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COMPACT OPERATORS AND INTEGRAL EQUATIONS IN THE $\mathcal{HK}$ SPACE

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Abstract. The space $\mathcal{HK}$ of Henstock-Kurzweil integrable functions on $[a,b]$ is the uncountable union of Fréchet spaces $\mathcal{HK}(X)$. In this paper, on each Fréchet space $\mathcal{HK}(X)$, an F-norm is defined for a continuous linear operator. Hence, many important results in functional analysis, like the Banach-Steinhaus theorem, the open mapping theorem and the closed graph theorem, hold for the $\mathcal{HK}(X)$ space. It is known that every control-convergent sequence in the $\mathcal{HK}$ space always belongs to a $\mathcal{HK}(X)$ space for some X. We illustrate how to apply results for Fréchet spaces $\mathcal{HK}(X)$ to control-convergent sequences in the $\mathcal{HK}$ space. Examples of compact linear operators are given. Existence of solutions to linear and Hammerstein integral equations is proved.

Keywords: compact operator; integral equation; controlled convergence; Henstock-Kurzweil integral

MSC 2020: 26A39, 26A42

1. Preliminaries

Let $\mathcal{HK}$ be the space of all Henstock-Kurzweil integrable functions defined on a compact interval $[a,b]$ of the real line. It is well-known that the $\mathcal{HK}$ space can be normed by the Alexiewicz norm $\|f\| = \sup\limits_{x\in[a,b]} |\int_a^x f(t)\,\mathrm{d}t|$. Unfortunately, it is not complete under this norm. Moreover, there is no natural Banach norm on the space $\mathcal{HK}$; for examples, see [8], [22]. In this paper we will overcome this shortcoming by considering subspaces $\mathcal{HK}(X)$, which are defined as follows:

Let $X = \{X_i\}$ be a sequence of closed subsets of $[a,b]$ such that $X_i \subset X_{i+1}$ and $a,b \in X_i$ for each $i \in \mathbb{N}$, while $[a,b] = \bigcup\limits_{i\in\mathbb{N}} X_i$. A function F is $AC^\star(X_i)$ if

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for each $\varepsilon > 0$ there is $\eta > 0$ such that for every partial partition $P = \{[u_j, v_j]\colon j = 1, 2, \dots, m\}$ of $[a, b]$ with u_j or $v_j \in X_i$ for each j, which satisfy $\sum\limits_{j=1}^{m} |v_j - u_j| < \eta$, we have $\sum\limits_{j=1}^{m} \omega(F; [u_j, v_j]) < \varepsilon$, where

$$\omega(F; [u_j, v_j]) = \sup\{|F(y) - F(x)|\colon\ x, y \in [u_j, v_j]\},$$

see [12], pages 27 and 32. Let $\mathbb{X}$ be the family of all X such that $X_i \subset X_{i+1}$ and $a, b \in X_i$, for each $i \in \mathbb{N}$, while $[a, b] = \bigcup\limits_{i \in \mathbb{N}} X_i$. For each $X \in \mathbb{X}$, let $\mathcal{HK}(X)$ be the space of all Henstock-Kurzweil integrable functions defined on $[a, b]$ such that its primitive function is $AC^\star(X_i)$ for each i. Clearly, $\mathcal{HK} = \bigcup\limits_{X \in \mathbb{X}} \mathcal{HK}(X)$. Suppose $f \in \mathcal{HK}(X)$ and $g \in \mathcal{HK}(Y)$. Then $f \in \mathcal{HK}(X) \subseteq \mathcal{HK}(X \cap Y)$ and $g \in \mathcal{HK}(Y) \subseteq \mathcal{HK}(X \cap Y)$, where $X \cap Y = \{X_i \cap Y_j\}_{i,j}$. Thus, $f, g \in \mathcal{HK}(X \cap Y)$.

For brevity, a partial division $P = \{[u_j, v_j]\colon\ j = 1, 2, \dots, m\}$ and $\sum\limits_{j=1}^{m} |F(v_j) - F(u_j)|$ in this paper are often written as $P = \{[u, v]\}$ and $\sum\limits_{P} |F(v) - F(u)|$, respectively.

Let a sequence $X = \{X_i\} \in \mathbb{X}$ be fixed. If $f \in \mathcal{HK}(X)$ and F is its primitive, then F is $AC^\star(X_i)$ for each $X_i \in X$ and

$$\|f\|_i^X = \sup_P \sum_P |F(v) - F(u)| = \sup_P \sum_P \left| \int_u^v f(t)\,\mathrm{d}t \right| < \infty,$$

where the supremum is taken over all partial partitions $P = \{[u, v]\}$ of $[a, b]$ with $u \in X_i$ or $v \in X_i$ for each subinterval $[u, v] \in P$, see [5], [22]. Thus, the sequence of semi-norms $\{\|f\|_i^X\}$ with $f \in \mathcal{HK}(X)$ is increasing since $X_i \subset X_{i+1}$ for all i. Define

$$\|f\|^X = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f\|_i^X}{1 + \|f\|_i^X}.$$

Then $\|\cdot\|^X$ is an F-norm and $\mathcal{HK}(X)$ is a metrisable locally convex space generated by the increasing sequence of norms $\{\|\cdot\|_i^X\}$, see [9], pages 202–205. In this paper, the space $\mathcal{HK}(X)$ is always supposed to be equipped with F-norm $\|\cdot\|^X$. We also denote the Alexiewicz norm of f by $\|f\| = \sup\limits_{x \in [a,b]} |\int_a^x f(t)\,\mathrm{d}t|$. Recall $a \in X_i$. Hence, $\|f\| \leqslant \|f\|_i^X$. Thus, $\|f\|_i^X$ in fact is a norm.

Definition 1.1 ([12], page 39). A sequence of functions $\{f_n\}$ is said to be *control-convergent* to f on $[a, b]$ if the following conditions are satisfied:

(i) $f_n(x) \to f(x)$ almost everywhere in $[a, b]$ as $n \to \infty$, where each f_n is Henstock integrable on $[a, b]$.

(ii) The primitives F_n of f_n are $ACG^\star$ uniformly in n, i.e., $[a,b]$ is the union of a sequence of closed sets X_i such that on each X_i the function F_n is $AC^\star(X_i)$ uniformly in n, in other words, $\eta > 0$ in the definition of $AC^\star(X_i)$ is independent of n.

(iii) The primitives F_n converge uniformly on $[a,b]$.

The following four theorems (Theorems 1.2–1.5) are given in [5]. However, these four theorems are proved for the space of Denjoy integrable functions. The Denjoy and $\mathcal{HK}$ integrals are equivalent, see [12], Theorems 6.12 and 6.13. Therefore, these four theorems also hold for the space $\mathcal{HK}$. In [5], the space of Denjoy integrable functions on $[a,b]$ is denoted by D and $X = \{X_i\}$ is denoted by Δ. Furthermore, in [5], page 520, if $f \in D(\Delta)$ and F is its primitive, then F is assumed to have bounded variation (in the restricted sense) on each X_i. In fact, in [5], page 518, F is $AC^\star(X_i)$ for each X_i. In this paper, we assume F is $AC^\star(X_i)$ instead of F being of bounded variation on each X_i.

Theorem 1.2 ([5], Theorem 3.1)**.** *A sequence $\{f_n\}$ is Cauchy (or convergent to f) in the space $\mathcal{HK}(X)$ with the norm $\|\cdot\|^X$ if and only if $\{f_n\}$ is Cauchy (or convergent to f) in the space $\mathcal{HK}(X)$ with the norm $\|\cdot\|_i^X$ for all i.*

Theorem 1.3 ([5], Theorem 3.3 (a))**.** *If a sequence $\{f_n\}$ is $\|\cdot\|^X$-convergent to f in the $\mathcal{HK}(X)$ space, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which is control-convergent to f in the $\mathcal{HK}$ space.*

Theorem 1.4 ([5], Theorem 3.3 (b))**.** *If a sequence $\{f_n\}$ is control-convergent to f in the $\mathcal{HK}$ space, then there are $X \in \mathbb{X}$ and $f \in \mathcal{HK}(X)$ such that $\{f_n\} \subseteq \mathcal{HK}(X)$ and $\{f_n\}$ tends to f in $\mathcal{HK}(X)$.*

Theorem 1.5 ([5], Theorem 3.4)**.** *The space $\mathcal{HK}(X)$ is complete under $\|\cdot\|^X$.*

2. Continuous linear operators in $\mathcal{HK}(X)$ and $\mathcal{HK}$ spaces

Let $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ be linear. Suppose T is continuous under $\|\cdot\|_i^X$ for each i. Define

$$\|T\|_i^X = \sup\Big\{\frac{\|T\varphi\|_i^X}{\|\varphi\|_i^X}\colon \varphi \in \mathcal{HK}(X)\Big\} \quad \text{and} \quad \|T\|^X = \sum_{i=1}^{\infty} \frac{1}{2^i}\frac{\|T\|_i^X}{1+\|T\|_i^X}.$$

Thus, $\|T\|^X$ is an F-norm of T. Note that $\|T\varphi\|_i^X \leqslant \|T\|_i^X\|\varphi\|_i^X$ for all φ if T is $\|\cdot\|_i^X$-continuous. The operator T is continuous under $\|\cdot\|^X$ if and only if T is continuous under $\|\cdot\|_i^X$ for each i. However, the inequality

$$\|T\varphi\|^X \leqslant \|T\|^X\|\varphi\|^X$$

may not be true. Because $\|\cdot\|^X$ is not a norm, it is an F-norm and $\|\alpha\varphi\|^X \neq |\alpha|\|\varphi\|^X$. In this paper, to prove results for $\|\cdot\|^X$, we always prove the corresponding results for norms $\|\cdot\|_i^X$ for each i first. Let $B(\mathcal{HK}(X))$ be the space of all continuous operators from $\mathcal{HK}(X)$ to $\mathcal{HK}(X)$ with F-norm $\|\cdot\|^X$. Let $(\mathcal{HK}(X))^\star$ be the space of continuous linear functionals defined on $\mathcal{HK}(X)$.

Theorem 2.1. *$B(\mathcal{HK}(X))$ and $(\mathcal{HK}(X))^\star$ are complete.*

P r o o f. The proof is standard, see [18], page 221, Proposition 3. $\square$

It is known that control-convergence is one of the best convergences in the $\mathcal{HK}$ space. On the other hand, from Theorems 1.3 and 1.4, control-convergence is related to $\|\cdot\|^X$-convergence. So we define the continuity of an operator $T\colon \mathcal{HK} \to \mathcal{HK}$ as follows. Let $T\colon \mathcal{HK} \to \mathcal{HK}$. Then T is said to be *control-continuous* if $\{T(\varphi_n)\}$ is control-convergent to $T\varphi$ whenever $\{\varphi_n\}$ is control-convergent to φ in the $\mathcal{HK}$ space. From Theorems 1.3 and 1.4, we have:

Theorem 2.2. *Let $T\colon \mathcal{HK} \to \mathcal{HK}$ be linear and $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ for all $X \in \mathbb{X}$. Then $T\colon \mathcal{HK} \to \mathcal{HK}$ is control-continuous if and only if for each $X \in \mathbb{X}$, $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is $\|\cdot\|^X$-continuous.*

Let $T\colon \mathcal{HK} \to (B, \|\cdot\|_B)$, where $(B, \|\cdot\|_B)$ is a Banach space. The operator T is said to be control-continuous if $\|T\varphi_n - T\varphi\|_B \to 0$ as $n \to \infty$ for each $\{\varphi_n\}$ which control-converges to φ in the space $\mathcal{HK}$. By Theorems 1.3 and 1.4, we have:

Theorem 2.3. *Let $T\colon \mathcal{HK} \to (B, \|\cdot\|_B)$ be linear. Then $T\colon \mathcal{HK} \to (B, \|\cdot\|_B)$ is control-continuous if and only if for each $X \in \mathbb{X}$, $T\colon \mathcal{HK}(X) \to (B, \|\cdot\|_B)$ is continuous.*

Theorem 2.4 (Banach-Steinhaus Theorem)**.** *For each n, let $T_n\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ for all $X \in \mathbb{X}$ and $T_n\colon \mathcal{HK} \to \mathcal{HK}$ be linear and control-continuous. If for each $\varphi \in \mathcal{HK}$, $\{T_n\varphi\}$ is control-convergent to $T\varphi$ in the $\mathcal{HK}$ space, then $T\colon \mathcal{HK} \to \mathcal{HK}$ is a control-continuous linear operator.*

P r o o f. First, we shall show that T is continuous in each Fréchet space $\mathcal{HK}(X)$ under the F-norm $\|\cdot\|^X$, by the classical Banach-Steinhaus Theorem for Fréchet spaces. Note that $T_n\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ for each n. Let $\varphi \in \mathcal{HK}(X)$. The sequence $\{T_n\varphi\}$ is control-convergent to $T\varphi$ in the $\mathcal{HK}(X)$ space. By Theorem 1.4, for each $\varphi \in \mathcal{HK}(X)$, $\|T_n\varphi - T\varphi\|^X \to 0$ as $n \to \infty$. By the classical Banach-Steinhaus Theorem, $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is $\|\cdot\|^X$-continuous. By Theorem 2.2, $T\colon \mathcal{HK} \to \mathcal{HK}$ is control-continuous. $\square$

By Theorem 2.2 and the classical open mapping theorem for Fréchet spaces we have:

Theorem 2.5 (Open Mapping Theorem). *Let $T\colon \mathcal{HK} \to \mathcal{HK}$ be a control-continuous linear operator and for each $X \in \mathbb{X}$, $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ be surjective. Then T is an open mapping in the sense that for each $X \in \mathbb{X}$, $T(G)$ is open in $\mathcal{HK}(X)$ whenever G is open in $\mathcal{HK}(X)$.*

By Theorems 2.2 and 2.5 we have:

Theorem 2.6 (Bounded Inverse Theorem). *Let $T\colon \mathcal{HK} \to \mathcal{HK}$ be a control-continuous linear operator and for each $X \in \mathbb{X}$, $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ be bijective. Then $T^{-1}\colon \mathcal{HK} \to \mathcal{HK}$ is control-continuous.*

Theorem 2.7 (Closed Graph Theorem). *Let $T\colon \mathcal{HK} \to \mathcal{HK}$ and $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ for each $X \in \mathbb{X}$. Suppose that T has the property that whenever $\{\varphi_n\}$ is control-convergent to φ in $\mathcal{HK}$ and $T\varphi_n$ is control-convergent to ψ, $T\varphi = \psi$. Then T is control-continuous.*

Proof. We shall apply the classical closed graph theorem to the $\mathcal{HK}(X)$ space. Let $\|f_n - f\|^X \to 0$ and $\|Tf_n - g\|^X \to 0$ as $n \to \infty$. Then by Theorem 1.3, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ is control-convergent to f and $\{Tf_{n_k}\}$ is control-convergent to g. By the given condition, $Tf = g$. By the classical closed graph theorem, $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is $\|\cdot\|^X$-continuous. By Theorem 2.2, $T\colon \mathcal{HK} \to \mathcal{HK}$ is control-continuous. $\square$

Remark 2.8. The above four theorems (Theorems 2.4–2.7) hold true for linear operators from the $\mathcal{HK}$ space to a Banach space $(B, \|\cdot\|_B)$. For example, by Theorem 2.3, we have:

Theorem 2.9 (Banach-Steinhaus Theorem). *For each n, let $T_n\colon \mathcal{HK} \to (B, \|\cdot\|_B)$ be a control-continuous linear operator. Suppose for each $\varphi \in \mathcal{HK}$, $\|T_n\varphi - T\varphi\|_B \to 0$ as $n \to \infty$. Then $T\colon \mathcal{HK} \to (B, \|\cdot\|_B)$ is control-continuous.*

Topology for the $\mathcal{HK}$ space has been discussed in [1], [3], [5], [8], [11], [13], [14], [17], [19], [22]. Let $BV[a,b]$ be the space of functions of bounded variation on $[a,b]$. If $g \in BV[a,b]$, then $V(g)$ denotes the total variation of g on $[a,b]$. Define $\|g\|_{BV} = V(g) + |g(b)|$. It is known that if $f \in \mathcal{HK}$ and $g \in BV[a,b]$, then $|\int_a^b f(x)g(x)\,\mathrm{d}x| \leqslant \|f\|\|g\|_{BV}$, see [12], page 74, where $\|f\| = \sup\limits_{x\in[a,b]} |\int_a^x f(t)\,\mathrm{d}t|$.

Recall that if $\varphi \in \mathcal{HK}(X)$, $\|\varphi\| \leqslant \|\varphi\|_i^X$ for all i. By Theorem 1.2 and the above inequality, $\|\cdot\|^X$-convergence implies $\|\cdot\|$-convergence. Thus, if $G\colon \mathcal{HK} \to \mathbb{R}$ is a $\|\cdot\|$-continuous linear functional, then G is $\|\cdot\|^X$-continuous. Hence, G is control-continuous by Theorem 1.4. The converse is also true, see [12], page 103. Hence, the norm continuity and control-continuity of a linear functional on the $\mathcal{HK}$ spaces are equivalent.

Example 2.10. Let $T_n\colon \mathcal{HK} \to \mathbb{R}$ be defined by $T_n(\varphi) = \int_a^b g_n(x)\varphi(x)\,\mathrm{d}x$, where $g_n \in BV[a,b]$. Then T_n is control-continuous. Suppose $\lim\limits_{n\to\infty} \int_a^b g_n(x)\varphi(x)\,\mathrm{d}x$ exists. Then by Theorem 2.9, $T\colon \mathcal{HK} \to \mathbb{R}$ defined by $T\varphi = \lim\limits_{n\to\infty} \int_a^b g_n(x)\varphi(x)\,\mathrm{d}x$ is control-continuous.

Example 2.10 is also given in [12], pages 70–71. However, the Banach-Steinhaus Theorem is applied to a Sargent space.

Example 2.11. Let $\{g_i\}$ be a sequence of functions in $BV[a,b]$ and $\{h_i\}$ a sequence of functions in $\mathcal{HK}$. Let $T_n\colon \mathcal{HK} \to \mathcal{HK}$ be defined by $(T_n\varphi)(x) = \sum\limits_{i=1}^{n} \int_a^b (h_i(x)g_i(t))\varphi(t)\,\mathrm{d}t$. Then each $T_n\colon \mathcal{HK} \to \mathcal{HK}$ is a linear operator of finite rank. Hence, T_n is control-continuous. Suppose $\{T_n\varphi\}$ is control-convergent to $T\varphi$ in $\mathcal{HK}$ for each $\varphi \in \mathcal{HK}$. Then by Theorem 2.4, $T\colon \mathcal{HK} \to \mathcal{HK}$ is control-continuous. We shall discuss the compactness of T in the next section.

Linear operators in the $\mathcal{HK}$ space have also been discussed in [6], [7], [15], [21], [23].

3. Compact operators in the $\mathcal{HK}(X)$ space

A sequence $\{\varphi_n\}$ in $\mathcal{HK}(X)$ is said to be bounded if $\{\varphi_n\}$ is bounded under $\|\cdot\|_i^X$ for each i. Let $B \subseteq \mathcal{HK}(X)$. The *set* B is said to be *compact* if for any bounded sequence in B there exists a $\|\cdot\|^X$-convergent subsequence.

An *operator* $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is said to be *compact* if for any bounded sequence $\{\varphi_n\}$ in $\mathcal{HK}(X)$ there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{T\varphi_{n_k}\}$ is convergent in $\mathcal{HK}(X)$.

Using subsequence argument as in Banach spaces, if $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is linear and compact, then T is $\|\cdot\|^X$-continuous.

The rank of an operator is the dimension of its range. It is well-known that every finite rank continuous linear operator acting between Banach spaces is compact. The space $\mathcal{HK}(X)$ is not a Banach space. It is a Fréchet space. Similar result holds for continuous linear operators of finite rank in Fréchet spaces, see [20], page 98. However, in [20], the result is for locally convex spaces. In the following Example 3.1, we shall give a proof for easy reference.

Example 3.1. Let $X \in \mathbb{X}$ be arbitrary, $h_j \in \mathcal{HK}(X)$ and $g_j \in BV[a,b]$ for $j = 1, 2, \ldots, n$. Suppose $K\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is defined by

$$(K\varphi)(x) = \int_a^b \Big(\sum_{j=1}^n h_j(x) g_j(t)\Big)\varphi(t)\,\mathrm{d}t.$$

Then

$$(K\varphi)(x) = \int_a^b \Big(\sum_{j=1}^n h_j(x) g_j(t)\Big)\varphi(t)\,\mathrm{d}t = \sum_{j=1}^n h_j(x)\int_a^b g_j(t)\varphi(t)\,\mathrm{d}t = \sum_{j=1}^n h_j(x)\alpha_j,$$

where $\alpha_j = \int_a^b g_j(t)\varphi(t)\,\mathrm{d}t \in \mathbb{R}$. Thus, K is a linear operator of finite rank.

Next, we shall prove that K is compact. Let $\{\varphi_k\}_{k=1}^\infty$ be a bounded sequence in $\mathcal{HK}(X)$. Hence, $\{\|\varphi_k\|_i^X\}_{k=1}^\infty$ is bounded in $\mathbb{R}$ for each i. Then for each fixed i,

$$|\alpha_{j,k}| = \left|\int_a^b g_j(t)\varphi_k(t)\,\mathrm{d}t\right| \leqslant \|\varphi_k\| V(g_j) \leqslant \|\varphi_k\|_i^X V(g_j),$$

i.e., $\{\alpha_{j,k}\}_{k=1}^\infty$ is bounded in $\mathbb{R}$ for $j = 1, 2, \ldots, n$. By the Bolzano-Weierstrass theorem, there exists a subsequence $\{\alpha_{j,k_l}\}$ of $\{\alpha_{j,k}\}_{k=1}^\infty$, converging in $\mathbb{R}$ for $j = 1, 2, \ldots, n$. Thus,

$$\|K\varphi_{k_p} - K\varphi_{k_q}\|_i^X = \left\|\sum_{j=1}^n h_j\alpha_{j,k_p} - \sum_{j=1}^n h_j\alpha_{j,k_q}\right\|_i^X \leqslant \sum_{j=1}^n \|h_j\|_i^X |\alpha_{j,k_p} - \alpha_{j,k_q}|.$$

Hence, $\{K\varphi_{k_l}\}$ is a Cauchy sequence in $\mathcal{HK}(X)$ under $\|\cdot\|_i^X$. Therefore, there exists $\psi \in \mathcal{HK}(X)$ such that $\|K\varphi_{k_l} - K\psi\|_i^X \to 0$ as $k_l \to \infty$. Note that $\|\cdot\| \leqslant \|\cdot\|_i^X$ for each i. Thus, ψ is independent of i. Hence, $\|K\varphi_{k_l} - K\psi\|^X \to 0$ as $k_l \to \infty$, i.e., K is compact. Therefore, K is $\|\cdot\|^X$-continuous. We remark that we can use the same idea to prove that K is $\|\cdot\|^X$-continuous without using the fact that compactness implies continuity.

Next we shall prove a result for a countably infinite dimensional rank.

Lemma 3.2. *Let $X \in \mathbb{X}$ be arbitrary, $\sum\limits_{j=1}^\infty \|g_j\|_{BV} < \infty$ and $\{h_j\}$ a sequence of functions such that for each x, $|h_j(x)| \leqslant A(x) < \infty$ for all j and $\sum\limits_{j=1}^\infty h_j(x)g_j(t)$ exists for any $x, t \in [a,b]$. Then for each $x \in [a,b]$, $\sum\limits_{j=1}^\infty h_j(x)g_j(t) \in BV[a,b]$ and for each $\varphi \in \mathcal{HK}$,*

$$\int_a^b \Big(\sum_{j=1}^\infty h_j(x)g_j(t)\Big)\varphi(t)\,\mathrm{d}t = \sum_{j=1}^\infty h_j(x)\int_a^b g_j(t)\varphi(t)\,\mathrm{d}t.$$

P r o o f. Let $m, n \in \mathbb{N}$ be fixed. For any fixed x we have

$$V\Bigl(\sum_{j=m}^{n} h_j(x)g_j(t)\Bigr) \leqslant \sum_{j=m}^{n} |h_j(x)| V(g_j(t)) \leqslant A(x) \sum_{j=m}^{n} V(g_j).$$

Then

$$\begin{aligned} V\Bigl(\sum_{j=m}^{\infty} h_j(x)g_j(t)\Bigr) &\leqslant \lim_{n\to\infty} V\Bigl(\sum_{j=m}^{n} h_j(x)g_j(t)\Bigr) \\ &\leqslant A(x) \lim_{n\to\infty} \sum_{j=m}^{n} V(g_j) \\ &= A(x) \sum_{j=m}^{\infty} V(g_j). \end{aligned}$$

Thus, $V(\sum\limits_{j=m}^{\infty} h_j(x)g_j(t)) \to 0$ as $m \to \infty$. Hence, for each $x \in [a,b]$,

$$V\Bigl(\sum_{j=1}^{\infty} h_j(x)g_j(t)\Bigr) < \infty.$$

Therefore, $\sum\limits_{j=1}^{\infty} h_j(x)g_j(t) \in BV[a,b]$ for each $x \in [a,b]$. We remark that this result, in fact, is a consequence of the completeness of $BV[a,b]$, see [15], page 14. Notice that

$$\begin{aligned} \Bigl|\int_a^b \Bigl(\sum_{j=n+1}^{\infty} h_j(x)g_j(t)\Bigr)\varphi(t)\,\mathrm{d}t\Bigr| &\leqslant \|\varphi\| \Bigl(V\Bigl(\sum_{j=n+1}^{\infty} h_j(x)g_j(t)\Bigr) + \sum_{j=n+1}^{\infty} h_j(b)g_j(b)\Bigr) \\ &< \|\varphi\| \Bigl(V\Bigl(\sum_{j=n+1}^{\infty} h_j(x)g_j(t)\Bigr) + A(b) \sum_{j=n+1}^{\infty} |g_j(b)|\Bigr). \end{aligned}$$

Hence, $\Bigl|\int_a^b \Bigl(\sum\limits_{j=n+1}^{\infty} h_j(x)g_j(t)\Bigr)\varphi(t)\,\mathrm{d}t\Bigr| \to 0$ as $n \to \infty$. Observe that

$$\int_a^b \sum_{j=1}^{\infty} h_j(x)g_j(t)\varphi(t)\,\mathrm{d}t - \sum_{j=1}^{n} \int_a^b h_j(x)g_j(t)\varphi(t)\,\mathrm{d}t = \int_a^b \Bigl(\sum_{j=n+1}^{\infty} h_j(x)g_j(t)\Bigr)\varphi(t)\,\mathrm{d}t.$$

So

$$\lim_{n\to\infty} \int_a^b \Bigl(\sum_{j=1}^{n} h_j(x)g_j(t)\Bigr)\varphi(t)\,\mathrm{d}t = \int_a^b \Bigl(\sum_{j=1}^{\infty} h_j(x)g_j(t)\Bigr)\varphi(t)\,\mathrm{d}t.$$

□

The following lemma is proved by Abel's transformation, see [2], Chapter page 365.

Lemma 3.3. *Let $\{a_j\}$ be a sequence in a normed spaces S with norm $\|\cdot\|_S$ and $\{b_j\}$ a real-valued sequence such that $\sum\limits_{j=1}^{\infty} a_j$ and $\sum\limits_{j=1}^{\infty} |b_{j+1} - b_j|$ exist. Then $\lim\limits_{j\to\infty} b_j$, $\sum\limits_{j=1}^{\infty} a_j b_j$ exist and for each $m = 1, 2, \ldots$,*

$$\left\|\sum_{j=m}^{\infty} a_j b_j\right\|_S \leqslant 2A \sum_{j=m}^{\infty} |b_{j+1} - b_j| + \left\|\sum_{k=m}^{\infty} a_k\right\|_S |b|, \tag{3.1}$$

where $A = \sup\limits_n \|\sum\limits_{j=1}^{n} a_j\|_S$ and $b = \lim\limits_{j\to\infty} b_j$.

Lemma 3.4. *Let $X \in \mathbb{X}$ be arbitrary, $\sum\limits_{j=1}^{\infty} \|g_j\|_{BV} < \infty$ and $\{h_j\}$ a sequence of functions in $\mathcal{HK}(X)$ with $\sum\limits_{j=1}^{\infty} h_j \in \mathcal{HK}(X)$.*

(i) *Then for each x, there exists $0 < A(x) < \infty$ such that $|h_j(x)| \leqslant A(x)$ for each j and $\sum\limits_{j=1}^{\infty} h_j(x) g_j(t)$ exists for any $x, t \in [a, b]$.*

(ii) *Let $\varphi \in \mathcal{HK}(X)$ and $\alpha_j = \int_a^b g_j(t)\varphi(t)\,\mathrm{d}t$. Then $\sum\limits_{j=1}^{\infty} |\alpha_{j+1} - \alpha_j| < \infty$ and $\sum\limits_{j=1}^{\infty} h_j \alpha_j \in \mathcal{HK}(X)$.*

Proof. (i) By given condition, $\sum\limits_{j=1}^{\infty} h_j$ exists. Hence, there exists $0 < A(x) < \infty$ such that $|h_j(x)| \leqslant A(x)$ for each j. Applying Lemma 3.3 to two real-valued sequences with $a_j = h_j(x)$ and $b_j = g_j(t)$ we have that $\sum\limits_{j=1}^{\infty} h_j(x) g_j(t)$ exists for any $x, t \in [a, b]$.

(ii) First,

$$\begin{aligned} |\alpha_{j+1} - \alpha_j| &= \left|\int_a^b (g_{j+1}(t) - g_j(t))\varphi(t)\,\mathrm{d}t\right| \\ &\leqslant (V(g_{j+1}) + |g_{j+1}(b)| + V(g_j) + |g_j(b)|)\|\varphi\| \\ &= (\|g_{j+1}\|_{BV} + \|g_j\|_{BV})\|\varphi\|. \end{aligned}$$

Hence, $\sum\limits_{j=1}^{\infty} |\alpha_{j+1} - \alpha_j| < \infty$, i.e., $\{\alpha_j\}$ is of bounded variation. For each $x \in [a, b]$, applying Lemma 3.3 to two real-valued sequences, $a_j = h_j(x)$ and $b_j = \alpha_j$, we have that $\sum\limits_{j=1}^{\infty} h_j(x)\alpha_j$ exists.

Now apply Lemma 3.3 to a real-valued sequence $\{\alpha_j\}$ and a sequence $\{h_j(x)\}$ in the normed space with norm $\|\cdot\|_i^X$. We have that $\sum\limits_{j=1}^{\infty} h_j\alpha_j$ exists under norm $\|\cdot\|_i^X$.

Thus, $\Big\{\sum\limits_{j=1}^{n} h_j\alpha_j\Big\}$ is Cauchy under $\|\cdot\|_i^X$ for each i. Hence, $\Big\{\sum\limits_{j=1}^{n} h_j\alpha_j\Big\}$ is Cauchy under $\|\cdot\|^X$. Therefore, there exists $q \in \mathcal{HK}(X)$ such that $\Big\|\sum\limits_{j=1}^{n} h_j\alpha_j - q\Big\|^X \to 0$ as $n \to \infty$.

By Theorem 1.3, there exists a subsequence of $\Big\{\sum\limits_{j=1}^{n} h_j\alpha_j\Big\}$ which converges pointwise to q almost everywhere. However, for each x, $\sum\limits_{j=1}^{\infty} h_j(x)\alpha_j$ exists. Thus, for almost all x, $q(x) = \sum\limits_{j=1}^{\infty} h_j(x)\alpha_j$. Hence, $\Big\|\sum\limits_{j=n}^{\infty} h_j\alpha_j\Big\|^X \to 0$ as $n \to \infty$. Therefore, $\sum\limits_{j=1}^{\infty} h_j\alpha_j \in \mathcal{HK}(X)$. $\square$

Theorem 3.5. *Let $X \in \mathbb{X}$ be arbitrary, $\varphi \in \mathcal{HK}(X)$, $\sum\limits_{j=1}^{\infty} \|g_j\|_{BV} < \infty$ and $\sum\limits_{j=1}^{\infty} h_j \in \mathcal{HK}(X)$. Let K, $K_n\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ be linear operators defined by*

$$(K_n\varphi)(x) = \int_a^b \Big(\sum_{j=1}^{n} h_j(x)g_j(t)\Big)\varphi(t)\,\mathrm{d}t$$

and

$$(K\varphi)(x) = \int_a^b \Big(\sum_{j=1}^{\infty} h_j(x)g_j(t)\Big)\varphi(t)\,\mathrm{d}t.$$

Then $\|K_n - K\|^X \to 0$ as $n \to \infty$. Furthermore, K is $\|\cdot\|^X$-continuous and compact.

Proof. By Lemmas 3.2 and 3.4 (i) for each $x, t \in [a,b]$, $\sum\limits_{j=1}^{\infty} h_j(x)g_j(t)$ exists and

$$K\varphi(x) = \int_a^b \Big(\sum_{j=1}^{\infty} h_j(x)g_j(t)\Big)\varphi(t)\,\mathrm{d}t = \sum_{j=1}^{\infty} h_j(x)\int_a^b g_j(t)\varphi(t)\,\mathrm{d}t = \sum_{j=1}^{\infty} h_j(x)\alpha_j,$$

where $\alpha_j = \int_a^b g_j(t)\varphi(t)\,\mathrm{d}t$. Thus, $(K - K_n)\varphi(x) = \sum\limits_{j=n+1}^{\infty} h_j(x)\alpha_j$.

Hence, $\|(K - K_n)\varphi\|_i^X = \Big\|\sum\limits_{j=n+1}^{\infty} h_j\alpha_j\Big\|_i^X$ for each i. By Lemma 3.4 (ii),

$$\|(K - K_n)(\varphi)\|^X \to 0 \quad \text{as} n \to \infty.$$

By the Banach-Steinhaus theorem, K is $\|\cdot\|^X$-continuous. Now, we shall prove that $\|K_n - K\|^X \to 0$ as $n \to \infty$. Recall that $\{\alpha_j\}$ is of bounded variation, i.e.,

$\sum\limits_{j=1}^{\infty} |\alpha_{j+1} - \alpha_j| < \infty$. Let $\alpha = \lim\limits_{n\to\infty} \alpha_n$. Apply Lemma 3.3 to $\{a_j\}$, where $a_j = h_j$ with norm $\|\cdot\|_i^X$ and $b_j = \alpha_j$, use inequality (3.1). We have

$$\begin{aligned}
\|(K-K_n)\varphi\|_i^X &= \Bigl\|\sum_{j=n+1}^{\infty} h_j\alpha_j\Bigr\|_i^X \\
&\leqslant 2\sup_j \Bigl\|\sum_{k=1}^{j} h_k\Bigr\|_i^X \sum_{j=n+1}^{\infty} |\alpha_j - \alpha_{j+1}| + \Bigl\|\sum_{k=n+1}^{\infty} h_k\Bigr\|_i^X |\alpha| \\
&\leqslant 2\sup_j \Bigl\|\sum_{k=1}^{j} h_k\Bigr\|_i^X \Bigl(\sum_{j=n+1}^{\infty} (\|g_{j+1}\|_{BV} + \|g_j\|_{BV})\Bigr) \|\varphi\|_i^X \\
&\quad + \Bigl\|\sum_{k=n+1}^{\infty} h_k\Bigr\|_i^X \Bigl(\sum_{j=1}^{\infty} \|g_j\|_{BV}\Bigr) \|\varphi\|_i^X.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(K-K_n)\|_i^X &= \sup_{\varphi} \frac{\|(K-K_n)\varphi\|_i^X}{\|\varphi\|_i^X} \\
&\leqslant 2\sup_j \Bigl\|\sum_{k=1}^{j} h_k\Bigr\|_i^X \sum_{j=n+1}^{\infty} (\|g_{j+1}\|_{BV} + \|g_j\|_{BV}) \\
&\quad + \Bigl\|\sum_{k=n+1}^{\infty} h_k\Bigr\|_i^X \Bigl(\sum_{j=1}^{\infty} \|g_j\|_{BV}\Bigr).
\end{aligned}$$

Hence, $\|K_n - K\|_i^X \to 0$ as $n \to \infty$ for each i. Therefore, $\|K_n - K\|^X \to 0$ as $n \to \infty$. Note that each K_n is a linear operator of finite rank. Hence, each K_n is compact. Thus, K is compact, i.e., for any bounded sequence $\{\varphi_n\}$ in $\mathcal{HK}(X)$ there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{K\varphi_{n_k}\}$ is convergent in $\mathcal{HK}(X)$ under $\|\cdot\|^X$. $\square$

4. Compact operators in the $\mathcal{HK}$ space

Let $T\colon \mathcal{HK} \to \mathcal{HK}$ and $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ for each $X \in \mathbb{X}$. The operator $T\colon \mathcal{HK} \to \mathcal{HK}$ is said to be compact if $T\colon \mathcal{HK}(X) \to \mathcal{HK}(X)$ is compact for each $X \in \mathbb{X}$. By Theorems 1.2 and 1.3, $T\colon \mathcal{HK} \to \mathcal{HK}$ is compact if for any fixed $X \in \mathbb{X}$ for any bounded sequence $\{\varphi_n\}$ under $\|\cdot\|^X$, there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{T\varphi_{n_k}\}$ is control-convergent. We remark that the space $\mathcal{HK}$ is the uncountable union of Fréchet spaces $\mathcal{HK}(X)$. In this paper we are unable to define a suitable F-norm on the space $\mathcal{HK}$. However, it is known that every control-convergent sequence in the space $\mathcal{HK}$ always belongs to a $\mathcal{HK}(X)$ space for some $X \in \mathbb{X}$. Therefore, we define the compactness of an operator $T\colon \mathcal{HK} \to \mathcal{HK}$ in the above way.

Theorem 4.1. *Let $X \in \mathbb{X}$ be arbitrary, $\sum\limits_{j=1}^{\infty} \|g_j\|_{BV} < \infty$, $\sum\limits_{j=1}^{\infty} h_j \in \mathcal{HK}(X)$ and $\Big\|\sum\limits_{j=n}^{\infty} h_j\Big\|_i^X \to 0$ as $n \to \infty$ for each i. Let $k(x,t) = \sum\limits_{j=1}^{\infty} h_j(x) g_j(t)$ and $K\colon \mathcal{HK} \to \mathcal{HK}$ be defined by*

$$(K\varphi)(x) = \int_a^b k(x,t)\varphi(t)\,\mathrm{d}t = \int_a^b \Big(\sum_{j=1}^{\infty} h_j(x) g_j(t)\Big)\varphi(t)\,\mathrm{d}t.$$

Then K is compact in the $\mathcal{HK}$ space.

Proof. Let $Y \in \mathbb{X}$ be fixed and $\varphi \in \mathcal{HK}(Y)$. Then $\varphi \in \mathcal{HK}(Y) \subseteq \mathcal{HK}(X \cap Y)$, $\{h_j\} \subseteq \mathcal{HK}(X) \subseteq \mathcal{HK}(X \cap Y)$ and $\sum\limits_{j=1}^{\infty} h_j \in \mathcal{HK}(X) \subseteq \mathcal{HK}(X \cap Y)$. Note that $\|\cdot\|_i^{X\cap Y} \leqslant \|\cdot\|_i^X$ for each i. Hence, $\Big\|\sum\limits_{j=n}^{\infty} h_j\Big\|_i^{X\cap Y} \to 0$ as $n \to \infty$ for each i. Applying Theorem 3.5 to the $\mathcal{HK}(X \cap Y)$ space, $K\varphi$ is well-defined and K is compact in the $\mathcal{HK}(X \cap Y)$ space. Let $\{\varphi_n\}$ be a bounded sequence in $\mathcal{HK}(Y)$. Then $\{\varphi_n\} \subseteq \mathcal{HK}(Y) \subseteq \mathcal{HK}(X\cap Y)$ and $\{\varphi_n\}$ is bounded in $(\mathcal{HK}(X\cap Y), \|\cdot\|^{X\cap Y})$. Therefore, $\{K\varphi_n\}$ has a subsequence which is convergent in $\mathcal{HK}(X \cap Y)$ under $\|\cdot\|^{X\cap Y}$. Thus, by Theorem 1.3, there exists a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ such that $\{K\varphi_{n_k}\}$ is control-convergent. We have proved that for any fixed $Y \in \mathbb{X}$, for any bounded sequence $\{\varphi_n\}$ in $(\mathcal{HK}(Y), \|\cdot\|^Y)$ there exists a subsequence $\{\varphi_{n_k}\}$ such that $\{K\varphi_{n_k}\}$ is control-convergent. Hence, K is compact in the $\mathcal{HK}$ space. $\square$

By Theorems 1.4 and 4.1 we have:

Corollary 4.2. *Let $\sum\limits_{j=1}^{\infty} \|g_j\|_{BV} < \infty$. Let $\{h_i\}$ be a sequence of functions in $\mathcal{HK}$ and $\Big\{\sum\limits_{j=1}^{n} h_j\Big\}_{n=1}^{\infty}$ be control-convergent to $\sum\limits_{j=1}^{\infty} h_j$. Let $k(x,t) = \sum\limits_{j=1}^{\infty} h_j(x) g_j(t)$ and $K\colon \mathcal{HK} \to \mathcal{HK}$ be defined by*

$$(K\varphi)(x) = \int_a^b k(x,t)\varphi(t)\,\mathrm{d}t = \int_a^b \Big(\sum_{j=1}^{\infty} h_j(x) g_j(t)\Big)\varphi(t)\,\mathrm{d}t.$$

Then K is compact in the $\mathcal{HK}$ space.

Now we shall consider nonlinear operators. Let $h(t,s)$ be a Carathéodory function from $[a,b] \times \mathbb{R}$ to $\mathbb{R}$, i.e., the function $h(t,\cdot)$ is continuous for almost all $t \in [a,b]$ and the function $h(\cdot,s)$ is measurable for every $s \in \mathbb{R}$.

Let u be a function defined on $[a,b]$ and Hu a function defined on $[a,b]$ and $(Hu)(t) = h(t,u(t))$. It is well-known, see [10], page 358, Lemma 17.6, that if

$H\colon L_1[a,b] \to L_1[a,b]$, i.e., $(Hu)(t) = h(t,u(t)) \in L_1[a,b]$ whenever $u \in L_1[a,b]$, then there exist $\alpha > 0$, p and $q \in L_1[a,b]$ such that

$$q(t) + \alpha|s| \leqslant h(t,s) \leqslant p(t) + \alpha|s|$$

for almost all $t \in [a,b]$ and $s \in \mathbb{R}$. It is known, see [4], Theorem 1.4 and its proof, that if $H\colon \mathcal{HK} \to \mathcal{HK}$, then H maps every control-convergent sequence to a control-convergent sequence.

Suppose $H\colon \mathcal{HK} \to \mathcal{HK}$. We conjecture that there exist $p, q \in \mathcal{HK}$ and $\alpha > 0$ such that

$$q(t) + \alpha s \leqslant h(t,s) \leqslant p(t) + \alpha s \tag{4.1}$$

for almost all $t \in [a,b]$ and all $s \in \mathbb{R}$.

In the following, we assume that condition (4.1) holds for $h(t,s)$. Let $k(x,t)$ be given as in Theorem 4.1. Define $K\colon \mathcal{HK} \to \mathcal{HK}$ as follows: Let $u \in \mathcal{HK}$. Then $Ku \in \mathcal{HK}$ and

$$(Ku)(x) = \int_a^b k(x,t)u(t)\,\mathrm{d}t.$$

Then the composite operator KH maps the $\mathcal{HK}$ space to the $\mathcal{HK}$ space. Let $u \in \mathcal{HK}$. Then $(KH)(u) \in \mathcal{HK}$ and for each $x \in [a,b]$

$$((KH)(u))(x) = (K(Hu))(x) = \int_a^b k(x,t)h(t,u(t))\,\mathrm{d}t.$$

The operator H is called a *Nemytskii operator*. The composite operator KH is called a *Hammerstein operator*.

Corollary 4.3. *The nonlinear Hammerstein operator $KH\colon \mathcal{HK} \to \mathcal{HK}$ given above is compact.*

P r o o f. First, using inequality (4.1), the Nemytskii operator H maps every bounded sequence under $\|\cdot\|_i^X$ to a bounded sequence under $\|\cdot\|_i^X$. By Theorem 4.1 the operator K is compact. Therefore, the composite Hammerstein operator KH from $\mathcal{HK}$ to $\mathcal{HK}$ is compact. $\square$

5. Integral equations

Fredholm integral equations of the second kind are equations of the form

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\varphi(t)\,\mathrm{d}t, \tag{5.1}$$

where $f\colon [a,b] \to \mathbb{R}$ and $k\colon [a,b]\times[a,b] \to \mathbb{R}$. The function k is known as the *integral kernel*.

In this section, let $f, \varphi \in \mathcal{HK}[a,b]$. We first discuss the case when the integral kernel is separable, i.e., $k(x,t) = h(x)g(t)$. Suppose that $h \in \mathcal{HK}[a,b]$ and $g \in BV[a,b]$. Then, as in the classical case,

$$\varphi(x) = f(x) + \lambda h(x) \frac{\int_a^b f(t)g(t)\,\mathrm{d}t}{1 - \lambda \int_a^b g(t)h(t)\,\mathrm{d}t}$$

is the unique solution of a Fredholm integral equation with separable integral kernel whenever $\lambda \int_a^b g(t)h(t)\,\mathrm{d}t \neq 1$.

For the case when $\lambda \int_a^b g(t)h(t)\,\mathrm{d}t = 1$, the equation has no solution if

$$\lambda \int_a^b f(t)g(t)\,\mathrm{d}t \neq 0.$$

When $\lambda \int_a^b f(t)g(t)\,\mathrm{d}t = 0$, as in the classical case, the general solution of the Fredholm integral equation is of the form

$$\varphi(x) = \beta h(x) + f(x)$$

for any real constant β.

Let $g_j \in BV[a,b]$, $f, h_j \in \mathcal{HK}$ for $j = 1, 2, \ldots, n$ and

$$k(x,t) = \sum_{j=1}^{n} h_j(x) g_j(t)$$

for $x, t \in [a,b]$. Then the corresponding Fredholm integral equation has properties analogous to those well known for the classical case. Let us recall that linear Fredholm equations with regulated Banach space valued solutions and nondegenerate kernel have been treated in [6].

Now, let us turn back to the case when the kernel k and the operator K are like in Theorem 4.1 or Corollary 4.2, the operator H is like in Corollary 4.3 and

$f \in \mathcal{HK}$. Put $T\varphi = \lambda K\varphi + f$ for $\varphi \in \mathcal{HK}$. Let $\{\alpha_j\}$ be of bounded variation, i.e., $\sum\limits_{j=1}^{\infty} |\alpha_{j+1} - \alpha_j| < \infty$. By Lemma 3.3, $\alpha = \lim\limits_{n\to\infty} \alpha_n$ exists. Recall that $\sum\limits_{j=1}^{\infty} h_j \in \mathcal{HK}(X)$ in Theorem 4.1. From (3.1) for each i we have

$$\begin{aligned}
\|K\varphi\|_i^X = \Big\|\sum_{j=1}^{\infty} h_j\alpha_j\Big\|_i^X &\leqslant 2\sup_j \Big\|\sum_{k=1}^{j} h_k\Big\|_i^X \sum_{j=1}^{\infty} |\alpha_j - \alpha_{j+1}| + \Big\|\sum_{k=1}^{\infty} h_k\Big\|_i^X |\alpha| \\
&\leqslant 2\sup_j \Big\|\sum_{k=1}^{j} h_k\Big\|_i^X \Big(\sum_{j=n+1}^{\infty} (\|g_{j+1}\|_{BV} + \|g_j\|_{BV})\Big)\|\varphi\|_i^X \\
&\quad + \Big\|\sum_{k=n+1}^{\infty} h_k\Big\|_i^X \Big(\sum_{j=1}^{\infty} \|g_j\|_{BV}\Big)\|\varphi\|_i^X \\
&\leqslant 3\sup_j \Big\|\sum_{k=1}^{j} h_k\Big\|_i^X \Big(\sum_{j=1}^{\infty} \|g_j\|_{BV}\Big)\|\varphi\|_i^X.
\end{aligned}$$

Therefore,

$$\|K\|_i^X = \sup_{\varphi} \frac{\|T\varphi\|_i^X}{\|\varphi\|_i^X} \leqslant 3\sup_j \Big\|\sum_{k=1}^{j} h_k\Big\|_i^X \sum_{j=1}^{\infty} \|g_j\|_{BV}.$$

We assume that $3\sup\limits_j \Big\|\sum\limits_{k=1}^{j} h_k\Big\|_i^X \sum\limits_{j=1}^{\infty} \|g_j\|_{BV} \leqslant \mu$ for all i. Then $\|K\|_i^X \leqslant \mu$ for all i.

Theorem 5.1. *Suppose $\|K\psi\|_i^X \leqslant \mu\|\psi\|_i^X$ for each i and ψ, i.e., $\|K\|_i^X \leqslant \mu$ for all i. If $0 < \lambda < 1/\mu$, then there exists a unique fixed point $u \in \mathcal{HK}(X)$, i.e.,*

$$u(x) = \lambda(Ku)(x) + f(x) = \lambda \int_a^b k(x,t)u(t)\,\mathrm{d}t + f(x).$$

This solution u is given by a convergent Neumann series $u(x) = \sum\limits_{j=1}^{\infty} \lambda^j K^j f$ and $\|u\|^X \leqslant \|f\|^X/(1-\mu u)$.

P r o o f. The proof is standard. Let $u_0(x) = f(x)$, $u_n(x) = \lambda K u_{n-1}(x) + f(x)$, $n = 1, 2, \ldots$ Then $u_{n+1}(x) = \sum\limits_{j=0}^{n+1} \lambda^j K^j f(x)$, $n = 1, 2, \ldots$ Since $\|K\|_i^X \leqslant \mu$ for all i we have

$$\|K^j f\|_i^X = \|KK^{j-1}f\|_i^X \leqslant \mu\|K^{j-1}f\|_i^X \leqslant \mu^j\|f\|_i^X.$$

Thus, for any $m, n \in \mathbb{N}$ we have

$$\|u_n - u_m\|_i^X = \Big\|\sum_{j=m+1}^{n} \lambda^j K^j f\Big\|_i^X \leqslant \sum_{j=m+1}^{n} \lambda^j \|K^j f\|_i^X = \Big(\sum_{j=m+1}^{n} (\lambda\mu)^j\Big)\|f\|_i^X.$$

Since $0 < \lambda\mu < 1$ by our assumptions, the sequence $\{u_n\}$ is Cauchy under $\|\cdot\|_i^X$ for each i. Thus, $\{u_n\}$ is Cauchy under $\|\cdot\|^X$. Hence, $\lim_{n\to\infty} u_n = \sum_{j=0}^{\infty} \mu^j K^j f$ exists in $\mathcal{HK}(X)$ if $0 < \lambda < 1/\mu$.

Let $u(x) = \sum_{j=0}^{\infty} \mu^j (K^j f)(x)$. Then $u(x) = \lim_{n\to\infty} u_n(x)$. By Theorem 3.5 the operator K is $\|\cdot\|^X$-continuous. From the iteration equation $u_n(x) = \lambda K u_{n-1}(x) + f(x)$, we have

$$\begin{aligned}
\lim_{n\to\infty} u_n(x) &= \lambda K\Big(\lim_{n\to\infty} u_{n-1}(t)\Big) + f(x) \\
&= \lambda \int_a^b k(x,t)\Big(\lim_{n\to\infty} u_{n-1}(t)\Big)\,\mathrm{d}t + f(x) \\
&= \lambda \int_a^b k(x,t)u(t)\,\mathrm{d}t + f(x).
\end{aligned}$$

Thus, if $0 < \lambda < 1/\mu$, we can find $u \in \mathcal{HK}(X)$ such that

$$u(x) = \lambda \int_a^b k(x,t)u(t)\,\mathrm{d}t + f(x).$$

Now we shall prove that the fixed point u is unique. Suppose that there are two fixed points, namely u and v. Then $u = \lambda K u + f$ and $v = \lambda K v + f$. Therefore, $u - v = \lambda K(u - v)$ and

$$\|u - v\|_i^X = \lambda \|K(u-v)\|_i^X \leqslant \lambda \|K\|_i^X \|(u-v)\|_i^X \leqslant \lambda\mu \|(u-v)\|_i^X.$$

Hence, $(1 - \lambda\mu)\|u - v\|_i^X \leqslant 0$. Recall that $1 - \lambda\mu > 0$. It implies that $\|u - v\|_i^X = 0$ for all i. Thus, $\|u - v\|^X = 0$. Consequently, $u = v$. Therefore, the fixed point u is unique.

Now we shall prove that $\|u\|^X \leqslant 1/(1-\lambda\mu)\|f\|^X$. First, $\|u\|_i^X \leqslant \|f\|_i^X/(1-\lambda\mu)$ for all i. Hence,

$$\begin{aligned}
\|u\|^X &= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|u\|_i^X}{1 + \|u\|_i^X} \\
&\leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f\|_i^X/(1-\lambda\mu)}{1 + \|f\|_i^X/(1-\lambda\mu)} \\
&\leqslant \Big(\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f\|_i^X}{1 + \|f\|_i^X/(1-\lambda\mu)}\Big)\Big(\frac{1}{1-\lambda\mu}\Big) \\
&\leqslant \Big(\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\|f\|_i^X}{1 + \|f\|_i^X}\Big)\Big(\frac{1}{1-\lambda\mu}\Big) \leqslant \|f\|^X\Big(\frac{1}{1-\lambda\mu}\Big).
\end{aligned}$$

□

Remark 5.2. We remark that although μ is not easy to find, Theorem 5.1 says that if λ is a small enough positive number, then a unique fixed point exists. Furthermore, it seems that Theorem 5.1 is a fixed point theorem for the $\mathcal{HK}(X)$ space only. In fact, if $\{h_j\}$ and $\sum\limits_{j=1}^{\infty} h_j$ are in the $\mathcal{HK}$ space, then they are in $\mathcal{HK}(X)$ for some $X \in \mathbb{X}$. Hence, Theorem 5.1 is also a fixed point theorem for the $\mathcal{HK}$ space.

Theorem 5.3 (Tychonoff's theorem, [16], Theorem A)**.** *Let A be a convex subset of a locally convex topological vector space. Suppose T is a continuous operator which maps A into a compact subset of A. Then T has a fixed point.*

Theorem 5.4. *Let $T\varphi = \lambda(KH)\varphi + f$, where KH is given before Corollary 4.3. Assume that $p, q, f \in \mathcal{HK}(X)$, $(\|p\|_i^X + \|q\|_i^X)/\|f\|_i^X \leqslant \beta$ and $\|K\|_i^X \leqslant \mu$ for all i and $0 < \lambda < 1/\mu(\beta + 2\alpha)$. Then $T\colon\ \mathcal{HK}(X) \to \mathcal{HK}(X)$ has a fixed point.*

Proof. From (4.1) we have

$$q(t) + \alpha\varphi(t) \leqslant H\varphi(t) = h(t, \varphi(t)) \leqslant p(t) + \alpha\varphi(t).$$

Then

$$\|H\varphi\|_i^X \leqslant \max\{\|q\|_i^X + \alpha\|\varphi\|_i^X,\ \|p\|_i^X + \alpha\|\varphi\|_i^X\} \leqslant \|q\|_i^X + \|p\|_i^X + \alpha\|\varphi\|_i^X.$$

Since K is $\|\cdot\|^X$-continuous, i.e., K is $\|\cdot\|_i^X$-continuous for each i, we have

$$\|KH\varphi\|_i^X \leqslant \|K\|_i^X\|H\varphi\|_i^X \leqslant \|K\|_i^X(\|q\|_i^X + \|p\|_i^X + \alpha\|\varphi\|_i^X).$$

Let $\alpha_i = \|f\|_i^X$ and $A = \{u \in \mathcal{HK}(X)\colon\ \|u\|_i^X \leqslant 2\alpha_i \text{ for all } i\}$. Then A is convex and bounded. Hence, for every $\varphi \in A$ we have

$$\begin{aligned}
\|T\varphi\|_i^X &\leqslant \lambda\|(KH)\varphi\|_i^X + \|f\|_i^X \\
&\leqslant \lambda\|K\|_i^X(\|q\|_i^X + \|p\|_i^X + \alpha\|\varphi\|_i^X) + \|f\|_i^X \\
&\leqslant \lambda\mu(\|q\|_i^X + \|p\|_i^X + \alpha\|\varphi\|_i^X) + \|f\|_i^X \\
&\leqslant \lambda\mu(\|q\|_i^X + \|p\|_i^X + \alpha(2\alpha_i)) + \alpha_i \\
&= \lambda\mu\alpha_i\Big(\frac{\|q\|_i^X + \|p\|_i^X}{\alpha_i} + 2\alpha\Big) + \alpha_i \\
&\leqslant \lambda\mu\alpha_i(\beta + 2\alpha) + \alpha_i \\
&< \frac{1}{\mu(\beta + 2\alpha)}\mu\alpha_i(\beta + 2\alpha) + \alpha_i = 2\alpha_i.
\end{aligned}$$

Then $TA \subseteq A$. The operator T is compact since KH is compact. Thus, TA is compact. Hence, by Tychonoff's theorem, T has a fixed point in A. $\square$

Remark 5.5. We remark that although $1/\mu(\beta+2\alpha)$ is not easy to find, Theorem 5.4 says that if λ is a small enough positive number, then $T\colon \mathcal{HK}(X)\to\mathcal{HK}(X)$ has a fixed point. Furthermore, similarly to Remark 5.2 of Theorem 5.1, Theorem 5.4 is also a fixed point theorem for the $\mathcal{HK}$ space, since if p, q, f are in the $\mathcal{HK}$ space, then p, q, f are in $\mathcal{HK}(X)$ for some $X\in\mathbb{X}$.

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