 7$, then $k = 6$, 8, 10 or 12. Furthermore, we see that if $k = 6$, then $T[V(H)] - x_6 \cong G_5$, if $k = 8$, then $T[V(H)] - x_8 \cong G_6$, if $k = 10$, then $T[V(H)] - x_{10} \cong G_6$ or G_7, and if $k = 12$, then $T[V(H)] - x_{12} \cong G_5$ or G_6. By Lemma 2.2 and Lemma 2.5 (1)–(7), there is a 4-packing of T in B_{n+3}.

If x_1, x_k and y_s are in the same partite, we may assume that $\{x_1, x_k, y_s\} \subseteq V_0$. Thus $x_{i_0} \in V_0$ or V_1. Without loss of generality, we assume that $x_{i_0} \in V_0$, then $|V(H)| = 2h + 1$ for some $h \geqslant 3$. If $h \geqslant 8$, let $H' = H - y_s$, then we prove the theorem as above. Therefore $h = 3$, 4, 5, 6 or 7. If $h = 3$, then $k = 5$, $s = 2$, and $T[V(H)] \cong G_8$. If $h = 4$, then $k = 5$, $s = 4$, or $k = 7$, $s = 2$, and $T[V(H)] \cong G_2$. If $h = 5$, then $k = 5$, $s = 6$, or $k = 7$, $s = 4$, or $k = 9$, $s = 2$, and $T[V(H)] \cong G_3$ or G_4. If $h = 6$, then $k = 5$, $s = 8$, or $k = 7$, $s = 6$, or $k = 9$, $s = 4$, or $k = 11$, $s = 2$, and $T[V(H)] \cong G_5$, G_6 or G_7. If $h = 7$, then $k = 5$, $s = 10$, or $k = 7$, $s = 8$, or $k = 9$, $s = 6$, or $k = 11$, $s = 4$, or $k = 13$, $s = 2$, and $T[V(H)] \cong G_9$, G_{10}, G_{11} or G_{12}. By Lemma 2.2 and Lemma 2.5 (2)–(12), there is a 4-packing of T in B_{n+3}.

Subcase 2.1.2: We suppose that T has exactly four distinct supernodes. In this case, without loss of generality, say $\{x_1, x_{i_0}\} \subseteq V_0$. As T does not contain four strongly independent endvertices of the same partities, we may assume that $y_s \in V_0$

and $z_t \in V_0$ or V_1. Let $H = Q_1 \cup Q_2 \cup Q_3$, and $t' = t$ if $z_t \in V_0$. Let $H = Q_1 \cup Q_2 \cup Q_3 - z_t$, and $t' = t - 1$ if $z_t \in V_1$. Then $|V(H)| = 2h$ for some $h \geqslant 5$. We can see that each vertex of $T - V(H)$ is an endvertex of T, for otherwise T would have four strongly independent endvertices in the same partite. Clearly, H is a linear forest. If $h \geqslant 8$, the proof is the same as that in Subcase 2.1.1. Therefore, $h = 5$, 6 or 7. If $h = 5$, then $k = 6$, $s = 2$ and $t' = 2$, and $T[V(H)] \cong G_{13}$. If $h = 6$, then $k = 6$ or $k = 8$. Furthermore, we see that if $k = 6$ then $s = 4$, $t' = 2$, or $s = 2$, $t' = 4$, and $T[V(H)] - x_6 \cong G_{14}$. If $k = 8$, then $s = 2$, $t' = 2$, and $T[V(H)] - x_8 \cong G_{14}$. If $h = 7$, then $k = 6$, 8 or 10. Furthermore, we see that if $k = 6$, then $s = 2$, $t' = 6$, or $s = 4$, $t' = 4$, or $s = 6$, $t' = 2$, and $T[V(H)] - x_6 \cong G_{15}$ or G_{16}. If $k = 8$, then $s = 4$, $t' = 2$, or $s = 2$, $t' = 4$, and $T[V(H)] - x_8 \cong G_{15}$. If $k = 10$, then $s = 2$, $t' = 2$, and $T[V(H)] - x_{10} \cong G_{15}$ or G_{16}. By Lemma 2.2 and Lemma 2.5 (13)–(16), there is a 4-packing of T in B_{n+3}.

Subcase 2.1.3: We suppose that T has exactly five distinct supernodes. Without loss of generality, say $\{x_1, x_{i_0}\} \subseteq V_0$. As T does not contain four strongly independent endvertices in the same partite, we may assume that $y_s \in V_0$, $z_t \in V_1$ and $a_p \in V_0$ or V_1. Let $H = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ if $a_p \in V_1$, and let $H = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 - z_t$ if $a_p \in V_0$. Then $|V(H)| = 2h$ for some $h \geqslant 6$. It is easy to see that each vertex of $T - V(H)$ is an endvertex of T, for otherwise T would have four strongly independent endvertices in the same partite. Clearly, H is a linear forest. If $h \geqslant 8$, the proof is the same as that in Subcase 2.1.1. Therefore, $h = 6$ or 7. If $h = 6$, then $k = 6$, $s = 2$, $t - 1 = 2$, $p = 2$, and $T[V(H)] \cong G_{17}$. If $h = 7$, then $k = 6$ or $k = 8$. Furthermore, we see that if $k = 6$, then $s = 2$, $t = 3$, $p = 3$, or $s = 2$, $t - 1 = 2$, $p = 4$, or $s = 2$, $t - 1 = 4$, $p = 2$, and $T[V(H)] - z_3 - a_3 \cong G_{17}$ or $T[V(H)] - x_6 \cong G_{18}$. If $k = 8$, then $s = 2$, $t - 1 = 2$, $p = 2$, and $T[V(H)] - x_8 \cong G_{18}$. By Lemma 2.2 and Lemma 2.5 (17) and (18), there is a 4-packing of T in B_{n+3}.

Subcase 2.1.4: We suppose that T has exactly six distinct supernodes. As T does not contain four strongly independent endvertices in the same partite, we may assume that $\{x_1, y_s, z_t\} \subseteq V_0$, and $\{x_k, a_p, b_q\} \subseteq V_1$. Let $H = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5$. Then $|V(H)| = 2h$ for some $h \geqslant 8$. The proof is the same as that in Subcase 2.1.1.

Subcase 2.2: $|U| = 2$. Say $U = \{u, v\}$. In this case, T has at least four supernodes, there are two pairwise vertex-disjoint paths $Q_1 = x_1 x_2 \ldots x_k$ and $Q_2 = y_1 y_2 \ldots y_s$ such that x_2, x_{k-1}, y_2 and y_{s-1} are supernodes of T. Without loss of generality, say $u = x_{i_0}$ and $v = y_{j_0}$ for some $i_0 \in \{3, 4, \ldots, k-2\}$ and $j_0 \in \{3, 4, \ldots, s-2\}$. Let $Q_3 = z_1 z_2 \ldots z_t$ be the path vertex-disjoint from $Q_1 \cup Q_2$ such that $\{x_{i_0} z_1, y_{j_0} z_t\} \subseteq E(T)$. We divide this case into the following three subcases, see Figure 2.

Subcase 2.2.1: We suppose that T has exactly four distinct supernodes. In this case, T has at most two another nodes. Set $m_1 = k + s + t$. Without loss of generality, we assume that $\{x_1, x_k\} \subseteq V_0$. Let $H = Q_1 \cup Q_2$ if $\{y_1, y_s\} \subseteq V_1$ and let

$H = Q_1 \cup Q_2 - y_1$ if $y_1 \in V_0$, $y_s \in V_1$. Then H is a linear forest and $|V(H)| = 2h$. Assume for the moment that $h \geqslant 8$, by Lemma 2.3 (1), there is a 4-packing (b, g, r, s) of H in B_{2h} such that each vertex of H is 4-placed. For even t, let $Q_3' = Q_3$. For odd t, let $Q_3' = Q_3 - z_t$ if z_1 and z_t are not nodes or z_1 is a node, let $Q_3' = Q_3 - z_1$ if z_t is a node, and let $Q_3' = Q_3 + d$ if z_1 and z_t are both nodes, where d is an endvertex which is adjacent to z_t. If there is a 4-packing (b_1, g_1, r_1, s_1) of Q_3' in $B_{|V(Q_3')|+3}$ such that each vertex of Q_3' is 4-placed, we can see that a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3)]$ in B_{m_1+3} is obtained from $(b \cup b_1, g \cup g_1, r \cup r_1, s \cup s_1)$ by defining $c(z_t) = z_t$ for each $c \in \{b_1, g_1, r_1, s_1\}$ when $Q_3' = Q_3 - z_t$ or by defining $c(z_1) = z_1$ for each $c \in \{b_1, g_1, r_1, s_1\}$ when $Q_3' = Q_3 - z_1$. Furthermore, each node of T is 4-placed in this packing. Then by Lemma 2.2 the theorem holds. Thus, there is no such a 4-packing of Q_3'.

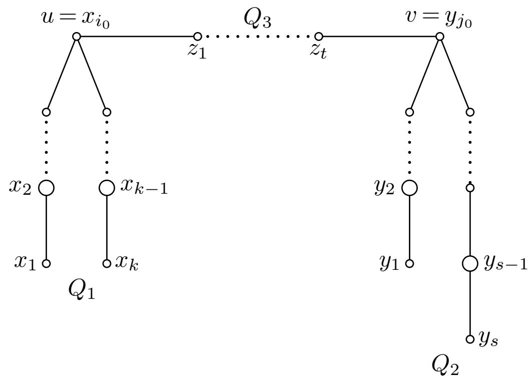

Figure 2. $|U| = 2$. (The larger dots are supernodes.)

Therefore, by Lemma 2.3, we see that $t \leqslant 9$ when z_1 and z_t are not nodes. If $t \in \{7, 9\}$, by Lemma 2.4 (3), there is a 4-packing (b_2, g_2, r_2, s_2) of Q_3 in B_{t+3} such that each vertex of Q_3 is 4-placed. If $t \in \{4, 6, 8\}$, by Lemma 2.4 (1), there is a 4-packing (b_2, g_2, r_2, s_2) of Q_3 in B_{t+3} such that each vertex of Q_3 is 4-placed except z_t. If $t = 5$, by Lemma 2.4 (2), there is a 4-packing (b_2, g_2, r_2, s_2) of Q_3 in B_{t+3} such that each vertex of Q_3 is 4-placed except z_1 and z_t. Then $(b \cup b_2, g \cup g_2, r \cup r_2, s \cup s_2)$ is a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3)]$ in B_{m_1+3} such that each node of T is 4-placed. By Lemma 2.2 the theorem holds. Hence, we must have $t \leqslant 3$. Let $w_0 = x_{i_0}$ if $t = 0$. Let $w_0 = z_1$ if $t = 1$. Let $w_0 = z_2$ if $t \in \{2, 3\}$. We define or redefine the values of $b(w_0)$, $g(w_0)$, $r(w_0)$ and $s(w_0)$ as: $b(w_0) = w_0$, $g(w_0) = w_1$, $r(w_0) = w_2$ and $s(w_0) = w_3$. Let $b(x) = g(x) = r(x) = s(x)$ for all $x \in V(T) - V(Q_1 \cup Q_2) - \{w_0\}$. Then (b, g, r, s) is a 4-packing of T in B_{n+3}.

When z_1 is a node, we see that $t \leqslant 9$ by Lemma 2.3. If $t \in \{1, 2, 4, 6, 7, 8, 9\}$, we prove the theorem as above. Let $Q_3'' = Q_3 + e$ when $t \in \{3, 5\}$, where e is an endvertex which is adjacent to z_1. Then by Lemma 2.4 (1), there is a 4-packing

(b_2, g_2, r_2, s_2) of Q_3'' in $B_{|V(Q_3'')|+3}$ such that each vertex of Q_3'' is 4-placed except e. Then $(b \cup b_2, g \cup g_2, r \cup r_2, s \cup s_2)$ is a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3)]$ in B_{m_1+3} such that each node of T is 4-placed. By Lemma 2.2 the theorem holds.

When z_t is a node, the proof is the same as the case when z_1 is a node.

When z_1 and z_t are both nodes, we see that $t \leqslant 8$ by Lemma 2.3. If $t \in \{1, 7\}$, we prove it as above. Let $Q_3'' = Q_3 + d + e$ if $t \in \{2, 4, 6, 8\}$, and let $Q_3'' = Q_3 + d$ if $t \in \{3, 5\}$, where d and e are the endvertices which are adjacent to z_t and z_1, respectively. Then by Lemma 2.3 (2) and Lemma 2.4 (1), there is a 4-packing (b_2, g_2, r_2, s_2) of Q_3'' in $B_{|V(Q_3'')|+3}$. Furthermore, each vertex of Q_3'' is 4-placed for $t \in \{2, 3, 4, 5, 6\}$ except d and each vertex of Q_3'' is 4-placed for $t = 8$. Then $(b \cup b_2, g \cup g_2, r \cup r_2, s \cup s_2)$ is a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3)]$ in B_{m_1+3} so that each node of T is 4-placed. By Lemma 2.2 the theorem holds.

Now, we conclude that $h = 5$, 6 or 7. Then if $h = 5$, each of Q_1 and Q_2 is a path of order 5. Thus $Q_1 = x_1x_2x_3x_4x_5$ and $u = x_3$. Rename $Q_2 = x_6x_7x_8x_9x_{10}$. Thus, $v = x_8$. As we already assumed $x_1 \in V_0$, we have $x_6 \in V_1$. Hence, the order of Q_3 must be even. Say $t = 2t'$. If $t' = 0$, i.e., $x_3x_8 \in E(T)$, let $H = Q_1 \cup Q_2 + x_3x_8$. Then $H \cong F_1$ and there is a 4-packing of H in B_{13} such that each vertex of H is 4-placed except x_{10}. If $t' = 1$, say $x_{12} = z_1$ and $x_{11} = z_t$. Let $H = Q_1 \cup Q_2 \cup Q_3 + x_3x_{12} + x_8x_{11}$. Then $H \cong F_2$. There is a 4-packing of H in B_{15} and each vertex of H is 4-placed. If $t' = 2$, say $Q_3 = x_{14}x_{13}x_{12}x_{11}$. Let $H = Q_1 \cup Q_2 \cup Q_3 + x_3x_{14} + x_8x_{11}$. Then $H \cong F_3$. There is a 4-packing of H in B_{17} and each vertex of H is 4-placed. If $t' \geqslant 3$, rename z_1, z_2, z_3, z_{t-2}, z_{t-1} and z_t as x_{16}, x_{15}, x_{14}, x_{13}, x_{12} and x_{11}, respectively. Let $H = Q_1 \cup Q_2 + x_3x_{16}x_{15}x_{14} + x_{13}x_{12}x_{11}x_8$. Then $H \cong F_4$. There is a 4-packing (b, g, r, s) of H in B_{16} such that each vertex of H is 4-placed except x_1 and x_5. We consider two situations $t' = 3$ and $t' > 3$. If $t' = 3$, we define or redefine $b(x_{14}) = x_{14}$, $g(x_{14}) = w_1$, $r(x_{14}) = w_2$ and $s(x_{14}) = w_3$. Let $c(x) = x$ for all $x \in V(T) - V(H) - \{x_{14}\}$ and $c \in \{b, g, r, s\}$. We can find that (b, g, r, s) is a 4-packing of T in B_{m_1+3} such that each node of T is 4-placed. Therefore, we have $t' > 3$, and let $Q_3'' = Q_3 - z_1z_2z_3 - z_tz_{t-1}z_{t-2}$. If $3 < t' < 8$, we have $|V(Q_3'')| \in \{2, 4, 6, 8\}$. Then there is a 4-packing (b_2, g_2, r_2, s_2) of Q_3'' in $B_{|V(Q_3'')|+3}$ such that each vertex of Q_3'' is 4-placed, and we can give the proof as above. If $t' \geqslant 8$, by Lemma 2.3, there is a 4-packing (b_2, g_2, r_2, s_2) of Q_3'' in $B_{|V(Q_3'')|+3}$ and each vertex of Q_3'' is 4-placed. Then when $t' > 3$, $(b \cup b_2, g \cup g_2, r \cup r_2, s \cup s_2)$ is a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3)]$ in B_{m_1+3} such that each node of T is 4-placed. By Lemma 2.2 and Lemma 2.6 (1)–(4) the theorem holds.

If $h = 6$ or 7, the proof is the same as that of $h = 5$.

Subcase 2.2.2: We suppose that T has exactly five distinct supernodes. In this case, T has at most one another node. Let $Q_4 = a_1a_2 \ldots a_p$ be a path vertex-disjoint from $Q_1 \cup Q_2 \cup Q_3$ such that a_{p-1} is a supernode and a_p is an endvertex, where a_1

is adjacent to a vertex of $Q_3 \cup \{u, v\}$. (If a_1 is adjacent to a vertex of Q_1 or Q_2, we can deal with the case in the same way.) Without loss of generality, we assume that $va_1 \in E(T)$. Set $m_2 = k + s + p + t$. We assume that $\{x_1, x_k\} \subseteq V_0, \{y_1, y_s\} \subseteq V_1$, since T does not have four strongly independent endvertices in the same partite. Let $H = Q_1 \cup Q_2 \cup Q_4$ if p is even and let $H = Q_1 \cup Q_2 \cup Q_4 - a_p$ if p is odd. Then H is a linear forest and $|V(H)| = 2h$ for some $h \geqslant 6$. Assume for the moment that $h \geqslant 8$. By Lemma 2.3 (1), there is a 4-packing (b, g, r, s) of H in B_{2h} such that each vertex of H is 4-placed. For even t, let $Q'_3 = Q_3$. For odd t, let $Q'_3 = Q_3 - z_t$ if z_1 and z_t are not nodes or z_1 is a node, and let $Q'_3 = Q_3 - z_1$ if z_t is a node. If there is a 4-packing (b_1, g_1, r_1, s_1) of Q'_3 in $B_{|V(Q'_3)|+3}$ such that each vertex of Q'_3 is 4-placed, we can see that a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3 \cup Q_4)]$ in B_{m_2+3} is obtained from $(b \cup b_1, g \cup g_1, r \cup r_1, s \cup s_1)$ by defining $c(z_t) = z_t$ for $c \in \{b_1, g_1, r_1, s_1\}$ when $Q'_3 = Q_3 - z_t$ or by defining $c(z_1) = z_1$ for each $c \in \{b_1, g_1, r_1, s_1\}$ when $Q'_3 = Q_3 - z_1$. Furthermore, each node of T is 4-placed in this packing. Then by Lemma 2.2 the theorem holds. Thus, there is no such a 4-packing of Q'_3. Therefore, by Lemma 2.3 we see that $t \leqslant 9$. At most one of z_1 and z_t is a node. The proof is the same as that in Subcase 2.2.1.

We conclude that $h = 6$ or 7. Then if $h = 6$, each of Q_1 and Q_2 is a path of order 5. Thus, $Q_1 = x_1x_2x_3x_4x_5$ and $u = x_3$. Rename $Q_2 = x_6x_7x_8x_9x_{10}$, $Q_4 = x_{11}x_{12}$. Thus, $v = x_8$. As we already assumed $x_1 \in V_0$, we have $x_6 \in V_1$. Hence, the order of Q_3 must be even. Say $t = 2t'$. If $t' = 0$, i.e., $x_3x_8 \in E(T)$, let $H = Q_1 \cup Q_2 \cup Q_4 + x_3x_8x_{11}$. Then $H \cong F_5$. Therefore, there is a 4-packing of H in B_{15} and each vertex of H is 4-placed. If $t' = 1$, say $x_{14} = z_1$ and $x_{13} = z_t$. Let $H = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 + x_3x_{14} + x_{13}x_8x_{11}$. Then $H \cong F_6$. There is a 4-packing of H in B_{17} and each vertex of H is 4-placed except x_6 and x_{12}. If $t' = 2$, say $Q_3 = x_{16}x_{15}x_{14}x_{13}$. Let $H = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 + x_3x_{16} + x_{11}x_8x_{13}$. Then $H \cong F_7$. There is a 4-packing of H in B_{19} and each vertex of H is 4-placed. If $t' \geqslant 3$, rename z_1, z_2, z_3, z_{t-2}, z_{t-1} and z_t as x_{18}, x_{17}, x_{16}, x_{15}, x_{14} and x_{13}, respectively. Let $H = Q_1 \cup Q_2 \cup Q_4 + x_3x_{18}x_{17}x_{16} + x_{15}x_{14}x_{13}x_8x_{11}$. Then $H \cong F_8$. There is a 4-packing (b, g, r, s) of H in B_{18} and each vertex of H is 4-placed. We consider two situations $t' = 3$ and $t' > 3$. If $t' = 3$, define or redefine $b(x_{15}) = x_{15}$, $g(x_{15}) = w_1$, $r(x_{15}) = w_2$ and $s(x_{15}) = w_3$. Let $c(x) = x$ for all $x \in V(T) - V(H) - \{x_{15}\}$ and $c \in \{b, g, r, s\}$. We can find that (b, g, r, s) is a 4-packing of T in B_{m_2+3} such that each node of T is 4-placed. Therefore, we have $t' > 3$. Let $Q''_3 = Q_3 - z_1z_2z_3 - z_tz_{t-1}z_{t-2}$. If $3 < t' < 8$, we can prove that there is a 4-packing (b_2, g_2, r_2, s_2) of Q''_3 in $B_{|V(Q''_3)|+3}$ and each vertex of Q''_3 is 4-placed as that in Subcase 2.2.1. If $t' \geqslant 8$, by Lemma 2.3, there is a 4-packing (b_2, g_2, r_2, s_2) of Q''_3 in $B_{|V(Q''_3)|+3}$ and each vertex of Q''_3 is 4-placed. Then when $t' > 3$, $(b \cup b_2, g \cup g_2, r \cup r_2, s \cup s_2)$ is a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3 \cup Q_4)]$ in B_{m_2+3} such that each node of T is 4-placed. By Lemma 2.2 and Lemma 2.6 (5)–(8) the theorem holds.

If $h = 7$, we prove the theorem as the case $h = 6$.

Subcase 2.2.3: We suppose that T has exactly six distinct supernodes. In this case, there exist two pairwise vertex-disjoint paths $Q_4 = a_1a_2 \dots a_p$ and $Q_5 = b_1b_2 \dots b_q$ whose vertices are also disjoint from $Q_1 \cup Q_2 \cup Q_3$. Furthermore, a_{p-1} and b_{q-1} are two supernodes while a_1 is adjacent to a vertex of $Q_1 \cup Q_3$ and b_1 is adjacent to a vertex of Q_2. Set $m_3 = k+s+p+q+t$. Without loss of generality, say $x_{i_0} \in V_0$, $y_{j_0} \in V_1$. As T does not have four strongly independent endvertices in the same partite, we assume that $\{x_1, y_1, a_p\} \subseteq V_0$ and $\{x_k, y_s, b_q\} \subseteq V_1$. Let $H = Q_1 \cup Q_2 \cup Q_4 \cup Q_5$ if $m_3 - t$ is even, and let $H = Q_1 \cup Q_2 \cup Q_4 \cup Q_5 - b_q$ if $m_3 - t$ is odd. Then H is a linear forest and $|V(H)| = 2h$ for some $h \geqslant 8$. By Lemma 2.3 (1), there is a 4-packing (b, g, r, s) of H in B_{2h} such that each vertex of H is 4-placed. If t is even, let $Q_3' = Q_3$. If t is odd, let $Q_3' = Q_3 - z_t$. If there is a 4-packing (b_1, g_1, r_1, s_1) of Q_3' in $B_{|V(Q_3')|+3}$ such that each vertex of Q_3' is 4-placed, we can see that a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5)]$ in B_{m_3+3} is obtained from $(b \cup b_1, g \cup g_1, r \cup r_1, s \cup s_1)$ by defining $c(z_t) = z_t$ for $c \in \{b_1, g_1, r_1, s_1\}$ when t is odd. Furthermore, each node of T is 4-placed in this packing. Then by Lemma 2.2 the theorem holds. Thus, there is no such a 4-packing of Q_3'. Therefore, by Lemma 2.3, we see that $t \leqslant 9$. Then we can give the proof as that in Subcase 2.2.1 when z_1 and z_t are not nodes. Thus, we can find a 4-packing of $T[V(Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5)]$ in B_{m_3+3} such that each node of T is 4-placed. By Lemma 2.2 the theorem holds.

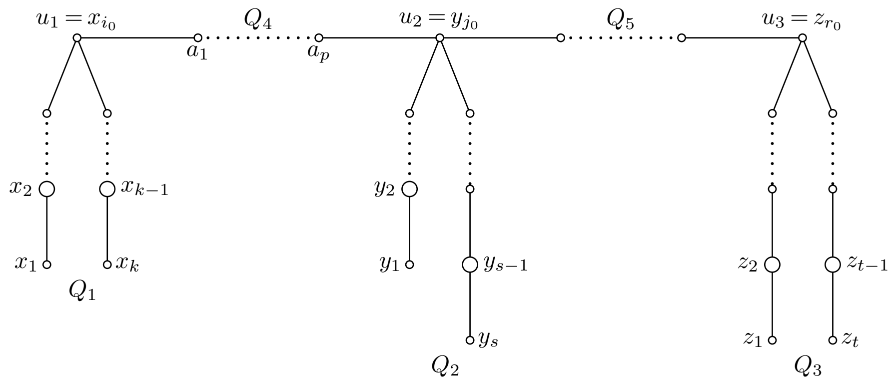

Figure 3. $|U| = 3$. (The larger dots are supernodes.)

Case 2.3: $|U| = 3$. Say $U = \{u_1, u_2, u_3\}$. In this case, T has exactly six distinct supernodes. There exist three vertex-disjoint paths $Q_1 = x_1x_2 \dots x_k$, $Q_2 = y_1y_2 \dots y_s$ and $Q_3 = z_1z_2 \dots z_t$ in T such that x_1, x_k, y_1, y_s, z_1 and z_t are six endvertices while x_2, x_{k-1}, y_2, y_{s-1}, z_2 and z_{t-1} are six distinct supernodes. Furthermore, $u_1 = x_{i_0}$ for some $i_0 \in \{3, 4, \dots, k-2\}$, $u_2 = y_{j_0}$ for some $j_0 \in \{3, 4, \dots, s-2\}$, and $u_3 = z_{r_0}$ for some $r_0 \in \{3, 4, \dots, t-2\}$. Let $Q_4 = a_1a_2 \dots a_p$ be a path vertex-disjoint from $Q_1 \cup Q_2 \cup Q_3$ such that $\{x_{i_0}a_1, y_{j_0}a_p\} \subseteq E(T)$. Thus, there exists a path

$Q_5 = b_1 b_2 \ldots b_q$ vertex-disjoint from $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ such that b_1 is adjacent to a vertex of $Q_1 \cup Q_2 \cup Q_4$ and $b_q z_{r_0} \in E(T)$, see Figure 3. Let $H = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5$. We can see that every vertex of $T - H$ is an endvertex of T. And T does not have other nodes besides the six supernodes, for otherwise T would have four strongly independent endvertices in the same partite. Let $H_1 = Q_1 \cup Q_2 \cup Q_3$. Set $m = k + s + t$. As T does not contain four strongly independent endvertices in the same partite, without loss of generality, we assume that $\{x_1, x_k, y_1\} \subseteq V_0$ and $\{y_s, z_1, z_t\} \subseteq V_1$. Thus, $|V(H_1)| = 2h$ for some $h \geqslant 8$. By Lemma 2.3 (1), there is a 4-packing (b, g, r, s) of H_1 in B_{2h} such that each vertex of H_1 is 4-placed. Let $H_2 = Q_4 \cup Q_5$. Set $l = p + q$. We can find that there is a 4-packing (b_1, g_1, r_1, s_1) of $T[V(H_2)]$ in B_{l+3} by Case 1 and Subcase 2.1.1. We can see that $(b \cup b_1, g \cup g_1, r \cup r_1, s \cup s_1)$ is a 4-packing of $T[V(H)]$ in B_{m+l+3}. Furthermore, each node of $T[V(H)]$ is 4-placed in this 4-packing. Then by Lemma 2.2, the theorem holds. This completes the proof of the theorem. □

In this theorem, $n + 3$ cannot be further reduced. A simple example is a star. Another example is a tree such that it is obtained from two vertex-disjoint stars by connecting two centers of them with a path of length 2.

We can see there are more cases in the proof of the conjecture (see [7]) when $k = 4$. Another purpose of this article is to improve the state of knowledge approaching the conjecture by determining the case $k = 4$.

4. Appendix (I): The proof of Lemma 2.5

For each case, we define the required 4-packing (b, g, r, s) with b as identity embedding as follows.

To prove (1), let

- ▷ $g(G_1) = x_1 x_4 x_9 x_2 x_7 x_6 \cup x_9 x_{10} x_{11}$ with $g(x_1) = x_1$ and $g(x_7) = x_{11}$,
- ▷ $r(G_1) = x_9 x_8 x_{11} x_6 x_3 x_{10} \cup x_{11} x_4 x_7$ with $r(x_1) = x_9$ and $r(x_7) = x_7$,
- ▷ $s(G_1) = x_7 x_{10} x_5 x_8 x_1 x_6 \cup x_5 x_2 x_{11}$ with $s(x_1) = x_7$ and $s(x_7) = x_{11}$.

To prove (2), let

- ▷ $g(G_2) = x_5 x_{10} x_1 x_6 x_9 x_2 x_{11} \cup x_1 x_4 x_7$ with $g(x_1) = x_5$ and $g(x_9) = x_7$,
- ▷ $r(G_2) = x_{11} x_8 x_5 x_2 x_7 x_{10} x_3 \cup x_5 x_{12} x_9$ with $r(x_1) = x_{11}$ and $r(x_9) = x_9$,
- ▷ $s(G_2) = x_9 x_4 x_{11} x_{12} x_1 x_8 x_7 \cup x_{11} x_6 x_3$ with $s(x_1) = x_9$ and $s(x_9) = x_3$.

To prove (3), let

- ▷ $g(G_3) = x_7 x_{14} x_{13} x_6 x_{11} \cup x_{13} x_4 x_9 x_{12} x_5 x_8 x_1$ with $g(x_1) = x_7$ and $g(x_{11}) = x_1$,
- ▷ $r(G_3) = x_3 x_8 x_{11} x_{12} x_{13} \cup x_{11} x_{14} x_1 x_4 x_7 x_2 x_5$ with $r(x_1) = x_3$ and $r(x_{11}) = x_5$,
- ▷ $s(G_3) = x_5 x_6 x_9 x_{14} x_3 \cup x_9 x_2 x_{13} x_{10} x_1 x_{12} x_7$ with $s(x_1) = x_5$ and $s(x_{11}) = x_7$.

To prove (4), let

- ▷ $g(G_4) = x_3x_6x_9x_{12}x_{13}x_{14}x_1 \cup x_9x_2x_{11}x_4x_7$ with $g(x_1) = x_3$ and $g(x_{11}) = x_7$,
- ▷ $r(G_4) = x_5x_{10}x_{13}x_2x_7x_8x_{11} \cup x_{13}x_6x_1x_{12}x_3$ with $r(x_1) = x_5$ and $r(x_{11}) = x_3$,
- ▷ $s(G_4) = x_{11}x_{14}x_7x_{10}x_1x_4x_9 \cup x_7x_{12}x_5x_8x_{13}$ with $s(x_1) = x_{11}$ and $s(x_{11}) = x_{13}$.

To prove (5), let

- ▷ $g(G_5) = x_3x_8x_{11}x_{14}x_{15} \cup x_{11}x_{16}x_{13}x_{10}x_1x_4x_7x_2x_9$ with $g(x_1) = x_3$ and $g(x_{13}) = x_9$,
- ▷ $r(G_5) = x_5x_6x_9x_{12}x_1 \cup x_9x_{14}x_3x_{16}x_{15}x_8x_{13}x_4x_{11}$ with $r(x_1) = x_5$ and $r(x_{13}) = x_{11}$,
- ▷ $s(G_5) = x_9x_{16}x_5x_{10}x_3 \cup x_5x_{12}x_{15}x_2x_{11}x_6x_1x_{14}x_7$ with $s(x_1) = x_9$ and $s(x_{13}) = x_7$.

To prove (6), let

- ▷ $g(G_6) = x_3x_6x_9x_{12}x_{15}x_{16}x_1 \cup x_9x_2x_{11}x_{14}x_{13}x_4x_7$ with $g(x_1) = x_3$ and $g(x_{13}) = x_7$,
- ▷ $r(G_6) = x_5x_8x_{11}x_6x_1x_{14}x_9 \cup x_{11}x_{16}x_3x_{12}x_7x_2x_{15}$ with $r(x_1) = x_5$ and $r(x_{13}) = x_{15}$,
- ▷ $s(G_6) = x_9x_{16}x_7x_8x_{13}x_2x_5 \cup x_7x_{10}x_1x_4x_{15}x_{14}x_3$ with $s(x_1) = x_9$ and $s(x_{13}) = x_3$.

To prove (7), let

- ▷ $g(G_7) = x_3x_6x_9x_{12}x_{15}x_{16}x_1x_4x_{11} \cup x_{15}x_2x_7x_{14}x_5$ with $(x_1) = x_3$ and $g(x_{13}) = x_5$,
- ▷ $r(G_7) = x_9x_{16}x_{11}x_{14}x_3x_{10}x_{13}x_8x_5 \cup x_3x_{12}x_1x_6x_{15}$ with $r(x_1) = x_9$ and $r(x_{13}) = x_{15}$,
- ▷ $s(G_7) = x_7x_{12}x_5x_2x_9x_{14}x_{15}x_{10}x_1 \cup x_9x_4x_{13}x_{16}x_3$ with $s(x_1) = x_7$ and $s(x_{13}) = x_3$.

To prove (8), let

- ▷ $g(G_8) = x_3x_8x_1x_9x_5 \cup x_1x_{10}x_7$ with $g(x_1) = x_3$ and $g(x_7) = x_7$,
- ▷ $r(G_8) = x_1x_6x_5x_{10}x_3 \cup x_5x_2x_7$ with $r(x_1) = x_1$ and $r(x_7) = x_7$,
- ▷ $s(G_8) = x_3x_9x_7x_8x_5 \cup x_7x_4x_1$ with $s(x_1) = x_3$ and $s(x_7) = x_1$.

To prove (9), let

- ▷ $g(G_9) = x_3x_6x_9x_{12}x_{15}x_{16}x_{17}x_{18}x_5x_{14}x_1x_{10}x_7 \cup x_9x_2x_{13}$ with $g(x_1) = x_3$ and $g(x_{13}) = x_7$,
- ▷ $r(G_9) = x_5x_{10}x_{15}x_2x_7x_{14}x_3x_{16}x_{11}x_4x_9x_{18}x_1 \cup x_{15}x_6x_{17}$ with $r(x_1) = x_5$ and $r(x_{13}) = x_1$,
- ▷ $s(G_9) = x_9x_{14}x_{17}x_{10}x_3x_{18}x_{11}x_6x_{13}x_{16}x_5x_8x_{15} \cup x_{17}x_{12}x_7$ with $s(x_1) = x_9$ and $s(x_{13}) = x_{15}$.

To prove (10), let

- ▷ $g(G_{10}) = x_3x_6x_9x_{12}x_{15}x_{16}x_{17}x_{18}x_1x_4x_7 \cup x_9x_{14}x_5x_{10}x_{13}$ with $g(x_1) = x_3$ and $g(x_{15}) = x_{13}$,
- ▷ $r(G_{10}) = x_5x_8x_{11}x_{14}x_{17}x_2x_9x_4x_{13}x_{16}x_3 \cup x_{11}x_6x_{15}x_{18}x_7$ with $r(x_1) = x_5$ and $r(x_{15}) = x_7$,
- ▷ $s(G_{10}) = x_7x_{12}x_{17}x_6x_{13}x_{18}x_3x_{10}x_{15}x_2x_5 \cup x_{17}x_8x_1x_{16}x_9$ with $s(x_1) = x_7$ and $s(x_{15}) = x_9$.

To prove (11), let

- ▷ $g(G_{11}) = x_{11}x_4x_9x_6x_{13}x_{10}x_1x_{14}x_3 \cup x_{13}x_2x_{17}x_4x_1x_{18}x_5$ with $g(x_1) = x_{11}$ and $g(x_{15}) = x_5$,
- ▷ $r(G_{11}) = x_5x_8x_1x_{16}x_{17}x_{18}x_9x_2x_{11} \cup x_{17}x_{12}x_3x_{10}x_7x_4x_{13}$ with $r(x_1) = x_5$ and $r(x_{15}) = x_{13}$,
- ▷ $s(G_{11}) = x_9x_{16}x_{11}x_{14}x_{15}x_2x_5x_{12}x_1 \cup x_{15}x_8x_{13}x_{18}x_3x_6x_{17}$ with $s(x_1) = x_9$ and $s(x_{15}) = x_{17}$.

To prove (12), let

- ▷ $g(G_{12}) = x_3x_6x_9x_{12}x_{15}x_{16}x_{17}x_{18}x_5 \cup x_9x_{14}x_1x_{10}x_7x_4x_{13}$ with $g(x_1) = x_3$ and $g(x_{15}) = x_{13}$,
- ▷ $r(G_{12}) = x_5x_8x_{15}x_2x_7x_{14}x_3x_{16}x_{11} \cup x_{15}x_4x_9x_{18}x_1x_6x_{17}$ with $r(x_1) = x_5$ and $r(x_{15}) = x_{17}$,
- ▷ $s(G_{12}) = x_7x_{12}x_{17}x_8x_3x_{18}x_{11}x_6x_{13} \cup x_{17}x_2x_5x_{16}x_9x_{10}x_5$ with $s(x_1) = x_7$ and $s(x_{15}) = x_5$.

To prove (13), let

- ▷ $g(G_{13}) = x_3x_6x_7x_{10}x_{11}x_{12} \cup x_1x_4x_7x_{13}x_9$ with $g(x_1) = x_3$ and $g(x_7) = x_9$,
- ▷ $r(G_{13}) = x_5x_{12}x_9x_8x_1x_{13} \cup x_1x_6x_9x_2x_{11}$ with $r(x_1) = x_5$ and $r(x_7) = x_{11}$,
- ▷ $s(G_{13}) = x_3x_{13}x_5x_2x_7x_{12} \cup x_{11}x_8x_5x_{10}x_1$ with $s(x_1) = x_3$ and $s(x_7) = x_1$.

To prove (14), let

- ▷ $g(G_{14}) = x_3x_6x_9x_{10}x_{13}x_{14}x_1 \cup x_7x_{12}x_9x_2x_5$ with $g(x_1) = x_3$ and $g(x_{11}) = x_7$,
- ▷ $r(G_{14}) = x_5x_8x_{11}x_{12}x_1x_4x_9 \cup x_{13}x_2x_{11}x_{14}x_7$ with $r(x_1) = x_5$ and $r(x_{11}) = x_{13}$,
- ▷ $s(G_{14}) = x_{11}x_4x_{13}x_8x_7x_{10}x_5 \cup x_1x_6x_{13}x_{12}x_3$ with $s(x_1) = x_{11}$ and $s(x_{11}) = x_1$.

To prove (15), let

- ▷ $g(G_{15}) = x_3x_6x_9x_{12}x_{11}x_{14}x_{15} \cup x_5x_2x_9x_{16}x_1x_4x_{13}$ with $g(x_{10}) = x_3$ and $g(x_{13}) = x_5$,
- ▷ $r(G_{15}) = x_5x_8x_{11}x_{16}x_3x_{10}x_{13} \cup x_1x_6x_{11}x_2x_{15}x_{12}x_7$ with $r(x_1) = x_5$ and $r(x_{13}) = x_1$,
- ▷ $s(G_{15}) = x_7x_{16}x_{15}x_6x_{13}x_8x_1 \cup x_{11}x_4x_{15}x_{10}x_5x_{14}x_3$ with $s(x_1) = x_7$ and $s(x_{13}) = x_{11}$.

To prove (16), let

- ▷ $g(G_{16}) = x_3x_6x_9x_{10}x_{13}x_{14}x_{15}x_{16}x_1 \cup x_7x_2x_9x_{12}x_5$ with $g(x_1) = x_3$ and $g(x_{13}) = x_7$,
- ▷ $r(G_{16}) = x_{15}x_8x_{11}x_{14}x_1x_{10}x_5x_2x_{13} \cup x_9x_4x_{11}x_{16}x_7$ with $r(x_1) = x_{15}$ and $r(x_{13}) = x_9$,
- ▷ $s(G_{16}) = x_9x_{16}x_{13}x_6x_{11}x_{12}x_7x_{14}x_5 \cup x_3x_8x_{13}x_4x_{15}$ with $s(x_1) = x_9$ and $s(x_{13}) = x_3$.

To prove (17), let

- ▷ $g(G_{17}) = x_3x_6x_9x_{12}x_{13}x_8 \cup x_5x_{15}x_9x_{14}x_1 \cup x_9x_4x_7$ with $g(x_1) = x_3$ and $g(x_7) = x_5$,
- ▷ $r(G_{17}) = x_5x_8x_{11}x_{14}x_7x_{10} \cup x_9x_2x_{11}x_4x_{13} \cup x_{11}x_{15}x_1$ with $r(x_1) = x_5$ and $r(x_7) = x_9$,

▷ $s(G_{17}) = x_7x_{15}x_{13}x_{10}x_1x_{12} \cup x_{11}x_6x_{13}x_2x_5 \cup x_{13}x_{14}x_3$ with $s(x_1) = x_7$ and $s(x_7) = x_{11}$.

To prove (18), let

▷ $g(G_{18}) = x_3x_6x_9x_{12}x_{15}x_{16}x_{13} \cup x_5x_{14}x_9x_4x_7 \cup x_9x_{10}x_1$ with $g(x_1) = x_3$ and $g(x_{13}) = x_1$,

▷ $r(G_{18}) = x_7x_8x_{11}x_{14}x_1x_{12}x_5 \cup x_3x_{16}x_{11}x_6x_{13} \cup x_{11}x_2x_{15}$ with $r(x_1) = x_7$ and $r(x_{13}) = x_{15}$,

▷ $s(G_{18}) = x_{13}x_{14}x_{15}x_{10}x_7x_2x_9 \cup x_1x_6x_{15}x_8x_5 \cup x_{15}x_4x_{11}$ with $s(x_1) = x_{13}$ and $s(x_{13}) = x_{11}$.

5. Appendix (II): The proof of Lemma 2.6

For each case, we define the required 4-packing (b, g, r, s) with b as identity embedding as follows.

To prove (1), let

▷ $g(F_1) = x_5x_8x_{13}x_{10}x_7 \cup x_{12}x_{11}x_6x_1x_4 \cup x_{13}x_6$,

▷ $r(F_1) = x_{11}x_4x_7x_2x_{13} \cup x_8x_1x_{12}x_3x_{10} \cup x_7x_{12}$,

▷ $s(F_1) = x_{13}x_{12}x_9x_6x_3 \cup x_{10}x_5x_2x_{11}x_8 \cup x_9x_2$.

To prove (2), let

▷ $g(F_2) = x_3x_8x_1x_{10}x_{13} \cup x_4x_9x_6x_{15}x_2 \cup x_1x_{14}x_{11}x_6$,

▷ $r(F_2) = x_7x_{12}x_5x_{14}x_3 \cup x_2x_{11}x_4x_{13}x_{12} \cup x_5x_8x_{15}x_4$,

▷ $s(F_2) = x_9x_{14}x_{15}x_{12}x_1 \cup x_8x_{13}x_2x_5x_6 \cup x_{15}x_{10}x_7x_2$.

To prove (3), let

▷ $g(F_3) = x_3x_6x_9x_{12}x_{15} \cup x_8x_{13}x_{16}x_{11}x_{14} \cup x_9x_2x_{17}x_{10}x_5x_{16}$,

▷ $r(F_3) = x_5x_8x_{15}x_{16}x_{17} \cup x_{10}x_{11}x_2x_{13}x_4 \cup x_{15}x_6x_1x_{14}x_7x_2$,

▷ $s(F_3) = x_7x_4x_{17}x_6x_{11} \cup x_{12}x_5x_{14}x_{15}x_2 \cup x_{17}x_8x_3x_{16}x_9x_{14}$.

To prove (4), let

▷ $g(F_4) = x_{13}x_8x_5x_{16}x_9 \cup x_{12}x_{15}x_{10}x_3x_{14} \cup x_5x_6x_1x_4 \cup x_{10}x_7x_2x_{11}$,

▷ $r(F_4) = x_1x_{16}x_{11}x_6x_3 \cup x_4x_9x_2x_{15}x_8 \cup x_{11}x_{14}x_7x_{12} \cup x_2x_{13}x_{10}x_5$,

▷ $s(F_4) = x_{11}x_{10}x_1x_8x_3 \cup x_{14}x_{13}x_4x_7x_{16} \cup x_1x_{12}x_5x_2 \cup x_4x_{15}x_6x_9$.

To prove (5), let

▷ $g(F_5) = x_3x_{12}x_9x_6x_{13} \cup x_4x_1x_{14}x_5x_8 \cup x_9x_{14}x_7x_{10}$,

▷ $r(F_5) = x_5x_{10}x_{13}x_8x_1 \cup x_2x_{11}x_4x_7x_{12} \cup x_{13}x_4x_{15}x_{14}$,

▷ $s(F_5) = x_{11}x_6x_1x_{10}x_3 \cup x_8x_{15}x_{12}x_{13}x_{14} \cup x_1x_{12}x_5x_2$.

To prove (6), let

- ▷ $g(F_6) = x_3x_6x_9x_{12}x_{15} \cup x_2x_{13}x_{10}x_1x_{14} \cup x_9x_4x_{17}x_{10}x_5x_{16}$,
- ▷ $r(F_6) = x_5x_8x_{15}x_2x_{17} \cup x_{14}x_{11}x_4x_{13}x_{12} \cup x_{15}x_6x_1x_4x_7x_{10}$,
- ▷ $s(F_6) = x_7x_{14}x_{17}x_6x_{11} \cup x_2x_9x_{16}x_{15}x_4 \cup x_{17}x_8x_3x_{16}x_1x_{12}$.

To prove (7), let

- ▷ $g(F_7) = x_3x_6x_9x_{12}x_{13} \cup x_8x_{15}x_{18}x_{17}x_2 \cup x_9x_{14}x_1x_{10}x_{19}x_{18}x_7x_4$,
- ▷ $r(F_7) = x_5x_8x_{17}x_{16}x_{11} \cup x_4x_9x_2x_7x_{14} \cup x_{17}x_6x_{19}x_{12}x_{15}x_2x_{13}x_{10}$,
- ▷ $s(F_7) = x_9x_{18}x_{11}x_6x_{15} \cup x_{12}x_{17}x_4x_{13}x_{16} \cup x_{11}x_{10}x_3x_8x_1x_4x_{19}x_2$.

To prove (8), let

- ▷ $g(F_8) = x_3x_6x_9x_{12}x_{13} \cup x_8x_{15}x_{18}x_1x_{16} \cup x_{17}x_4x_{11}x_{18}x_5x_{10} \cup x_9x_2x_7x_{14}$,
- ▷ $r(F_8) = x_5x_8x_{17}x_6x_{11} \cup x_{12}x_1x_{10}x_3x_{18} \cup x_{13}x_2x_{15}x_{10}x_7x_{16} \cup x_{17}x_{14}x_9x_4$,
- ▷ $s(F_8) = x_{17}x_{10}x_{13}x_{18}x_9 \cup x_2x_{11}x_{16}x_5x_{14} \cup x_7x_{12}x_3x_{16}x_{15}x_6 \cup x_{13}x_4x_1x_8$.

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