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ON THE CONVERGENCE THEORY OF DOUBLE K-WEAK SPLITTINGS OF TYPE II

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Abstract. Recently, Wang (2017) has introduced the K-nonnegative double splitting using the notion of matrices that leave a cone $K \subseteq \mathbb{R}^n$ invariant and studied its convergence theory by generalizing the corresponding results for the nonnegative double splitting by Song and Song (2011). However, the convergence theory for K-weak regular and K-nonnegative double splittings of type II is not yet studied. In this article, we first introduce this class of splittings and then discuss the convergence theory for these sub-classes of matrices. We then obtain the comparison results for two double splittings of a K-monotone matrix. Most of these results are completely new even for $K = \mathbb{R}^n_+$. The convergence behavior is discussed by performing numerical experiments for different matrices derived from the discretized Poisson equation.

Keywords: linear system; iterative method; K-nonnegativity; double splitting; convergence theorem; comparison theorem

MSC 2020: 15A09, 65F10

1. Introduction

Consider a real large linear system of the form

$$Ax = b, \tag{1.1}$$

where A is a real non-singular matrix of order n that appears in many scientific and engineering problems. In many cases, the iterative methods are preferred to the

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direct methods. Woźnicki [19] proposed the iteration scheme

$$x^{k+1} = P^{-1}Rx^k + P^{-1}Sx^{k-1} + P^{-1}b, \quad k = 1, 2, \ldots, \tag{1.2}$$

based on the double splitting $A = P - R - S$. The notion of double splitting originates from the basic iterative methods like Jacobi, Gauss-Seidel and successive over-relaxation methods. Following the idea of Golub and Varga [7], Woźnicki [19] expressed the above scheme in the equivalent form

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & O \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ O \end{pmatrix}, \quad k = 1, 2, \ldots, \tag{1.3}$$

where I denotes the identity matrix and O denotes the null matrix of required size. Then, the iterative scheme (1.3) converges to $A^{-1}b$ for any starting vectors x^0, x^1 if and only if the spectral radius of the iteration matrix

$$W = \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & O \end{pmatrix}$$

is less than 1. The convergence of (1.2) is guaranteed in [14], [17] for different sub-classes of double splittings when A^{-1} is nonnegative. (A matrix $B \in \mathbb{R}^{n\times n}$ is called *nonnegative* if $B \geqslant 0$, where the inequality is entry-wise.) But, for

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \ngeqslant 0.$$

In such a case, the existing double splitting theory fails. To avoid this, the usual matrix nonnegativity is replaced by a more general cone nonnegativity. Then $A^{-1} \geqslant_K 0$, where $K = \{x \in \mathbb{R}^3;\ (x_1^2 + x_2^2)^{1/2} \leqslant x_3\}$ is a particular proper cone called as the *ice cream cone*. The symbol $\geqslant_K$ represents cone nonnegativity and we refer to the next section for more details on it.

In 2014, Hou [8] introduced two sub-classes of double splittings called K-regular and K-weak regular double splittings using the notion of cone nonnegativity that extends corresponding double splittings [14] in the usual nonnegativity setting (i.e., in $K = \mathbb{R}^n_+$ setting). A double splitting $A = P - R - S$ is called a *double K-regular splitting* (or K-regular double splitting) [8] if $P^{-1} \geqslant_K 0$, $R \geqslant_K 0$ and $S \geqslant_K 0$. Similarly, a double splitting $A = P - R - S$ is called a *double K-weak regular splitting of type I* (or K-weak regular double splitting) [8] if $P^{-1} \geqslant_K 0$, $P^{-1}R \geqslant_K 0$ and $P^{-1}S \geqslant_K 0$. In 2017, Wang [18] introduced a more general class of double splittings that is called a *K-nonnegative double splitting*. We call the same class of

double splittings as *double K-weak splitting of type I*, since the conditions are weak compared to the conditions in the earlier two subclasses of double splittings, and it is defined next. A double splitting $A = P - R - S$ is called a *double K-weak splitting of type I* [18] if $P^{-1}R \geqslant_K 0$ and $P^{-1}S \geqslant_K 0$. The authors of [8] and [18] then presented the convergence theory for this class of double splittings generalizing the work of [14]. The same authors also obtained several results comparing the spectral radii of the iteration matrices that help to find an iterative solution in a faster way.

A real $n \times n$ matrix A is called *monotone* if $Ax \geqslant 0 \Rightarrow x \geqslant 0$. The book by Collatz [5] discusses the natural occurrence of monotone matrices in finite difference approximation methods for certain type of partial differential equations. This class of matrices also arises in linear complementary problems in operations research, input-output production and growth models in economics, and Markov processes in probability and statistics, to name a few. A well-known characterization of a monotone matrix is: A is monotone if and only if A^{-1} exists and $A^{-1} \geqslant 0$. Several generalizations and characterizations of monotone matrices are reported in the book by Berman and Plemmons [1]. An important subclass of monotone matrices is the set of non-singular M-matrices. Another generalization of a monotone matrix is recalled next. A real $n \times n$ matrix A is K-monotone [8] if A^{-1} exists and $A^{-1} \geqslant_K 0$. The objective of this paper is to

(i) widen the convergence theory of double splittings by introducing two new subclasses of double splittings and establishing their convergence theory,

(ii) present a new characterization of K-monotone matrices,

(iii) compare the rate of convergence of two different iteration schemes arising out of two different double splittings.

To do this, the rest of the paper is sectioned as follows. In Section 2, we introduce our notations and definitions, and some preliminary results which are helpful in proving the main results. In Section 3, we propose the definition of two new subclasses of double splittings which we call double K-weak regular and double K-weak splittings of type II. We discuss their convergence theory. We then establish a new characterization of a K-monotone matrix. Several comparison results are also provided that help to find an iterative solution in a faster way. The double splittings based on the SOR method and the corresponding preconditioned conjugate gradient method are considered for the numerical experiments in Section 4. The roles of the right-hand side vectors are compared for different systems obtained from the discretized Poisson equations.

2. Preliminaries

In this section, we first briefly explain some of the terminologies. The notation $\mathbb{R}^{n\times n}$ represents the set of all real matrices of order n, $A^\top$ denotes the transpose of $A \in \mathbb{R}^{n\times n}$, and $\sigma(A)$ denotes the set of all eigenvalues of A. It is well known that for any two matrices A and B, $\sigma(AB) = \sigma(BA)$. The *spectral radius* of $A \in \mathbb{R}^{n\times n}$, denoted by $\varrho(A)$, is defined as $\varrho(A) = \max\limits_{1\leqslant j\leqslant n} |\lambda_j|$, where $\lambda_j \in \sigma(A)$. Calling a matrix A *convergent*, we mean $\lim\limits_{k\to\infty} A^k = 0$. A matrix A is convergent if and only if $\varrho(A) < 1$. We write K and $\mathrm{int}(K)$ to denote a proper cone and the interior of K in $\mathbb{R}^n$, respectively. A nonempty subset K in $\mathbb{R}^n$ is a *cone* if $0 \leqslant \lambda$ implies $\lambda K \subseteq K$. A cone K is *closed* if and only if it coincides with its closure. A cone is a *convex cone* if $K + K \subseteq K$, a *pointed cone* if $K \cap (-K) = \{0\}$ and a *solid cone* if $\mathrm{int}(K) \neq \varphi$. A closed, pointed, solid convex cone is called a *proper cone*. A proper cone induces a partial order in $\mathbb{R}^n$ via $x \geqslant_K y$ if and only if $x - y \geqslant_K 0$ (see [1] for more details). The symbol $\pi(K)$ denotes the set of all matrices in $\mathbb{R}^{n\times n}$ which leave a proper cone $K \subseteq \mathbb{R}^n$ invariant (i.e., $AK \subseteq K$). We next recall a result of [8] in this direction.

Lemma 2.1 ([8], Lemma 2.11)**.** *Let $K_{2n} = \{(x^\top, y^\top);\ \forall\, x, y \in K\}$. Then K_{2n} is a proper cone in $\mathbb{R}^{2n}$.*

We now move to the notion of K-nonnegativity of a matrix which generalizes the usual nonnegativity. A real $n \times n$ matrix A is called *K-nonnegative (K-positive)* if $AK \subseteq K$ ($A(K - \{0\}) \subseteq \mathrm{int(K)}$) and is denoted by $A \geqslant_K 0$ ($A >_K 0$). $A \geqslant_K 0$ is equivalent to $A \in \pi(K)$. For $A, B \in \mathbb{R}^{n\times n}$, $A \geqslant_K B$ ($A >_K B$) if $A - B \geqslant_K 0$ ($A - B >_K 0$). A vector $x \in \mathbb{R}^n$ is called *K-nonnegative (K-positive)* if $x \in K$ ($x \in \mathrm{int}(K)$) and is denoted by $x \geqslant_K 0$ ($x >_K 0$). Similarly, for $x, y \in \mathbb{R}^n$, $x \geqslant_K y$ ($x >_K y$) if $x - y \geqslant_K 0$ ($x - y >_K 0$).

Example 2.1. A proper cone K_n of $\mathbb{R}^n$ of the form $K_n = \{x \in \mathbb{R}^n;\ (x_1^2 + x_2^2 + \ldots + x_{n-1}^2)^{1/2} \leqslant x_n\}$ is called the ice cream cone. If $n = 3$, then the vector $x = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{R}^3$ is K_3-nonnegative as $(x_1^2 + x_2^2)^{1/2} = 1.4142 \leqslant 2 = x_3$.

If $A = \begin{pmatrix} B & C \\ I & O \end{pmatrix}$ such that $B \geqslant_K 0$ and $C \geqslant_K 0$, then by Lemma 2.1, A leaves the proper cone $K_{2n} \subseteq \mathbb{R}^{2n}$ invariant, i.e., $A \geqslant_{K_{2n}} 0$. The following results deal with the K-nonnegativity of a matrix and its spectral radius. Analogous results can also be found in [1] and [3].

Lemma 2.2 ([9], Corollary 3.2 & Lemma 3.3)**.** *Let $A \geqslant_K 0$. Then*

(i) *$Ax \geqslant_K \alpha x$, $x \geqslant_K 0$, implies $\alpha \leqslant \varrho(A)$. Moreover, if $Ax >_K \alpha x$, then $\alpha < \varrho(A)$.*
(ii) *$\beta x \geqslant_K Ax$, $x >_K 0$, implies $\varrho(A) \leqslant \beta$. Moreover, if $\alpha x >_K Ax$, then $\alpha > \varrho(A)$.*

Theorem 2.1 ([1], Theorem 1.3.2)**.** *Let $A \geqslant_K 0$. Then*

(i) *$\varrho(A)$ is an eigenvalue.*
(ii) *K contains an eigenvector of A corresponding to $\varrho(A)$.*

Lemma 2.3 ([18], Lemma 2)**.** *If $B \geqslant_K 0$, then $\varrho(B) < \alpha$ if and only if $\alpha I - B$ is non-singular and $(\alpha I - B)^{-1} \geqslant_K 0$.*

Theorem 2.2 ([1], Corollary 1.3.30)**.** *If $A \geqslant_K B \geqslant_K 0$, then $\varrho(A) \geqslant \varrho(B)$.*

Lemma 2.4 ([8], Lemma 2.13)**.** *Let $A = \begin{pmatrix} B & C \\ I & O \end{pmatrix} \geqslant_{K_{2n}} 0$ and $\varrho(B + C) < 1$. Then $\varrho(A) < 1$.*

The next result generalizes [16], Lemma 2.2 for an arbitrary proper cone K.

Lemma 2.5. *Let $A_i \geqslant_K 0$, $i = 1, 2$, be convergent. If there exists β, $0 < \beta \leqslant 1$, such that $\beta(I - A_2)^{-1} \geqslant_K (I - A_1)^{-1}$, then $\varrho(A_1) \leqslant \varrho(A_2)$ whenever $\beta = 1$ and $\varrho(A_1) < \varrho(A_2)$ whenever $\beta < 1$.*

P r o o f. Since $A_1 \geqslant_K 0$, there exists an eigenvector $x \geqslant_K 0$ corresponding to the eigenvalue $\varrho(A)$ such that $A_1 x = \varrho(A_1)x$ by Theorem 2.1. This further gives $(I - A_1)^{-1}x = x/(1 - \varrho(A_1))$. Now, $\beta(I - A_2)^{-1} \geqslant_K (I - A_1)^{-1}$ and $x \geqslant_K 0$ imply $\beta(I - A_2)^{-1}x \geqslant_K (I - A_1)^{-1}x = x/(1 - \varrho(A_1))$. By Lemma 2.2, we have $\varrho(A_2) \geqslant 1 - \beta(1 - \varrho(A_1))$. Clearly, if $\beta = 1$, we get $\varrho(A_1) \leqslant \varrho(A_2)$, and if $\beta < 1$, then $\varrho(A_1) < \varrho(A_2)$. □

3. Main results

To address the first objective, we introduce below two new sub-classes of double splittings using cone nonnegativity.

Definition 3.1. A double splitting $A = P - R - S$ is called a double *K-weak regular splitting of type II* if $P^{-1} \geqslant_K 0$, $RP^{-1} \geqslant_K 0$ and $SP^{-1} \geqslant_K 0$.

We next produce an example illustrating the importance of the above class of double splittings with respect to the ice cream cone $K_3 = \{x \in \mathbb{R}^3;\ (x_1^2 + x_2^2)^{1/2} \leqslant x_3\}$.

Example 3.1. Let

$$A = \begin{pmatrix} 4 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -5/2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} - \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= P - R - S$$

be a double splitting. Then,

$$P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \pi(K_3), \quad RP^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in \pi(K_3)$$

and

$$SP^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \pi(K_3),$$

but

$$R = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \notin \pi(K_3) \quad \text{as} \quad \begin{pmatrix} -3 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \notin K_3.$$

Hence, the double splitting $A = P - R - S$ is a double K-weak regular splitting of type II, but not a double K-regular splitting.

The next example motivates to introduce another new sub-class of double splittings.

Example 3.2. Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 1/2 & 1/2 & -1/2 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= P - R - S$$

be a double splitting. Then,

$$P^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \notin \pi(K_3) \quad \text{as} \quad \begin{pmatrix} -1 \\ 8 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \notin K_3,$$
$$RP^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \pi(K_3) \quad \text{and} \quad SP^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \pi(K_3),$$

but

$$P^{-1}R = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \notin \pi(K_3)$$

because

$$\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \notin K_3.$$

Hence, the double splitting $A = P - R - S$ is neither a double K-weak regular splitting of type I (or II) nor a double K-weak splitting of type I.

Observe that $RP^{-1} \in \pi(K_3)$ and $SP^{-1} \in \pi(K_3)$ in the above example. So, we have the following definition.

Definition 3.2. A double splitting $A = P - R - S$ is called a double K-*weak splitting of type II* if $RP^{-1} \geqslant_K 0$ and $SP^{-1} \geqslant_K 0$.

We note that a double K-weak splitting of type II contains a double K-weak regular splitting of type II and a double K-regular splitting. Introduction of such sub-classes of double splittings is meaningless unless we show that the iteration (1.2) converges for these splittings with or without some additional assumptions. The next subsection discusses the above stated problem and its solution. In the meantime, we also present a new characterization of a K-monotone matrix which is our second objective.

3.1. Convergence of double K-weak splittings of type II. From the double iteration scheme (1.3), we have

$$W = \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & O \end{pmatrix}. \tag{3.1}$$

If the splitting $A = P - R - S$ is a double K-weak splitting of type II, then the matrix $\widetilde{T} = (R+S)P^{-1} \in \pi(K)$ and the matrix

$$\widetilde{W} = \begin{pmatrix} RP^{-1} & SP^{-1} \\ I & O \end{pmatrix} \in \pi(K_{2n}). \tag{3.2}$$

Using the notion of similar matrices, Shekhar et al. [13] showed that W and $\widetilde{W}$ have the same spectral radius when $K = \mathbb{R}^n_+$ (see [13], Lemma 3.1). We now provide an alternative proof of the above result that does not use the similarity concept. Moreover, this proof can be easily generalized to rectangular matrices replacing the usual inverse by the Moore-Penrose inverse [1].

Lemma 3.1. *Let the matrices W and $\widetilde{W}$ be as defined in equations* (3.1) *and* (3.2)*, respectively. Then, $\varrho(W) = \varrho(\widetilde{W})$.*

P r o o f. Let $\lambda(\neq 0)$ be an eigenvalue of $\widetilde{W}$. Then, there exists an eigenvector x corresponding to λ, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, such that $\widetilde{W}x = \lambda x$, i.e.,

$$\begin{pmatrix} RP^{-1} & SP^{-1} \\ I & O \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Therefore,

$$RP^{-1}x_1 + SP^{-1}x_2 = \lambda x_1, \tag{3.3}$$

$$x_1 = \lambda x_2. \tag{3.4}$$

Pre-multiplying (3.3) and (3.4) by P^{-1}, we get

$$P^{-1}R(P^{-1}x_1) + P^{-1}S(P^{-1}x_2) = \lambda P^{-1}x_1,$$
$$P^{-1}x_1 = \lambda P^{-1}x_2.$$

Let $y_1 = P^{-1}x_1$ and $y_2 = P^{-1}x_2$. Then

$$P^{-1}Ry_1 + P^{-1}Sy_2 = \lambda y_1,$$
$$y_1 = \lambda y_2,$$

i.e., $Wy = \lambda y$, where $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Clearly, if $y = 0$, then $P^{-1}x_1 = 0$ and $P^{-1}x_2 = 0$. From equation (3.3), we get $\lambda x_1 = 0$, but $\lambda \neq 0$, so $x_1 = 0$ which further gives $x_2 = 0$ by (3.4), a contradiction as x is an eigenvector. Thus $y \neq 0$. Hence, $\sigma(\widetilde{W}) \setminus \{0\} \subseteq \sigma(W) \setminus \{0\}$.

Conversely, let $\mu \neq 0$ be an eigenvalue of W, then there exists an eigenvector corresponding to μ, $x^\top = (x_1^\top, x_2^\top)$, such that $x^\top W = \mu x^\top$, i.e.,

$$(x_1^\top, x_2^\top) \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & O \end{pmatrix} = \mu(x_1^\top, x_2^\top).$$

Therefore,

$$x_1^\top P^{-1}R + x_2^\top = \mu x_1^\top, \tag{3.5}$$

$$x_1^\top P^{-1}S = \mu x_2^\top. \tag{3.6}$$

Post-multiplying (3.5) and (3.6) by P^{-1}, we get

$$x_1^\top P^{-1}RP^{-1} + x_2^\top P^{-1} = \mu x_1^\top P^{-1},$$
$$x_1^\top P^{-1}SP^{-1} = \mu x_2^\top P^{-1}.$$

Let $z_1^\top = x_1^\top P^{-1}$ and $z_2^\top = x_2^\top P^{-1}$. Then

$$z_1^\top R P^{-1} + z_2^\top = \mu z_1^\top,$$
$$z_1^\top S P^{-1} = \mu z_2^\top,$$

i.e., $z^\top \widetilde{W} = \mu z^\top$, where $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Clearly, $z \neq 0$, otherwise $x_1^\top P^{-1} = 0$ and $x_2^\top P^{-1} = 0$. From equation (3.5), we get $\mu x_1^\top = 0$, but $\mu \neq 0$, so $x_1^\top = 0$ which further gives $x_2^\top = 0$ by (3.6), a contradiction. So, $z \neq 0$. Therefore, $\sigma(W) \setminus \{0\} \subseteq \sigma(\widetilde{W}) \setminus \{0\}$. Thus, $\sigma(W) \setminus \{0\} = \sigma(\widetilde{W}) \setminus \{0\}$. Hence, $\varrho(W) = \varrho(\widetilde{W})$. □

The double splitting $A = P - R - S$ reduces to the splitting $A = U - V$ by taking $V = R + S$ and $U = P$. This in turn yields the iteration scheme

$$x^{k+1} = Hx^k + c, \quad k = 1, 2, \ldots, \tag{3.7}$$

where $H = U^{-1}V$ and $c = U^{-1}b$, that converges to $A^{-1}b$ for any initial vector x^0 if and only if $\varrho(H) < 1$. If $A = P - R - S$ is a double K-weak regular splitting of type I, then $U^{-1} = P^{-1} \geqslant_K 0$ and $U^{-1}V = P^{-1}(R + S) \geqslant_K 0$, and the resulting $A = U - V$ is a double K-weak regular splitting of type I. Similarly, the other types of double splittings reduce to the respective types of single splittings by setting $U = P$ and $V = R + S$. Interested readers are referred to [3], [4], and [2] for the convergence criteria of this class of single splittings. At this juncture, the following question arises: Is there any relation between the convergence of a double splitting and a single splitting of a particular type? In 2017, Wang [18] answered this question for the double K-weak splitting of type I. We next answer the same question by showing the equivalence of the convergence of (3.7) and (1.3) for the double K-weak splitting of type II.

Theorem 3.1. *Let $A = P - R - S$ be a double K-weak splitting of type* II. *Then $\varrho(W) < 1$ if and only if $\varrho(H) < 1$.*

P r o o f. Suppose that $\varrho(W) < 1$. So, by Lemma 3.1, $\varrho(\widetilde{W}) < 1$. We have

$$(I - \widetilde{W})^{-1} = \begin{pmatrix} [I - (R+S)P^{-1}]^{-1} & [I - (R+S)P^{-1}]^{-1}SP^{-1} \\ [I - (R+S)P^{-1}]^{-1} & [I - (R+S)P^{-1}]^{-1}(I - RP^{-1}) \end{pmatrix}. \tag{3.8}$$

Since $\widetilde{W} \geqslant_{K_{2n}} 0$, we obtain $(I - \widetilde{W})^{-1} \geqslant_{K_{2n}} 0$ by Lemma 2.3. So, $[I - (R + S)P^{-1}]^{-1} \geqslant_K 0$. By Lemma 2.3, we thus have

$$\varrho((R + S)P^{-1}) = \varrho(P^{-1}(R + S)) < 1.$$

Conversely, if $\varrho(P^{-1}(R + S)) < 1$, then $\varrho((R + S)P^{-1}) < 1$. By Theorem 2.4, we have $\varrho(\widetilde{W}) < 1$. Again, using Lemma 3.1, we get $\varrho(W) < 1$. □

The following example demonstrates the above result.

Example 3.3. Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 1/10 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} = P - R - S$ be a double splitting, where

$$P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ -1/20 & 0 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 0 & 0 & 0 \\ -1/20 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$RP^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1/20 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \in \pi(K_3) \quad \text{and} \quad SP^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1/20 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \in \pi(K_3).$$

Thus, the splitting $A = P - R - S$ is a double K-weak splitting of type II. We have $\varrho(H) = \varrho(P^{-1}(R+S)) = 0.8333 < 1$. Hence, by Theorem 3.1, $\varrho(W) < 1$.

We say a double splitting is convergent (or called a *convergent double splitting*) if (1.2) is convergent for that double splitting. Note that the above theorem reduces to [13], Theorem 3.2 for $K = \mathbb{R}^n_+$. We next move to address the second objective. Before that let us recall a characterization of K-monotonicity of a matrix A by Climent and Perea [3].

Theorem 3.2 ([3], Theorem 2)**.** *Let $A = U - V$ be a K-weak regular splitting of type* II. *Then, $\varrho(U^{-1}V) < 1$ if and only if A is K-monotone.*

We now present a new characterization of K-monotonicity using a double splitting that belongs to a class of double splittings we introduced in the beginning of this section.

Theorem 3.3. *Let $A = P - R - S$ be a double K-weak regular splitting of type* II. *Then, $\varrho(W) < 1$ if and only if A is K-monotone.*

Proof. We have

$$\widetilde{W} = \begin{pmatrix} RP^{-1} & SP^{-1} \\ I & O \end{pmatrix} \geqslant_{K_{2n}} 0.$$

Since $A = P - R - S$ is a double K-weak regular splitting of type II, we find that $\varrho(P^{-1}(R+S)) < 1$ as $A^{-1} \geqslant_K 0$ by Theorem 3.2. Thus, $\varrho(W) < 1$ by Theorem 3.1.

Conversely, assume that $\varrho(W) < 1$. By Theorem 3.1, we get $\varrho(P^{-1}(R+S)) < 1$, i.e., $\varrho((R+S)P^{-1}) < 1$, which further yields $[I-(R+S)P^{-1}]^{-1} \geqslant_K 0$ by Lemma 2.3. Now, $P^{-1} \geqslant_K 0$ and $[I-(R+S)P^{-1}]^{-1} \geqslant_K 0$ imply that

$$P^{-1}[I-(R+S)P^{-1}]^{-1} \geqslant_K 0,$$

i.e., $A^{-1} \geqslant_K 0$. Hence, A is K-monotone. □

Similarly, we obtain another characterization of K-monotonicity using a double K-weak regular splitting of type I. But, the sufficient part is proved in [8] and the necessary part can be proved as above.

Theorem 3.4. *Let $A = P - R - S$ be a double K-weak regular splitting of type* I. *Then, $\varrho(W) < 1$ if and only if A is K-monotone.*

Corollary 3.1. *Let $A = P - R - S$ be a double K-regular splitting. Then, $\varrho(W) < 1$ if and only if A is K-monotone.*

However, K-monotonicity of a matrix is not an equivalent criterion for the convergence of a double K-weak splitting of type II. Next, we discuss a few equivalent criteria for the convergence of a double K-weak splitting of type II.

Theorem 3.5. *Let $A = P - R - S$ be a double K-weak splitting of type* II. *Then the following conditions are equivalent:*

(a) $\varrho(W) < 1$.
(b) $\varrho(P^{-1}(R+S)) = \varrho((R+S)P^{-1}) < 1$.
(c) $PA^{-1} \geqslant_K 0$.
(d) $(R+S)A^{-1} \geqslant_K -I$.
(e) $[I - (R+S)P^{-1}]^{-1} \geqslant_K 0$.

P r o o f. By Theorem 3.1, (a) and (b) are equivalent. To show that (b) $\Longrightarrow$ (c), assume that (b) holds. By Theorem 2.3, we then have $[I - (R+S)P^{-1}] \geqslant_K 0$. Since $A^{-1} = P^{-1}[I - (R+S)P^{-1}]^{-1}$, we obtain $PA^{-1} = [I - (R+S)P^{-1}]^{-1} \geqslant_K 0$. If (c) holds, then $I + (R+S)A^{-1} = I + (P-A)A^{-1} = PA^{-1} \geqslant_K 0$. Thus, (c) $\Longrightarrow$ (d). Similarly (d) implies that $I + (R+S)P^{-1}[I - (R+S)P^{-1}]^{-1} \geqslant_K 0$ which by further simplification yields that $[I - (R+S)P^{-1}]^{-1} \geqslant_K 0$. Now, to show that (e) $\Longrightarrow$ (a). Assume that (e) holds. Then by Lemma 2.3, we get $\varrho((R+S)P^{-1}) < 1$ which further implies $\varrho(W) < 1$ by Theorem 3.1. □

3.2. Comparison results. Comparison theorems between the spectral radii of the iteration matrices are useful tools in the analysis of the rate of convergence of iterative methods or for judging the efficiency of pre-conditioners. A matrix may have two different double splittings

$$A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2 \tag{3.9}$$

whose corresponding iteration matrices are

$$W_1 = \begin{pmatrix} P_1^{-1}R_1 & P_1^{-1}S_1 \\ I & O \end{pmatrix}, \quad W_2 = \begin{pmatrix} P_2^{-1}R_2 & P_2^{-1}S_2 \\ I & O \end{pmatrix}.$$

In practice, we seek such a W which not only makes the computation of y^{i+1} (given y^i) simpler but also yields the spectral radius of W (which is of course less than 1) as small as possible for the faster rate of convergence of the scheme (1.3). An accepted rule for preferring one iteration scheme to another is to choose the scheme having the smaller spectral radius of the iteration matrix. We refer the interested reader to [10], [14], [15], and [17] for several comparison results. In this direction, this subsection gathers a few comparison results for double weak splittings of type II. In such a case, we have

$$\widetilde{W}_1 = \begin{pmatrix} R_1P_1^{-1} & S_1P_1^{-1} \\ I & O \end{pmatrix}, \quad \widetilde{W}_2 = \begin{pmatrix} R_2P_2^{-1} & S_2P_2^{-1} \\ I & O \end{pmatrix}.$$

Our first main result of this subsection is presented next.

Theorem 3.6. *Let $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two convergent double K-weak splittings of type* II. *If $(R_2 + S_2)P_2^{-1} \geqslant_K I$, then $\varrho(W_1) \leqslant \varrho(W_2)$.*

P r o o f. Clearly, $\widetilde{W}_1 \geqslant_{K_{2n}} 0$. So, there exists an eigenvector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_{2n}$ such that $\widetilde{W}_1 x = \varrho(W_1)x$, i.e.,

$$\begin{aligned} R_1P_1^{-1}x_1 + S_1P_1^{-1}x_2 &= \varrho(W_1)x_1, \\ x_1 &= \varrho(W_1)x_2. \end{aligned}$$

Then,

$$\begin{aligned} \widetilde{W}_2 x - \varrho(W_1)x &= \begin{pmatrix} R_2P_2^{-1}x_1 + S_2P_2^{-1}x_2 - \varrho(W_1)x_1 \\ x_1 - \varrho(W_1)x_2 \end{pmatrix} \\ &= \begin{pmatrix} R_2P_2^{-1}x_1 + S_2P_2^{-1}x_2 - \varrho(W_1)x_1 \\ O \end{pmatrix} \\ &= \begin{pmatrix} \varrho(W_1)R_2P_2^{-1}x_2 + S_2P_2^{-1}x_2 - \varrho(W_1)^2x_2 \\ O \end{pmatrix} = \begin{pmatrix} \nabla \\ O \end{pmatrix}, \end{aligned}$$

where

$$\nabla = \varrho(W_1)R_2P_2^{-1}x_2 + S_2P_2^{-1}x_2 - \varrho(W_1)^2x_2.$$

Now, $\nabla - [\varrho(W_1)^2R_2P_2^{-1}x_2 + \varrho(W_1)^2S_2P_2^{-1}x_2 - \varrho(W_1)^2x_2] \geqslant_K 0$, i.e.,

$$\begin{aligned} \nabla &\geqslant_K \varrho(W_1)^2R_2P_2^{-1}x_2 + \varrho(W_1)^2S_2P_2^{-1}x_2 - \varrho(W_1)^2x_2 \\ &= \varrho(W_1)^2[(R_2 + S_2)P_2^{-1} - I]x_2. \end{aligned}$$

If $(R_2 + S_2)P_2^{-1} \geqslant_K I$, we have

$$\nabla \geqslant_K O.$$

Hence, $\widetilde{W}_2 x - \varrho(W_1)x \geqslant_{K_{2n}} 0$. By Lemma 2.2, we have $\varrho(\widetilde{W}_1) \leqslant \varrho(W_2)$ and Lemma 3.1 yields $\varrho(W_1) \leqslant \varrho(W_2)$. □

We present below an example to show that the converse of Theorem 3.6 is not true.

Example 3.4. Let $A = \begin{pmatrix} 4 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be two double splittings, where

$$P_1 = \begin{pmatrix} 4 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad R_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 4 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9/10 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -11/10 \end{pmatrix}.$$

Then,

$$R_1P_1^{-1} = \begin{pmatrix} -1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \in \pi(K_3), \quad S_1P_1^{-1} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \in \pi(K_3),$$

and

$$R_2P_2^{-1} = \begin{pmatrix} -1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3/10 \end{pmatrix} \in \pi(K_3), \quad S_2P_2^{-1} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 11/30 \end{pmatrix} \in \pi(K_3).$$

Hence, the splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ are double K-weak splittings of type II. We have $\varrho(W_1) = 0.6404 \leqslant 0.7738 = \varrho(W_2) < 1$, but

$$(R_2 + S_2)P_2^{-1} - I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \notin \pi(K_3)$$

because

$$\begin{pmatrix} 1 \\ 0 \\ -1/3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \notin K_3.$$

If $A = P - R - S$ is a convergent double K-weak splitting of type II, then $I - RP^{-1} \geqslant_K I - RP^{-1} - SP^{-1} = I - (R+S)P^{-1}$. Since $[I - (R+S)P^{-1}]^{-1} \geqslant_K 0$, pre-multiplying by $[I - (R+S)P^{-1}]^{-1}$ on both sides, we get

$$[I - (R+S)P^{-1}]^{-1}(I - RP^{-1}) \geqslant_K [I - (R+S)P^{-1}]^{-1}[I - (R+S)P^{-1}] = I.$$

Now, using this relation and (3.8), we obtain

$$(I - \widetilde{W})^{-1} \geqslant_{K_{2n}} \begin{pmatrix} [I - (R+S)P^{-1}]^{-1} & O \\ O & I \end{pmatrix}$$

and $A = P - R - S$ is a convergent double K-weak splitting of type II. Now, by Theorem 2.2, we have $\varrho((I - \widetilde{W})^{-1}) \geqslant \max\{\varrho([I - (R + S)P^{-1}]^{-1}), 1\}$, i.e., $\varrho((I - \widetilde{W})^{-1}) \geqslant \varrho([I - (R+S)P^{-1}]^{-1})$ and hence

$$\varrho(\widetilde{W}) \geqslant \varrho((R + S)P^{-1}).$$

We thus have $\varrho(P^{-1}(R + S)) \leqslant \varrho(W)$ by Lemma 3.1. Hence, the iterative scheme (1.2) does not converge faster than the iterative scheme (3.7) when $A = P - R - S$ is a convergent double K-weak splitting of type II.

We next move our attention to comparing the rate of convergence of two different linear systems. Such problems appear while choosing an effective preconditioner among a few ones for solving linear system in a faster way. Let

$$A_1 = P_1 - R_1 - S_1, \quad A_2 = P_2 - R_2 - S_2 \tag{3.10}$$

be two double splittings. Then, we put

$$W_1 = \begin{pmatrix} P_1^{-1}R_1 & P_1^{-1}S_1 \\ I & O \end{pmatrix}, \quad W_2 = \begin{pmatrix} P_2^{-1}R_2 & P_2^{-1}S_2 \\ I & O \end{pmatrix},$$

and

$$\widetilde{W}_1 = \begin{pmatrix} R_1P_1^{-1} & S_1P_1^{-1} \\ I & O \end{pmatrix}, \quad \widetilde{W}_2 = \begin{pmatrix} R_2P_2^{-1} & S_2P_2^{-1} \\ I & O \end{pmatrix}.$$

For $i = 1, 2$, let us introduce block matrices and their single splittings of the form

$$\mathbb{A}_i = \begin{pmatrix} A_i & -S_i \\ O & P_i \end{pmatrix} = \mathbb{U}_i - \mathbb{V}_i, \tag{3.11}$$

where $\mathbb{U}_i = \begin{pmatrix} P_i & O \\ P_i & P_i \end{pmatrix}$ and $\mathbb{V}_i = \begin{pmatrix} R_i + S_i & S_i \\ P_i & O \end{pmatrix}$. Then,

$$\mathbb{V}_i\mathbb{U}_i^{-1} = \begin{pmatrix} R_i + S_i & S_i \\ P_i & O \end{pmatrix} \begin{pmatrix} P_i^{-1} & O \\ -P_i^{-1} & P_i^{-1} \end{pmatrix} = \begin{pmatrix} R_iP_i^{-1} & S_iP_i^{-1} \\ I & O \end{pmatrix} = \widetilde{W}_i.$$

Note that the single splittings $\mathbb{A}_i = \mathbb{U}_i - \mathbb{V}_i$ for $i = 1, 2$ are K_{2n}-weak splittings of type II whenever the corresponding double splittings $A_i = P_i - R_i - S_i$ for $i = 1, 2$ are double K-weak splittings of type II or double K-weak regular splittings of type II. Since $\mathbb{V}_i\mathbb{U}_i^{-1} = \widetilde{W}_i$, the problem of comparing double splittings reduces to the problem of comparing single splittings. To do this, we need the following results for single splittings. The first one is for the K-weak splitting of type II that extends [3], Theorem 2.2 to an arbitrary proper cone. The proof is similar to the proof provided in [3] and hence is omitted.

Theorem 3.7. *If $A = U - V$ be a convergent K-weak splitting of type* II *of a K-monotone matrix A, then*

$$\varrho(U^{-1}V) = \frac{\varrho(VA^{-1})}{1 + \varrho(VA^{-1})}.$$

We next produce a result that compares the rate of convergence of K-weak splittings of two different linear systems.

Theorem 3.8. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two convergent K-weak splittings of type* II. *If $A_2^{-1} \geqslant_K A_1^{-1}$ and any of the conditions*
(i) *$U_2U_1^{-1} \geqslant_K I$ and $U_2 \geqslant_K 0$,*
(ii) *$I \geqslant_K U_1U_2^{-1}$ and $U_1 \geqslant_K 0$*
holds, then $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$.

P r o o f. Since $A_2^{-1} - A_1^{-1} \geqslant_K 0$ and $U_2 \geqslant_K 0$, we get $U_2(A_2^{-1} - A_1^{-1}) \geqslant_K 0$, i.e., $U_2A_2^{-1} \geqslant_K U_2A_1^{-1}$, which implies $(I - V_2U_2^{-1})^{-1} \geqslant_K U_2U_1^{-1}(I - V_1U_1^{-1})^{-1} \geqslant_K (I - V_1U_1^{-1})^{-1} \geqslant_K 0$ as $U_2U_1^{-1} \geqslant_K I$. Hence, by Lemma 2.5, $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$. Similarly, if $I \geqslant_K U_1U_2^{-1}$ and $U_1 \geqslant_K 0$, then $(I - V_2U_2^{-1})^{-1} \geqslant_K U_1U_2^{-1}(I - V_2U_2^{-1})^{-1} = U_1A_2^{-1} \geqslant_K U_1A_1^{-1} \geqslant_K (I - V_1U_1^{-1})^{-1}$. Again, $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$ by Lemma 2.5. □

We provide the following result with a different set of sufficient conditions.

Theorem 3.9. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two convergent K-weak splittings of type* II *of K-monotone matrices A_1 and A_2. If $A_2^{-1} \geqslant_K A_1^{-1}$ and $V_2 \geqslant_K V_1 \geqslant_K 0$, then $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$.*

P r o o f. Clearly, $A_2^{-1} \geqslant_K A_1^{-1} \geqslant_K 0$ and $V_2 \geqslant_K V_1 \geqslant_K 0$ imply $V_2A_2^{-1} \geqslant_K V_1A_2^{-1} \geqslant_K 0$ and $V_1A_2^{-1} \geqslant_K V_1A_1^{-1} \geqslant_K 0$. Since the partial order relation obeys transitivity, we have $V_2A_2^{-1} \geqslant_K V_1A_1^{-1} \geqslant_K 0$. Applying Theorem 2.2, we get $\varrho(V_2A_2^{-1}) \geqslant \varrho(V_1A_1^{-1})$. Now,

$$\varrho(U_i^{-1}V_i) = \frac{\varrho(V_iA_i^{-1})}{1 + \varrho(V_iA_i^{-1})}$$

for $i = 1, 2$ by Theorem 3.7. Hence, $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$. □

The next result compares the rate of convergence of K-weak splittings of the same or different types.

Theorem 3.10. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two convergent K-weak splittings of the same or different types of K-monotone matrices A_1 and A_2. If $A_2^{-1} \geqslant_K A_1^{-1}$, $U_2 \geqslant_K U_1 \geqslant_K 0$, then $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$.*

P r o o f. We consider two K-weak splittings of different types. The other case can be proved similarly. Assume that $A_1 = U_1 - V_1$ is a convergent K-weak splitting of type II and $A_2 = U_2 - V_2$ is a convergent K-weak splitting of type I. Then, $A_2^{-1} \geqslant_K A_1^{-1} \geqslant_K 0$ and $U_2 \geqslant_K U_1 \geqslant_K 0$ imply $U_2A_2^{-1} \geqslant_K U_2A_1^{-1} \geqslant_K 0$ and $U_2A_1^{-1} \geqslant_K U_1A_1^{-1} \geqslant_K 0$. By transitivity, we have $U_2A_2^{-1} \geqslant_K U_1A_1^{-1} \geqslant_K 0$ which further yields $(I+V_2U_2^{-1})^{-1} \geqslant_K (I+V_1U_1^{-1})^{-1}$. Hence, by Lemma 2.5, $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$. □

In the next result, we consider K-weak splittings of different types.

Theorem 3.11. *Let $A_1 = U_1 - V_1$ and $A_2 = U_2 - V_2$ be two convergent K-weak splittings of different types of K-monotone matrices A_1 and A_2. If $A_2^{-1} \geqslant_K A_1^{-1}$ and $U_1^{-1} \geqslant_K U_2^{-1}$, then $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$.*

P r o o f. Assume that $A_1 = U_1 - V_1$ is a convergent K-weak splitting of type II and $A_2 = U_2 - V_2$ is a convergent K-weak splitting of type I. Since $(I - U_2^{-1}V_2)^{-1} \geqslant_K 0$ as $\varrho(U_2^{-1}V_2) < 1$, the condition $U_1^{-1} \geqslant_K U_2^{-1}$ implies $(I - U_2^{-1}V_2)^{-1}U_1^{-1} \geqslant_K (I - U_2^{-1}V_2)^{-1}U_2^{-1} = A_2^{-1}$. Now, $(I - U_2^{-1}V_2)^{-1}A_1^{-1} = (I - U_2^{-1}V_2)^{-1}U_1^{-1}(I - V_1U_1^{-1})^{-1} \geqslant_K (I - U_2^{-1}V_2)^{-1}U_2^{-1}(I - V_1U_1^{-1})^{-1} = A_2^{-1}(I - V_1U_1^{-1})^{-1} \geqslant_K A_1^{-1}(I - V_1U_1^{-1})^{-1}$. Since $V_1U_1^{-1} \geqslant_K 0$, there exists an eigenvector $x \geqslant_K 0$ corresponding to $\varrho(U_1^{-1}V_1)$ such that $V_1U_1^{-1}x = \varrho(U_1^{-1}V_1)x$ by Theorem 2.1. Post-multiplying by x the inequality $(I - U_2^{-1}V_2)^{-1}A_1^{-1} \geqslant_K A_1^{-1}(I - V_1U_1^{-1})^{-1}$, we get

$$(I - U_2^{-1}V_2)^{-1}A_1^{-1}x \geqslant_K A_1^{-1}(I - V_1U_1^{-1})^{-1}x = \frac{A_1^{-1}x}{1 - \varrho(U_1^{-1}V_1)}.$$

Again,

$$(I - U_2^{-1}V_2)^{-1}y \geqslant_K \frac{1}{1 - \varrho(U_1^{-1}V_1)}y,$$

where $y = A_1^{-1}x \geqslant_K 0$ and $y \neq 0$ imply that

$$\frac{1}{1 - \varrho(U_1^{-1}V_1)} \leqslant \frac{1}{1 - \varrho(U_2^{-1}V_2)}$$

by Lemma 2.2. Hence, $\varrho(U_1^{-1}V_1) \leqslant \varrho(U_2^{-1}V_2)$. □

Using the results proved for single splittings and (3.11), we now provide our first comparison result for double K-weak regular splittings of type II.

Theorem 3.12. *Let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be two convergent double K-weak splittings of type* II. *If $A_2^{-1} \geqslant_K A_1^{-1}$, $A_2^{-1}S_2P_2^{-1} \geqslant_K A_1^{-1}S_1P_1^{-1}$ and $P_2^{-1} \geqslant_K P_1^{-1}$, and any of the conditions*

(i) $P_2P_1^{-1} \geqslant_K I$ *and* $P_2 \geqslant_K 0$,
(ii) $I \geqslant_K P_1P_2^{-1}$ *and* $P_1 \geqslant_K 0$

holds, then $\varrho(W_1) \leqslant \varrho(W_2)$.

P r o o f. The conditions $A_2^{-1} \geqslant_K A_1^{-1}$, $A_2^{-1}S_2P_2^{-1} \geqslant_K A_1^{-1}S_1P_1^{-1}$ and $P_2^{-1} \geqslant_K P_1^{-1}$ imply

$$\mathbb{A}_2^{-1} = \begin{pmatrix} A_2^{-1} & A_2^{-1}S_2P_2^{-1} \\ O & P_2^{-1} \end{pmatrix} \geqslant_{K_{2n}} \begin{pmatrix} A_1^{-1} & A_1^{-1}S_1P_1^{-1} \\ O & P_1^{-1} \end{pmatrix} = \mathbb{A}_1^{-1}.$$

By (3.11) and the condition $P_2 \geqslant_K 0$, we have $\mathbb{U}_2 \geqslant_K 0$ and

$$\mathbb{U}_2\mathbb{U}_1^{-1} = \begin{pmatrix} P_2 & O \\ P_2 & P_2 \end{pmatrix} \begin{pmatrix} P_1^{-1} & O \\ -P_1^{-1} & P_1^{-1} \end{pmatrix} = \begin{pmatrix} P_2P_1^{-1} & O \\ O & P_2P_1^{-1} \end{pmatrix}.$$

Now, $P_2P_1^{-1} \geqslant_K I$ implies that $\mathbb{U}_2\mathbb{U}_1^{-1} \geqslant_{K_{2n}} I$. As $\mathbb{A}_i = \mathbb{U}_i - \mathbb{V}_i$ for $i = 1, 2$ are K_{2n}-weak splittings of type II, we then have $\varrho(\widetilde{W}_1) = \varrho(\mathbb{V}_1\mathbb{U}_1^{-1}) \leqslant \varrho(\mathbb{V}_2\mathbb{U}_2^{-1}) = \varrho(\widetilde{W}_2)$ by Theorem 3.8. Applying Lemma 3.1, we finally get $\varrho(W_1) \leqslant \varrho(W_2)$. Similarly, using the conditions $I \geqslant_K P_1P_2^{-1}$ and $P_1 \geqslant_K 0$, we can easily show that $I \geqslant_{K_{2n}} \mathbb{U}_1\mathbb{U}_2^{-1}$ and $\mathbb{U}_1 \geqslant_{K_{2n}} 0$, which inequalities imply $\varrho(\widetilde{W}_1) = \varrho(\mathbb{V}_1\mathbb{U}_1^{-1}) \leqslant \varrho(\mathbb{V}_2\mathbb{U}_2^{-1}) = \varrho(\widetilde{W}_2)$ by Theorem 3.8. Hence, $\varrho(W_1) \leqslant \varrho(W_2)$ by Lemma 3.1. □

Next, we prove a comparison result for double K-weak splittings of different types without using the single splittings defined by (3.11).

Theorem 3.13. *Let* $A_1 = P_1 - R_1 - S_1$ *be a convergent double* K*-weak splitting of type* I *and* $A_2 = P_2 - R_2 - S_2$ *be a convergent double* K*-weak splitting of type* II *of* K*-monotone matrices* A_1 *and* A_2. *If* $P_1^{-1}A_1 \geqslant_K A_2P_2^{-1}$ *and any of the conditions*

(1) $P_1^{-1}R_1 \geqslant_K R_2P_2^{-1}$,
(2) $S_2P_2^{-1} \geqslant_K P_1^{-1}S_1$

holds, then $\varrho(W_1) \leqslant \varrho(W_2)$.

P r o o f. We have $W_1 \geqslant_{K_{2n}} 0$, then by Theorem 2.1, there exists a vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in K_{2n}$ such that $W_1x = \varrho(W_1)x$, i.e.,

$$P_1^{-1}R_1x_1 + P_1^{-1}S_1x_2 = \varrho(W_1)x_1, \tag{3.12}$$

$$x_1 = \varrho(W_1)x_2. \tag{3.13}$$

Now,

$$\begin{aligned}
&\widetilde{W}_2x - \varrho(W_1)x \\
&= \begin{pmatrix} R_2P_2^{-1}x_1 + S_2P_2^{-1}x_2 - \varrho(W_1)x_1 \\ x_1 - \varrho(W_1)x_2 \end{pmatrix} \\
&= \begin{pmatrix} R_2P_2^{-1}x_1 + \dfrac{1}{\varrho(W_1)}S_2P_2^{-1}x_1 - P_1^{-1}R_1x_1 - \dfrac{1}{\varrho(W_1)}P_1^{-1}S_1x_1 \\ O \end{pmatrix} = \begin{pmatrix} \nabla \\ O \end{pmatrix},
\end{aligned}$$

where

$$\nabla = R_2P_2^{-1}x_1 + \frac{1}{\varrho(W_1)}S_2P_2^{-1}x_1 - P_1^{-1}R_1x_1 - \frac{1}{\varrho(W_1)}P_1^{-1}S_1x_1.$$

If the conditions $P_1^{-1}R_1 \geqslant_K R_2P_2^{-1}$ and $P_1^{-1}A_1 \geqslant_K A_2P_2^{-1}$ hold, we then have

$$\begin{aligned}
\nabla - \frac{1}{\varrho(W_1)}(R_2P_2^{-1} - P_1^{-1}R_1)x_1 + \frac{1}{\varrho(W_1)}(S_2P_2^{-1} - P_1^{-1}S_1)x_1 \\
= \Big(\frac{1}{\varrho(W_1)} - 1\Big)(P_1^{-1}R_1 - R_2P_2^{-1})x_1 \geqslant_K 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla &\geqslant_K \frac{1}{\varrho(W_1)}((R_2P_2^{-1} - P_1^{-1}R_1)x_1 + (S_2P_2^{-1} - P_1^{-1}S_1)x_1) \\
&= \frac{1}{\varrho(W_1)}((R_2 + S_2)P_2^{-1}x_1 - P_1^{-1}(R_1 + S_1)x_1) \\
&= \frac{1}{\varrho(W_1)}((P_2 - A)P_2^{-1} - P_1^{-1}(P_1 - A)) \\
&= \frac{1}{\varrho(W_1)}(P_1^{-1}A_1 - A_2P_2^{-1})x_1 \geqslant_K 0.
\end{aligned}$$

Thus, $\widetilde{W}_2x - \varrho(W_1)x \geqslant_{K_{2n}} 0$. By Lemma 2.2, $\varrho(W_1) \leqslant \varrho(\widetilde{W}_2)$ which further yields $\varrho(W_1) \leqslant \varrho(W_2)$ by Lemma 3.1. Similarly, if $S_2P_2^{-1} \geqslant_K P_1^{-1}S_1$ and $P_1^{-1}A_1 \geqslant_K A_2P_2^{-1}$, we then have

$$\begin{aligned}
\nabla - (R_2P_2^{-1} - P_1^{-1}R_1)x_1 - (S_2P_2^{-1} - P_1^{-1}S_1)x_1 \\
= \Big(\frac{1}{\varrho(W_2)} - 1\Big)(S_2P_2^{-1} - P_1^{-1}S_1)x_1 \geqslant_K 0.
\end{aligned}$$

Hence,

$$\nabla \geqslant_K (R_2P_2^{-1} - P_1^{-1}R_1)x_1 + (S_2P_2^{-1} - P_1^{-1}S_1)x_1 = (P_1^{-1}A_1 - A_2P_2^{-1})x_1 \geqslant_K 0.$$

Thus, $\widetilde{W}_2x - \varrho(W_1)x \geqslant_{K_{2n}} 0$. Applying Lemma 2.2, we get $\varrho(W_1) \leqslant \varrho(\widetilde{W}_2)$. Hence, $\varrho(W_1) \leqslant \varrho(W_2)$ by Lemma 3.1. □

Similarly, one can prove the following result that provides some sufficient conditions under which a double K-weak splitting of type II converges faster than a double K-weak splitting of type I.

Theorem 3.14. *Let $A_1 = P_1 - R_1 - S_1$ be a convergent double K-weak splitting of type* II *and $A_2 = P_2 - R_2 - S_2$ be a convergent double K-weak splitting of type* I *of K-monotone matrices A_1 and A_2. If $A_1 P_1^{-1} \geqslant_K P_2^{-1} A_2$ and any of the conditions*

(1) $R_1 P_1^{-1} \geqslant_K P_2^{-1} R_2$,
(2) $P_2^{-1} S_2 \geqslant_K S_1 P_1^{-1}$

holds, then $\varrho(W_1) \leqslant \varrho(W_2)$.

For $i = 1, 2$, let us again introduce block matrices and their single splittings of the form:

$$\mathbb{A}_i = \begin{pmatrix} P_i - R_i & -S_i \\ -P_i & P_i \end{pmatrix} = \mathbb{U}_i - \mathbb{V}_i, \tag{3.14}$$

where $\mathbb{U}_i = \begin{pmatrix} P_i & O \\ O & P_i \end{pmatrix}$ and $\mathbb{V}_i = \begin{pmatrix} R_i & S_i \\ P_i & O \end{pmatrix}$. Then,

$$\mathbb{V}_i \mathbb{U}_i^{-1} = \begin{pmatrix} R_i & S_i \\ P_i & O \end{pmatrix} \begin{pmatrix} P_i^{-1} & O \\ O & P_i^{-1} \end{pmatrix} = \begin{pmatrix} R_i P_i^{-1} & S_i P_i^{-1} \\ I & O \end{pmatrix} = \widetilde{W}_i$$

and

$$\mathbb{U}_i^{-1} \mathbb{V}_i = \begin{pmatrix} P_i^{-1} & O \\ O & P_i^{-1} \end{pmatrix} \begin{pmatrix} R_i & S_i \\ P_i & O \end{pmatrix} = \begin{pmatrix} P_i^{-1} R_i & P_i^{-1} S_i \\ I & O \end{pmatrix} = W_i.$$

We observe that $\mathbb{A}_i = \mathbb{U}_i - \mathbb{V}_i$ for $i = 1, 2$ are convergent K_{2n}-weak splittings of type II (type I) whenever the corresponding double splittings are convergent double K-weak splittings of type II (type I). $\mathbb{A}_i = \mathbb{U}_i - \mathbb{V}_i$ for $i = 1, 2$ become K_{2n}-weak splittings of the same types if both the splittings are either K-weak splittings of type II or K-weak splittings of type I. Similarly, these are K_{2n}-weak splittings of different types if one of the splittings is a K_{2n}-weak splitting of type I and the other one is a K_{2n}-weak splitting of type II. Using the splittings defined by (3.14), we now prove our first comparison result.

Theorem 3.15. *Let $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ be two convergent double K-weak splittings of type II of K-monotone matrices A_1 and A_2 such that $(I + A_2^{-1}) P_2^{-1} \geqslant_K (I + A_1^{-1}) P_1^{-1} \geqslant_K 0$. If $A_2^{-1} \geqslant_K A_1^{-1}$, $S_2 P_2^{-1} \geqslant_K S_1 P_1^{-1}$, $R_2 \geqslant_K R_1 \geqslant_K 0$, and $P_2 \geqslant_K P_1 \geqslant_K 0$, then $\varrho(W_1) \leqslant \varrho(W_2)$.*

P r o o f. We consider the splittings $\mathbb{A}_1 = \mathbb{U}_1 - \mathbb{V}_1$ and $\mathbb{A}_2 = \mathbb{U}_2 - \mathbb{V}_2$ as defined in (3.14). Since $A_1 = P_1 - R_1 - S_1$ and $A_2 = P_2 - R_2 - S_2$ are convergent double K-weak splittings of type II, we get $\mathbb{A}_1 = \mathbb{U}_1 - \mathbb{V}_1$ and $\mathbb{A}_2 = \mathbb{U}_2 - \mathbb{V}_2$ are convergent K_{2n}-weak splittings of type II. The conditions $P_2 \geqslant_K P_1 \geqslant_K 0$ and $S_2P_2^{-1} \geqslant_K S_1P_1^{-1}$ imply $S_2 \geqslant_K S_1 \geqslant_K 0$, which further yields $\mathbb{V}_2 \geqslant_{K_{2n}} \mathbb{V}_1 \geqslant_{K_{2n}} 0$. Now, $A_2^{-1} \geqslant_K A_1^{-1} \geqslant_K 0$ and $S_2P_2^{-1} \geqslant_K S_1P_1^{-1}$ imply $A_2^{-1}S_2P_2^{-1} \geqslant_K A_2^{-1}S_1P_1^{-1} \geqslant_K 0$ and $A_2^{-1}S_1P_1^{-1} \geqslant_K A_1^{-1}S_1P_1^{-1} \geqslant_K 0$ which by transitivity yields $A_2^{-1}S_2P_2^{-1} \geqslant_K A_1^{-1}S_1P_1^{-1} \geqslant_K 0$. Since $A_2^{-1} \geqslant_K A_1^{-1} \geqslant_K 0$, $A_2^{-1}S_2P_2^{-1} \geqslant_K A_1^{-1}S_1P_1^{-1} \geqslant_K 0$ and $(I + A_2^{-1})P_2^{-1} \geqslant_K (I + A_1^{-1})P_1^{-1} \geqslant_K 0$, we find that

$$\mathbb{A}_2^{-1} = \begin{pmatrix} A_2^{-1} & A_2^{-1}S_2P_2^{-1} \\ A_2^{-1} & (I + A_2^{-1})P_2^{-1} \end{pmatrix} \geqslant_{K_{2n}} \begin{pmatrix} A_1^{-1} & A_1^{-1}S_1P_1^{-1} \\ A_1^{-1} & (I + A_1^{-1})P_1^{-1} \end{pmatrix} = \mathbb{A}_1^{-1} \geqslant_{K_{2n}} 0.$$

By Theorem 3.9, we thus have $\varrho(W_1) \leqslant \varrho(W_2)$. □

The next result compares the spectral radii of the iterative schemes corresponding to two convergent double K-weak splittings of different types.

Theorem 3.16. *Let $A_1 = P_1 - R_1 - S_1$ be a convergent double K-weak splitting of type II and $A_2 = P_2 - R_2 - S_2$ be a convergent double K-weak splitting of type I of K-monotone matrices A_1 and A_2 such that $(I + A_2^{-1})P_2^{-1} \geqslant_K (I + A_1^{-1})P_1^{-1} \geqslant_K 0$. If $A_2^{-1} \geqslant_K A_1^{-1}$, $P_2 \geqslant_K P_1 \geqslant_K 0$ and $S_2P_2^{-1} \geqslant_K S_1P_1^{-1}$, then $\varrho(W_1) \leqslant \varrho(W_2)$.*

P r o o f. Let $\mathbb{A}_1 = \mathbb{U}_1 - \mathbb{V}_1$ and $\mathbb{A}_2 = \mathbb{U}_2 - \mathbb{V}_2$ be as defined in (3.14). Since $A_1 = P_1 - R_1 - S_1$ is a convergent double K-weak splitting of type II and $A_2 = P_2 - R_2 - S_2$ is a convergent double K-weak splitting of type I, we obtain that $\mathbb{A}_1 = \mathbb{U}_1 - \mathbb{V}_1$ is a convergent K_{2n}-weak splitting of type II and $\mathbb{A}_2 = \mathbb{U}_2 - \mathbb{V}_2$ is a convergent K_{2n}-weak splitting of type I. Since $A_2^{-1} \geqslant_K A_1^{-1} \geqslant_K 0$, $S_2P_2^{-1} \geqslant_K S_1P_1^{-1}$, and $(I + A_2^{-1})P_2^{-1} \geqslant_K (I + A_1^{-1})P_1^{-1} \geqslant_K 0$, we have $\mathbb{A}_2^{-1} \geqslant_{K_{2n}} \mathbb{A}_2^{-1} \geqslant_{K_{2n}} 0$. Now, the condition $P_2 \geqslant_K P_1$ implies that

$$\mathbb{U}_2 = \begin{pmatrix} P_2 & O \\ P_2 & P_2 \end{pmatrix} \geqslant_{K_{2n}} \begin{pmatrix} P_1 & O \\ P_1 & P_1 \end{pmatrix} = \mathbb{U}_1 \geqslant_{K_{2n}} 0.$$

By Theorem 3.10, we have $\varrho(\mathbb{U}_1^{-1}\mathbb{V}_1) \leqslant \varrho(\mathbb{U}_2^{-1}\mathbb{V}_2)$. Hence, $\varrho(W_1) \leqslant \varrho(W_2)$. □

Note that the above result is also true in the case when both the splittings are double K-weak splittings of type II. The last result of this paper compares the rate of convergence of iteration matrices formed by double K-weak splittings of different types.

Theorem 3.17. *Let $A_1 = P_1 - R_1 - S_1$ be a convergent double K-weak splitting of type* II *and $A_2 = P_2 - R_2 - S_2$ be a convergent double K-weak splitting of type* I *of*

K-monotone matrices A_1 and A_2 such that $(I + A_2^{-1})P_2^{-1} \geqslant_K (I + A_1^{-1})P_1^{-1} \geqslant_K 0$. If $A_2^{-1} \geqslant_K A_1^{-1}$, $P_1^{-1} \geqslant_K P_2^{-1}$ and $S_2P_2^{-1} \geqslant_K S_1P_1^{-1}$, then $\varrho(W_1) \leqslant \varrho(W_2)$.

P r o o f. Since $A_1 = P_1 - R_1 - S_1$ is a convergent double K-weak splitting of type II and $A_2 = P_2 - R_2 - S_2$ is a convergent double K-weak splitting of type I, then $\mathbb{A}_1 = \mathbb{U}_1 - \mathbb{V}_1$ is a convergent K_{2n}-weak splitting of type II and $\mathbb{A}_2 = \mathbb{U}_2 - \mathbb{V}_2$ is a convergent K_{2n}-weak splitting of type I. The conditions $A_2^{-1} \geqslant_K A_1^{-1} \geqslant_K 0$, $S_2P_2^{-1} \geqslant_K S_1P_1^{-1}$ and $(I + A_2^{-1})P_2^{-1} \geqslant_K (I + A_1^{-1})P_1^{-1} \geqslant_K 0$ imply that $\mathbb{A}_2^{-1} \geqslant_{K_{2n}} \mathbb{A}_1^{-1} \geqslant_{K_{2n}} 0$. Now, $P_1^{-1} \geqslant_K P_2^{-1}$ yields

$$\mathbb{U}_1^{-1} = \begin{pmatrix} P_1^{-1} & O \\ O & P_1^{-1} \end{pmatrix} \geqslant_{K_{2n}} \begin{pmatrix} P_2^{-1} & O \\ O & P_2^{-1} \end{pmatrix} = \mathbb{U}_2^{-1}.$$

By Theorem 3.11, we thus have $\varrho(\mathbb{U}_1^{-1}\mathbb{V}_1) \leqslant \varrho(\mathbb{U}_2^{-1}\mathbb{V}_2)$. Hence, $\varrho(W_1) \leqslant \varrho(W_2)$. □

The converse of all the above stated results may not be true. The following example shows that the converse of the above theorem is not true by considering a particular ice cream cone. For simplicity, we take $P_1 = P_2 = P$.

E x a m p l e 3.5. Let

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1/20 & 1 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} = P - R_1 - S_1 \text{ and } A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1/30 & 1 & 0 \\ 0 & 0 & 40/61 \end{pmatrix} = P - R_2 - S_2$$

be two double splittings, where

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 0 & 0 \\ -1/20 & 0 & 0 \\ 0 & 0 & 14/87 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5/29 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1/30 & 0 & 0 \\ 0 & 0 & 21/122 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 21/122 \end{pmatrix}.$$

Then,

$$A_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1/20 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} \in \pi(K_3), \quad A_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1/30 & 1 & 0 \\ 0 & 0 & 61/40 \end{pmatrix} \in \pi(K_3),$$
$$R_1P^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ -1/20 & 0 & 0 \\ 0 & 0 & 14/87 \end{pmatrix} \in \pi(K_3), \quad S_1P^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5/29 \end{pmatrix} \in \pi(K_3),$$

$$P^{-1}R_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1/30 & 0 & 0 \\ 0 & 0 & 21/122 \end{pmatrix} \in \pi(K_3), \quad P^{-1}S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 21/122 \end{pmatrix} \in \pi(K_3),$$

$$(I + A_1^{-1})P^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ -1/20 & 2 & 0 \\ 0 & 0 & 5/2 \end{pmatrix} \in \pi(K_3),$$

$$(I + A_2^{-1})P^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ -1/30 & 2 & 0 \\ 0 & 0 & 101/40 \end{pmatrix} \in \pi(K_3)$$

and

$$(I + A_2^{-1})P^{-1} - (I + A_1^{-1})P^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1/60 & 0 & 0 \\ 0 & 0 & 1/40 \end{pmatrix} \in \pi(K_3).$$

Also,

$$A_2^{-1} - A_1^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1/60 & 0 & 0 \\ 0 & 0 & 1/40 \end{pmatrix} \in \pi(K_3).$$

Hence, $A_1 = P - R_1 - S_1$ is a double K-weak splitting of type II and $A_2 = P - R_2 - S_2$ is a double K-weak splitting of type I of K-monotone matrices A_1 and A_2 such that $(I + A_1^{-1})P^{-1} \geqslant_K (I + A_2^{-1})P^{-1} \geqslant_K 0$ and $A_2^{-1} \geqslant_K A_1^{-1}$. We have $\varrho(W_1) = 0.5034 \leqslant 0.5098 = \varrho(W_2) < 1$, but $S_2P^{-1} - S_1P^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/3538 \end{pmatrix} \notin \pi(K_3)$ because $\begin{pmatrix} 0 \\ 0 \\ -1/3538 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/3538 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin K_3$.

4. Numerical computations

Consider the two-dimensional Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad \forall\, (x, y) \in \Omega \subset \mathbb{R}^2 \tag{4.1}$$

with the boundary conditions

$$u(x, y)|_{\partial\Omega} = g(x, y).$$

The finite difference method using the $\mathcal{O}(h^2)$ central difference discretization on non-uniform grids with $N_h \times N_k$ interior nodes generates the linear system $Ax = b$, where

the right-hand side vector b is derived from the Dirichlet boundary conditions and discrete values of $f(x, y)$. The coefficient matrix A is of the form

$$A = I_k \otimes J_h + J_k \otimes I_h. \tag{4.2}$$

Here $\otimes$ is the Kronecker product, and the matrices J_h and J_k are tridiagonal matrices of order N_h and N_k, respectively, i.e.,

$$J_h = \text{tridiagonal}\Big(\frac{1}{h^2}, \frac{-2}{h^2}, \frac{1}{h^2}\Big)$$

and

$$J_k = \text{tridiagonal}\Big(\frac{1}{k^2}, \frac{-2}{k^2}, \frac{1}{k^2}\Big),$$

where h and k are the non-uniform step sizes along x and y directions, respectively. Similarly, the identity matrices I_h and I_k are of dimension N_h and N_k, respectively. Let $A = D - L - U$, where $D = \text{diag}(A)$, and L and U are strictly lower and upper triangular matrices, respectively. Let $P = (D - \omega L)$, $R = \frac{1}{2}((1-\omega)D + \omega U)$, $S = P - R - \omega A$ be the double SOR splitting of the coefficient matrix A, where $\omega = 2/\big[1 + \sqrt{1 - \varrho(D^{-1}(D-A))^2}\big]$. When $\omega = 1$, then the above splitting reduces to the double Gauss-Seidel splitting. Note that the matrix A varies with interior nodes, and we therefore have different values of ω for different grid sizes.

Choice of source functions. Let us consider the two-dimensional Poisson equation (4.1) with two different source functions in a common domain $\Omega \cup \partial\Omega = [0, 1] \times [0, 1]$,

$$f_1(x, y) = \frac{5}{4} \exp\Big(x + \frac{y}{2}\Big) \quad \text{and} \quad f_2(x, y) = xy.$$

The Dirichlet boundary conditions are the restrictions of the corresponding exact solution to the boundaries given by

$$u_1(x, y) = \exp\Big(x + \frac{y}{2}\Big) \quad \text{and} \quad u_2(x, y) = \frac{1}{6}xy^3 + \sin x \sinh\, y.$$

All the convergence and comparison results in the theoretical section are mostly based on monotone matrices, hence we have considered the self-adjoint differential operator in equation (4.1) such that the second order finite difference discretization generates a linear system with a monotone coefficient matrix. The general form of the coefficient matrix with unequal mesh sizes can be verified to be an M-matrix. Our computational experiments aim neither to compare the existing iterative methods nor to verify the conditions of the results. It is well known that (non-stationary) Krylov subspace methods are comparatively better than the stationary methods including

the double splitting iterative method for a large class of matrices (see [12]). Further, the theorem assumptions are verified in our previous work [11]. Here, we would like to address some interesting computational observations on stationary iterative methods, which may need a careful study in future. In particular,

- ▷ the influence of the right-hand side vector on double splitting iterations and
- ▷ the large M-matrices generated with smaller mesh aspect ratio and the very fine uniform grids.

To have two different right-hand side vectors, we consider two different source functions $f_1(x)$ and $f_2(x)$ for all the tables and figures generated for the computational analysis. All the computations were carried out by MacBook Pro: 2.9 GHz Dual-Core Intel Core i5, 8 GB 2133 MHz LPDDR3. The stopping criteria of the iteration methods are based on the 2-norm of the error. The error at each iteration is measured by comparing with the exact solution $x_{\text{exact}} = A\backslash b$, where the backslash ($\backslash$) operator is the linear solver facilitated by Matlab. In the Table 1 computations, we consider the matrices from the uniform mesh discretizations. Here we can see that as N_h and N_k are gradually increasing towards higher values, the spectral radii of the matrices are also going closer to one. The spectral radius of the GS (Gauss-Seidel) splitting is always larger than the optimal SOR splitting, hence its convergence is slower as per the convergence and comparison results. The important observation is that as the grid size increases, the iteration number difference is larger and the difference of the iterations is larger in the GS method than in the optimal SOR method, which is about 700 iterations in comparison with only 25 iterations for the optimal SOR method. The same can be seen in Figure 1, but the iteration growth figure is convex upward which shows that the difference in iterations is not of exponential type.

$N_h \times N_k$ of A	IT-GS (f_1, f_2)	IT-SOR (f_1, f_2)	IT-CG (f_1, f_2)	IT-PCG (f_1, f_2)	$\varrho_{(W_{\text{GS}})}, \varrho_{(W_{\text{SOR}})}$
10×10	392, 405	71, 74	35, 36	16, 17	0.9466, 0.6877
20×20	1477, 1522	139, 145	70, 73	23, 24	0.9852, 0.8212
30×30	3277, 3370	208, 216	105, 109	28, 28	0.9932, 0.8747
40×40	5804, 5963	277, 288	140, 145	32, 33	0.9961, 0.9036
50×50	9067, 9309	347, 361	176, 182	36, 37	0.9975, 0.9216
60×60	13071, 13415	417, 435	211, 218	40, 40	0.9982, 0.9340
70×70	17824, 18284	488, 508	247, 255	43, 44	0.9987, 0.9430
80×80	23326, 23927	559, 583	283, 292	46, 47	0.9990, 0.9498

Table 1. Comparison table for the convergence of the double splitting in case of the two different source functions (f_1, f_2) and uniform grid discretization where both N_h and N_k are equal.

Now, let us fix the matrix size to 3600 and vary the order pair (N_h, N_k) such that the grid aspect ratio $\lambda = N_h/N_k$ will be in $(0, 1]$. In this experiment, the unequal mesh size varies as 10×360, 20×180, 30×120, 40×90, 50×72, and 60×60. Since the solutions are symmetric or nearly symmetric about the $x = y$ line, we have less interest for the grids of sizes 72×50, 90×40, 120×30, 180×20, and 360×10. In Table 2, as the grid aspect ratio is smaller, the iteration numbers are going higher and the spectral radii are also close to one, especially the spectral radii of the iteration matrix of the GS method are almost one.

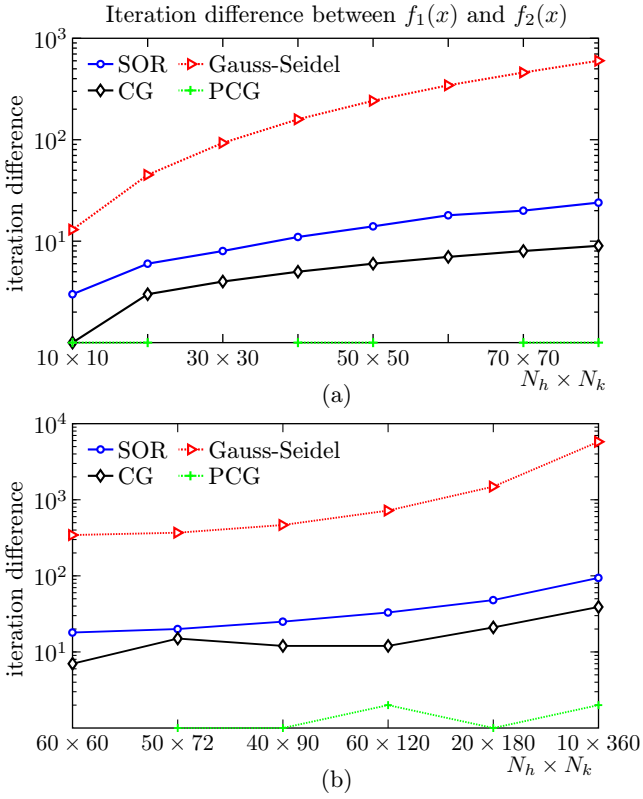

Figure 1. Iteration number plotted against the different grid pairs $(N_h \times N_k)$: Figure (a) is for the uniform grids with the grid aspect ratio 1.0 where the matrix size varies from 100 to 6400, and Figure (b) is for the matrices generated with the grid aspect ratio other than 1.0 with the fixed matrix size 3600.

The difference between the iterations due to the change in the right-hand side vector is more significant than the higher grid size like 6400 in Table 1. The differences in the iterations in these unequal meshes generated metrics are plotted in Figure 1(b). Here the iteration growth curve is concave upward and also monotoni-

cally increasing, which is exponential. In particular, for the grid size 10×360, the GS method required about 230 thousand iterations, whereas only 23 thousand iterations for the 80×80 grid size in uniform discretization.

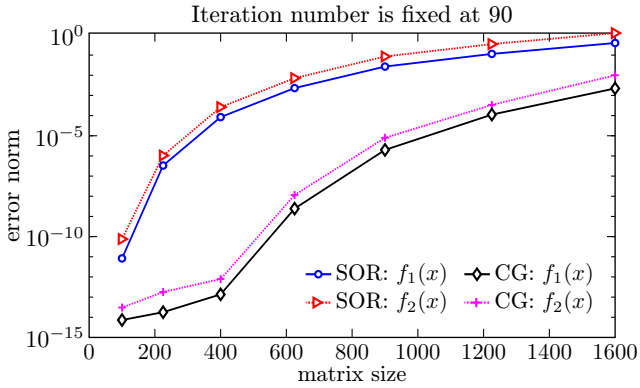

Figure 2. Behavior of the error norm for different matrix size.

Finally, in Figure 2, the iteration number is fixed at 90 and the location of the error is plotted for different sizes of matrices. Also these matrices are collected by fixing the grid aspect ratio 1 and their size varies from 100 to 1600. The SOR and CG (conjugate gradient) methods are applied to two different source functions. The error difference is not significant but the error curve corresponding to the f_1 source is continuously below the curve corresponding to the f_2 source. Although the error difference between SOR and CG is almost the same the corresponding iteration number difference is larger in the case of the SOR method than the CG method, due to the slow convergence of the SOR method. So, the impact of RHS is more significant for the SOR method than the CG method. The superlinear convergence and quadratic convergence of the CG method can be seen from this error graph.

$N_h \times N_k$	IT-GS (f_1, f_2)	IT-SOR (f_1, f_2)	IT-CG (f_1, f_2)	IT-PCG (f_1, f_2)	$\varrho_{(W_{\mathrm{GS}})}, \varrho_{(W_{\mathrm{SOR}})}$
10×360	230046, 235835	1732, 1826	1070, 1109	45, 47	0.9999, 0.9838
20×180	58380, 59868	874, 922	531, 552	58, 59	0.9996, 0.9681
30×120	27408, 28129	600, 633	362, 374	49, 51	0.9992, 0.9539
40×90	17497, 17963	481, 506	285, 297	44, 45	0.9987, 0.9426
50×72	13928, 14297	430, 450	247, 262	41, 42	0.9983, 0.9360
60×60	13071, 13415	417, 435	211, 218	40, 40	0.9982, 0.9340

Table 2. Comparison table for the convergence of the double splitting in case of the two different source functions (f_1, f_2) and uniform grid discretization where both N_h and N_k vary but the product is equal.

Preconditioned conjugate gradient method. The iterative methods based on matrix splittings give slower convergence than the Krylov subspace method, as they do not guarantee the convergence when the iteration number reaches the size of the matrix unlike Krylov subspace methods, assuming the infinite precision arithmetic. Further, the matrix generated by the Poisson equations is symmetric positive definite for which the conjugate gradient method is the suitable Krylov subspace method. The CG method can be made faster converging if we can make use of a suitable preconditioning matrix. Mostly, the preconditioning matrices are derived from the splitting iterations. If M_P is a preconditioner, then the preconditioned system is

$$M_P^{-1}Ax = M_P^{-1}b \tag{4.3}$$

and the CG method requires less number of iterations when the M_P matrix is closer to A. As the matrix A is symmetric positive definite, we can have M_P as a symmetric matrix. Hence, we can use the symmetric SOR (SSOR) preconditioning

$$M_{\mathrm{SSOR}} = (I + \omega LD^{-1})\frac{1}{\omega(2-\omega)}D(I + \omega D^{-1}U). \tag{4.4}$$

In our computations, we have used the M_{SSOR} preconditioner for the preconditioned conjugate gradient (PCG) algorithm (see [6] and [12]). The CG iteration minimizes the energy norm of the error, but in our computation, we have used the 2-norm of the error as the stopping criterion for the CG and PCG algorithms like GS and SOR methods. Table 1 presents the performance of the CG and PCG methods in its 4th and 5th columns. The CG method takes almost half the number of iterations required for the double SOR iteration scheme. The double SOR based preconditioner accelerates the CG iteration approximately six times, especially for fine grids. Further, the preconditioning matrix changes each time when the grid distribution changes. We can see the advantages in Table 2 where the size of the matrix remains the same and the mesh aspect ratio changes. For the GS, SOR, and CG method, the iteration number gradually increases when the aspect ratio reduces. But, the PCG method performs in almost uniform manner for all aspect ratios. Also, the impact of the right-hand side is not significant even for matrices of size 3600. In this table, the PCG method is 20 times faster than the CG method at the lowest aspect ratio.

Table 1 and Table 2 are summarized in Figure 1 to compare the change in the iteration number due to the change in source functions. The difference in the iteration numbers of the CG method applied to different source functions is plotted and found to have a similar pattern like the GS and SOR methods. The difference in the iteration numbers grows higher when the mesh aspect ratio reduces or the matrix size increases. In the case of the PCG method, the difference is almost negligible.

And the line representing PCG has few not visible regions that show the difference in iteration number is less than one.

5. Concluding remarks

In this paper, we have further studied the problem of convergence of the double iteration scheme for double K-weak splittings of type II. The important findings are summarized as follows:

- ▷ The notion of double K-weak splittings of type II is proposed first. Convergence theory for this class of double splittings is then established. This theory generalizes the existing theory for double weak splittings of type II that appeared in [13] to an arbitrary cone.
- ▷ Some new comparison results are examined next in Subsection 3.2. They are useful in detecting the matrix splitting which gives a faster convergence rate. Comparisons of the rate of convergence of iterative schemes of two different linear systems are then provided. More importantly, the results obtained in Subsection 3.2 are completely new even in the case of $K = \mathbb{R}^n_+$.
- ▷ Finally, we have applied the double splitting for the symmetric and positive definite matrices generated by the finite difference discretization of the Poisson equation with different source functions. The b vector of the linear system plays a significant role in the rate of convergence of classical matrix splitting methods. Further, the double splitting preconditioner used in the PCG method yields faster convergence and hence the influence of the b vector is negligible.

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