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AN INTERIOR-POINT ALGORITHM FOR SEMIDEFINITE LEAST-SQUARES PROBLEMS

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Abstract. We propose a feasible primal-dual path-following interior-point algorithm for semidefinite least squares problems (SDLS). At each iteration, the algorithm uses only full Nesterov-Todd steps with the advantage that no line search is required. Under new appropriate choices of the parameter β which defines the size of the neighborhood of the central-path and of the parameter θ which determines the rate of decrease of the barrier parameter, we show that the proposed algorithm is well defined and converges to the optimal solution of SDLS. Moreover, we obtain the currently best known iteration bound for the algorithm with a short-update method, namely, $\mathcal{O}(\sqrt{n}\log(n/\varepsilon))$. Finally, we report some numerical results to illustrate the efficiency of our proposed algorithm.

Keywords: semidefinite least-squares problem; interior-point method; polynomial complexity

MSC 2020: 65K05, 90C22, 90C25, 90C51

1. Introduction

Let $\mathbb{S}^n$ denote the space of $n \times n$ real symmetric matrices and $\mathbb{S}^n_+$ the cone of symmetric positive semidefinite matrices. The semidefinite least-squares (SDLS) problem is expressed as the following optimization problem:

$$(\mathcal{P}) \qquad p^* = \min_X f(X) = \frac{1}{2}\|X - C\|_F^2 \quad \text{s.t. } A_i \bullet X = b_i,\ i = 1, \ldots, m,\ X \succeq 0$$

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and its Lagrangian dual problem:

$$(\mathcal{D}) \qquad d^* = \max_{(y,Z)} g(y,Z) = b^\top y - \frac{1}{2}\|X\|_F^2 + \frac{1}{2}\|C\|_F^2$$
$$\text{s.t. } X - Z - \sum_{i=1}^{m} y_i A_i = C,\ Z \succeq 0,\ y \in \mathbb{R}^m.$$

Here, $b \in \mathbb{R}^m$, $C, A_i \in \mathbb{S}^n$, $i = 1, \ldots, m$, and the inequality $M \succeq 0$ means that $M \in \mathbb{S}^n_+$. The trace inner-product and the Frobenius norm in $\mathbb{S}^n$ are defined, respectively, by

$$A \bullet B = \mathrm{Tr}(AB) = \sum_{i,j} a_{ij} b_{ij} \text{ and } \|A\|_F = (\mathrm{Tr}(A^2))^{1/2} \text{ for } A = (a_{ij}),\ B = (b_{ij}) \in \mathbb{S}^n.$$

The SDLS problem provides an attractive class of nonlinear convex programming [10], [22], [23], [24] which has been proven to be useful in the domain of applied mathematics and numerical linear algebra. For instance, the nearest correlation matrix, preconditioning of linear systems, and error analysis of such iterative methods can be reformulated as SDLS [13]. Until now there have been some solution approaches for SDLS. In [17], Higham proposed an alternating projections method to solve particular instances of the SDLS (and it could be generalized to any semidefinite least-squares). Henrion and Malick [16] presented a Matlab package for solving conic least squares problems. In addition, Krislock [20] presented a numerical solution for semidefinite constrained least squares problems based on interior-point algorithms. Later on, Malick [21], proposed a Lagrangian dualization of SDLS and then solved the latter with a quasi-Newton algorithm.

Interior-point methods (IPMs) play an important role for solving wide problems of convex programming (see for example [29]). Among them, the so-called primal-dual IPMs are most efficient from a computational point of view. These methods were first used for solving linear optimization (LO) [3], [6], [29] and then were extended successfully for convex quadratic optimization (CQO) and complementarity problems (CP) [1], [7], [4], [19], semidefinite optimization (SDO) and CQSDO problems [2], [8], [15], [18], [14], [23], [25], [26], [28], [30].

In this paper, we propose a new feasible primal-dual path-following interior point algorithm for SDLS. For its analysis, we propose new appropriate choices of the parameter β which defines the size of the neighborhood of the central-path and of the parameter θ which determines the rate of decrease of the barrier parameter. Under these two defaults, we show that the algorithm is well defined and converges to the optimal solution of SDLS. Moreover, we prove that the polynomial complexity of this short-step algorithm is $\mathcal{O}(\sqrt{n}\log(n/\varepsilon))$. Finally, some numerical tests on several different examples of SDLS problems are included to illustrate the effectiveness of this algorithm. Next, some notations used throughout the paper are specified. Let $\mathbb{S}^n_{++}$ denote the set of $n \times n$ symmetric positive definite matrices. Furthermore $X \succeq 0$ ($X \succ 0$) means that $X \in \mathbb{S}^n_+$ ($X \in \mathbb{S}^n_{++}$), $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ stand for the largest and the smallest eigenvalue of the matrix X. The trace of an $n \times n$ matrix $A = (a_{ij})$ is denoted by $\mathrm{Tr}(A) = \sum\limits_{i=1}^{n} a_{ii}$ and the determinant is denoted by $\det A$. The symmetric positive definite square root of any symmetric positive definite matrix X is denoted by $X^{1/2}$. Finally, if $h(x) \geqslant 0$ is a real valued function of a real nonnegative variable, the notation $h(x) = \mathcal{O}(x)$ means that $h(x) \leqslant kx$ for some positive constant k. The abbreviation $A \sim B$ means that A and B are similar, i.e., $A = PBP^{-1}$ for some nonsingular matrix P. Finally, I, $0_{n\times n}$, and e denote the identity and zero matrices of order n, and the vector of ones, respectively.

The remaining part of the paper is organized as follows. In Section 2, the generic primal-dual path-following interior-point (IP) algorithm for SDLS is presented. In Section 3, the detailed proofs of the complexity results are stated. In Section 4, some numerical results are reported. A conclusion and future works end Section 5.

2. The primal-dual IP algorithm for SDLS

Before describing our generic algorithm for solving SDLS, some auxiliary results are required.

In the sequel, we will denote by

$$\begin{aligned}
F_{\mathcal{P}} &= \{X \in \mathbb{S}^n_+ \colon\ A_i \bullet X = b_i,\ i = 1, \dots, m\},\\
F^0_{\mathcal{P}} &= \{X \in F_{\mathcal{P}} \colon\ X \in \mathbb{S}^n_{++}\},\\
F_{\mathcal{D}} &= \left\{(y, Z) \in \mathbb{R}^m \times \mathbb{S}^n_+ \colon\ \sum_{i=1}^{m} y_i A_i + Z - X = -C\right\},\\
F^0_{\mathcal{D}} &= \{(y, Z) \in F_{\mathcal{D}} \colon\ Z \in \mathbb{S}^n_{++}\},
\end{aligned}$$

the feasible and the strictly feasible sets of $\mathcal{P}$ and $\mathcal{D}$, respectively.

Theorem 2.1 (Weak duality [11]). *Let $X \in F_{\mathcal{P}}$ and $(y, Z) \in F_{\mathcal{D}}$. Then*

$$f(X) - g(y, Z) = X \bullet Z \geqslant 0,$$

where the expressions $f(X) - g(y, Z)$ and $X \bullet Z$ are called, respectively, the duality gap and the complementarity condition for $\mathcal{P}$ and $\mathcal{D}$. Moreover, if $X \bullet Z = 0$, then X is an optimal solution of $\mathcal{P}$ and (y, Z) is an optimal solution of $\mathcal{D}$.

Theorem 2.2 (Strong duality [11]). *If $F^0_{\mathcal{P}} \neq \emptyset$ and $F^0_{\mathcal{D}} \neq \emptyset$, then the sets of optimal solutions of $\mathcal{P}$ and $\mathcal{D}$ are nonempty and bounded and we have*

$$p^* = d^*.$$

Lemma 2.1 ([14]). *If $X, Z \in \mathbb{S}^n_+$ and $X \bullet Z = 0$, then $XZ = 0$.*

Moreover, $X \in F_{\mathcal{P}}$ and $(y, Z) \in F_{\mathcal{D}}$ are optimal solutions for $\mathcal{P}$ and $\mathcal{D}$ if they satisfy the following optimality conditions:

$$\begin{cases} X - Z - \sum\limits_{i=1}^{m} y_i A_i = C, \quad X \succeq 0,\ Z \succeq 0, \\ A_i \bullet X = b_i, \quad i = 1, \dots, m, \\ XZ = 0. \end{cases} \tag{2.1}$$

In the sequel of this paper, we assume that both problems $\mathcal{P}$ and $\mathcal{D}$ satisfy the following conditions:

▷ Interior-Point-Condition (IPC), i.e., $F^0_{\mathcal{P}} \times F^0_{\mathcal{D}} \neq \emptyset$.
▷ The matrices A_i, $i = 1, \dots, m$, are linearly independent.

2.1. The central-path. To the problem $\mathcal{P}$, we associate the following perturbed optimization problem:

$$(\mathcal{P}_\mu) \qquad \min_X f_\mu(X) \quad \text{s.t. } X \in F_{\mathcal{P}},$$

where

$$f_\mu(X) = \begin{cases} f(X) - \mu \ln\det X & \text{if } X \in F^0_{\mathcal{P}}, \\ \infty & \text{otherwise.} \end{cases}$$

The function $(-\ln\det X)$ is called the logarithmic barrier function associated with the cone $\mathbb{S}^n_+$ and $\mu > 0$ is the barrier parameter. This function has been studied by many authors in the context of continuous convex programming [9], [12], [17]. The study of SDLS by IPMs presents a great similarity with SDO and CQSDO. By incorporating the concept of recession cones and if the IPC condition holds, then the problem $\mathcal{P}_\mu$ has a unique optimal solution $X(\mu)$ for all $\mu > 0$. The problem $\mathcal{P}_\mu$ is a continuous convex optimization problem. So the necessary and sufficient conditions for $X(\mu)$ to be an optimal solution for $\mathcal{P}_\mu$ is the existence of a vector $y(\mu) \in \mathbb{R}^m$ such that:

$$\begin{cases} X - C - \mu X^{-1} - \sum\limits_{i=1}^{m} y_i A_i = 0, \\ A_i \bullet X = b_i, \quad i = 1, \dots, m. \end{cases} \tag{2.2}$$

Set

$$Z(\mu) = \mu X^{-1}.$$

Then

$$X(\mu)Z(\mu) = \mu I.$$

Therefore the system (2.2) can be rewritten equivalently as

$$\begin{cases} X(\mu) - Z(\mu) - \sum\limits_{i=1}^{m} y_i(\mu)A_i = C, \quad X \succ 0,\ Z \succ 0, \\ A_i \bullet X(\mu) = b_i, \quad i = 1, \ldots, m, \\ X(\mu)Z(\mu) = \mu I, \quad \mu > 0. \end{cases} \tag{2.3}$$

The μ-central of $\mathcal{P}$ and $\mathcal{D}$ is defined by the solution $(X(\mu), y(\mu), Z(\mu))$ of the system (2.3) for all $\mu > 0$. The set

$$\mathcal{C} = \{(X(\mu), y(\mu), Z(\mu)) \mid \mu > 0\}$$

of μ-centers of (2.3) is called the central-path of $\mathcal{P}$ and $\mathcal{D}$. If $\mu \mapsto 0$ then the limit of the central-path exists and since the limit points satisfy the complementarity condition in (2.1), it yields an optimal solution of both $\mathcal{P}$ and $\mathcal{D}$, see [9], [12], [24].

2.2. The Nesterov-Todd (NT) search directions. Now we describe a full NT-step produced by the proposed algorithm for a given $\mu > 0$. Applying Newton's method for system (2.3) for a given strictly feasible point $(X \succ 0, y, Z \succ 0)$. Then the Newton direction at this point is the unique solution of the following linear system:

$$\begin{cases} \Delta X - \Delta Z - \sum\limits_{i=1}^{m} \Delta y_i A_i = 0, \\ A_i \bullet \Delta X = 0, \quad i = 1, \ldots, m, \\ \Delta X Z + X \Delta Z = \mu I - XZ, \quad \mu > 0. \end{cases} \tag{2.4}$$

The system (2.4) may give as a solution a search direction which is not necessarily symmetric. Since we want X and Z to be symmetric matrices, one must symmetrize the perturbed complementarity equation $XZ = \mu I$. Based on different symmetrization schemes, several search directions have been proposed in the literature of SDO problems such as Kojima et al. [19], Helmberg et al. [15], Monteiro [23], Nesterov-Todd (NT) [25], [27] and Alizadeh et al. [8]. Here we will use an invertible matrix P introduced by Nesterov and Todd, and replacing the equation:

$$\Delta X Z + X \Delta Z = \mu I - XZ$$

in system (2.4) by the following equation:

$$\Delta X + P \Delta Z P^{\top} = \mu Z^{-1} - X,$$

we then obtain the system

$$
\begin{cases}
\Delta X - \Delta Z - \sum\limits_{i=1}^{m} \Delta y_i A_i = 0, \\
A_i \bullet \Delta X = 0, \quad i = 1, \ldots, m, \\
\Delta X + P \Delta Z P^\top = \mu Z^{-1} - X, \quad \mu > 0,
\end{cases} \tag{2.5}
$$

where

$$
P = X^{1/2}(X^{1/2} Z X^{1/2})^{-1/2} X^{1/2} \quad (= Z^{-1/2}(Z^{1/2} X Z^{1/2})^{1/2} Z^{-1/2}), \quad P \in \mathbb{S}^n_{++}.
$$

Moreover, we define $D = P^{1/2}$. The matrix D can be used to rescale X and Z to the same symmetric positive definite matrix V, defined by

$$
V = \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} D Z D. \tag{2.6}
$$

Moreover, the scaled directions D_X and D_Z are defined by

$$
D_X = \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \quad D_Z = \frac{1}{\sqrt{\mu}} D \Delta Z D. \tag{2.7}
$$

Set $(D_y)_i = \Delta y_i$. Now, due to (2.6) and (2.7), the system (2.5) becomes

$$
\begin{cases}
D^2 D_X D^2 - D_Z - \sum\limits_{i=1}^{m} (D_y)_i \bar{A}_i = 0, \\
\bar{A}_i \bullet D_X = 0, \quad i = 1, \ldots, m, \\
D_X + D_Z = P_V,
\end{cases} \tag{2.8}
$$

where

$$
\bar{A}_i = \frac{1}{\sqrt{\mu}} D A_i D, \ i = 1, \ldots, m, \ \text{and} \quad P_V = V^{-1} - V. \tag{2.9}
$$

Note that:
- ▷ The matrices $\bar{A}_i$, $i = 1, \ldots, m$, are linearly independent.
- ▷ The directions D_X and D_Z are symmetric with $D_X \bullet D_Z \geqslant 0$. The latter follows from the first and the second equations of the system (2.5) and the expression in (2.7) that
$$
D_X \bullet D_Z = \frac{1}{\mu} \Delta X \bullet \Delta X = \frac{1}{\mu} \|\Delta X\|_F^2 \geqslant 0.
$$
  Hence, the directions D_X and D_Z are not orthogonal, unlike the cases of LO and SDO. This makes the analysis of the proposed algorithm slightly different.

The resolution of the system (2.8) gives us the symmetric directions D_X and D_Z. Then, via (2.7), we compute $(\Delta X, \Delta y, \Delta Z)$. Hence, a new full NT-step iteration is constructed according to

$$X_+ = X + \Delta X, \quad y_+ = y + \Delta y, \quad Z_+ = Z + \Delta Z.$$

Also, according to (2.9), we introduce a proximity measure as follows:

$$\delta(V) := \delta(X, Z; \mu) = \frac{\|P_V\|_F}{2}.$$

It is clear that $\delta(V) = 0 \Leftrightarrow V^{-1} = V \Leftrightarrow XZ = \mu I$. Hence, the value of $\delta(V)$ can be considered as a measure for the distance between a given point $(X \succ 0, y, Z \succ 0)$ and the central-path $\mathcal{C}$.

2.3. Algorithm. The primal-dual IP algorithm for solving SDLS works as follows. First, we use a suitable threshold (default) value $\beta > 0$ with $0 < \beta < 1$ and we suppose that a strictly feasible initial point $(X_0 \succ 0, y_0, Z_0 \succ 0)$ such that $\delta(X_0, Z_0; \mu_0) \leqslant \beta$ for certain μ_0 is known. Using the obtained search directions from (2.5) and taking a full NT-step, the algorithm produces a new iterate $(X_+, y_+, Z_+) = (X + \Delta X, y + \Delta y, Z + \Delta Z)$. Then, the barrier parameter value μ is reduced by the factor $(1 - \theta)$ with $0 < \theta < 1$ and solves system (2.5), and so targets a new μ-center and so on. This procedure is repeated until the stopping criterion $n\mu \leqslant \varepsilon$ is satisfied for a given accuracy parameter ε. The generic form of the algorithm is stated in Algorithm 1.

Begin Algorithm 1

Initialization

a precision parameter $\varepsilon > 0$, a proximity parameter β, $0 < \beta < 1$ (default $\beta = 1/\sqrt{2}$);
a parameter θ, $0 < \theta < 1$ (default $\theta = 1/(3\sqrt{n})$);
a strictly feasible point (X_0, y_0, Z_0) and $\mu_0 = (X_0 \bullet Z_0)/n$ s.t. $\delta(X_0, Z_0; \mu_0) \leqslant \beta$;
$k = 0$;

While $n\mu \geqslant \varepsilon$ **do**

- $\triangleright$ $\mu := (1 - \theta)\mu$;
- $\triangleright$ compute $(\Delta X, \Delta y, \Delta Z)$ via system (2.8) and use (2.7);
- $\triangleright$ update $X := X + \Delta X$, $y := y + \Delta y$, $Z := Z + \Delta Z$;
- $\triangleright$ $k := k + 1$;

endWhile

endAlgorithm

---

3. The analysis of the algorithm

In this section, we will prove with the defaults $\beta = 1/\sqrt{2}$ and $\theta = 1/(3\sqrt{n})$ that the algorithm solves the SDLS in polynomial time. For the analysis of the algorithm, we introduce the following symmetric matrix:

$$D_{XZ} = \frac{1}{2}(D_X D_Z + D_Z D_X).$$

We cite some useful lemmas in [5], [14] which will be used later.

Lemma 3.1 ([14], Lemma 6.1). *Let $X(\alpha) = X + \alpha\Delta X$, $Z(\alpha) = Z + \alpha\Delta Z$ such that $X \succ 0$ and $Z \succ 0$. If*

$$\det(X(\alpha)Z(\alpha)) > 0 \quad \forall\, 0 \leqslant \alpha \leqslant \overline{\alpha},$$

then

$$X(\overline{\alpha}) \succ 0 \quad \textit{and} \quad Z(\overline{\alpha}) \succ 0.$$

Lemma 3.2 ([14], Lemma 6.2). *Let $A \in \mathbb{S}^n$ and let $B \in \mathbb{R}^{n\times n}$ be an anti-symmetric matrix, i.e. $B = -B^\top$. If $A \succ 0$ then $\det(A+B) > 0$. Moreover, if $\lambda_i(A+B)$ are real for $i = 1, \ldots, n$, then*

$$0 < \lambda_{\min}(A) \leqslant \lambda_{\min}(A+B) \leqslant \lambda_{\max}(A+B) \leqslant \lambda_{\max}(B)$$

which implies that $(A+B) \succ 0$.

Lemma 3.3 ([5]). *Let (D_X, D_Z) be a solution of the system (2.8) and $\mu > 0$. If $\delta = \delta(X, Z; \mu)$ then*

$$0 \leqslant D_X \bullet D_Z \leqslant 2\delta^2. \tag{3.1}$$

In addition the spectral norm of D_{XZ} satisfies

$$\|D_{XZ}\|_2 \leqslant \frac{1}{4}\|D_X + D_Z\|_F^2 = \delta^2 \tag{3.2}$$

and

$$\|D_{XZ}\|_F^2 \leqslant \frac{\|P_V\|_F^4}{8}. \tag{3.3}$$

3.1. Feasibility and local quadratic convergence of the algorithm. In this subsection, a sufficient condition of the strict feasibility of the full NT-step and the local quadratic convergence of the proximity measure near the central-path $\mathcal{C}$ are stated.

Lemma 3.4. *If $\delta = \delta(X, Z; \mu) < 1$, then the full NT-step is strictly feasible.*

Proof. We will show that $X(1) \succ 0$ and $Z(1) \succ 0$. Then according to Lemma 3.1, we just show that $\det(X(\alpha)Z(\alpha)) > 0$ with $0 \leqslant \alpha \leqslant 1$. We have

$$X(\alpha)Z(\alpha) = (X + \alpha\Delta X)(Z + \alpha\Delta Z) = XZ + \alpha(\Delta XZ + X\Delta Z) + \alpha^2\Delta X\Delta Z.$$

By using (2.6) and (2.7), we have

$$\begin{aligned} X(\alpha)Z(\alpha) &= \mu DV^2D^{-1} + \alpha(\mu DD_XVD^{-1} + \mu DVD_ZD^{-1}) + \alpha^2\mu DD_XD_ZD^{-1} \\ &= \mu D[V^2 + \alpha(D_XV + VD_Z) + \alpha^2D_XD_Z]D^{-1} \\ &\sim \mu[V^2 + \alpha(D_XV + VD_Z) + \alpha^2D_XD_Z]. \end{aligned}$$

Then

$$X(\alpha)Z(\alpha) \sim B(\alpha) + M(\alpha), \tag{3.4}$$

where

$$B(\alpha) = \mu\Big[V^2 + \frac{1}{2}\alpha(D_XV + VD_Z + VD_X + D_ZV) + \frac{1}{2}\alpha^2(D_XD_Z + D_ZD_X)\Big]$$

and

$$M(\alpha) = \mu\Big[\frac{1}{2}\alpha(D_XV + VD_Z - VD_X - D_ZV) + \frac{1}{2}\alpha^2(D_XD_Z - D_ZD_X)\Big].$$

It is easy to see that the matrix $B(\alpha)$ is symmetric and the matrix $M(\alpha)$ is anti-symmetric. By Lemma 3.2, one has $\det(X(\alpha)Z(\alpha)) > 0$ if the matrix $B(\alpha) \succ 0$ with $0 \leqslant \alpha \leqslant 1$. For this, we write $B(\alpha)$ in the form

$$\begin{aligned} B(\alpha) &= \mu\Big[V^2 + \frac{1}{2}\alpha((D_X + D_Z)V + V(D_Z + D_X)) + \alpha^2D_{XZ}\Big] \\ &= \mu\Big[V^2 + \frac{1}{2}\alpha(P_VV + VP_V) + \alpha^2D_{XZ}\Big]. \end{aligned}$$

Using (2.9), we have

$$B(\alpha) = \mu[(1-\alpha)V^2 + \alpha(I + \alpha D_{XZ})]. \tag{3.5}$$

By Lemma 3.2, $B(\alpha) \succ 0$ if $\alpha < 1$ and $\|D_{XZ}\|_2 < 1$. As $\delta < 1$, by (3.2) in Lemma 3.3, it follows that $\|D_{XZ}\|_2 < 1$. Next, since $X(0) \succ 0$, $Z(0) \succ 0$, Lemma 3.1, implies that $X(1) \succ 0$ and $Z(1) \succ 0$. But since $X(1) = X_+$ and $Z(1) = Z_+$, we have $X_+ \succ 0$ and $Z_+ \succ 0$. This completes the proof. $\square$

Next, we proceed to show the local quadratic convergence of the full NT-step on the proximity measure. First, we mention that the expression of V_+ is defined as

$$V_+ = \frac{1}{\sqrt{\mu}} D^{-1} X_+ D^{-1} = \frac{1}{\sqrt{\mu}} D Z_+ D.$$

From this we have $V_+^2 = \mu^{-1} D^{-1} X_+ Z_+ D$ and $X_+ Z_+ \sim \mu V_+^2$.

Lemma 3.5. *One has*

$$\lambda_{\min}(V_+^2) \geqslant (1 - \delta^2),$$

where $\lambda_{\min}(V_+^2)$ is the smallest eigenvalue of V_+^2.

Proof. From (3.4) in the proof of Lemma 3.4, setting $\alpha = 1$ and as $X_+ = X(1)$ and $Z_+ = Z(1)$ we deduce that

$$X_+ Z_+ \sim \mu V_+^2 \sim B(1) + M(1).$$

Also setting $\alpha = 1$ in (3.5), we have

$$X_+ Z_+ \sim \mu(I + D_{XZ}) + M(1).$$

Hence,

$$\lambda_{\min}(V_+^2) = \lambda_{\min}\Big((I + D_{XZ}) + \frac{1}{\mu} M(1)\Big).$$

Since the matrix $(I + D_{XZ})$ is symmetric and the matrix $M(1)$ is anti-symmetric, then by Lemma 3.2 we get

$$\lambda_{\min}(V_+^2) \geqslant \lambda_{\min}(I + D_{XZ}) \geqslant 1 - |\lambda_{\min}(D_{XZ})| \geqslant 1 - |\lambda_{\max}(D_{XZ})| = 1 - \|D_{XZ}\|_2.$$

Due to (3.2) in Lemma 3.3, we conclude that

$$\lambda_{\min}(V_+^2) \geqslant 1 - \delta^2.$$

This gives the required result. $\square$

Lemma 3.6. *If $\delta = \delta(X, Z; \mu) < 1$, then*

$$\delta_+ := \delta(V_+) = \delta(X_+, Z_+; \mu) \leqslant \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

Moreover, if $\delta(X, Z; \mu) < 1/\sqrt{2}$, then $\delta(X_+, Z_+; \mu) < \delta^2$, which shows the local quadratic convergence of the proximity measure.

P r o o f. One has

$$\begin{aligned}\delta_+^2 = \frac{1}{4}\|P_{V_+}\|_F^2 = \frac{1}{4}\|V_+^{-1} - V_+\|_F^2 \leqslant \frac{1}{4}\|V_+^{-1}\|_2^2\|(I - V_+^2)\|_F^2 \\ = \frac{1}{4}\lambda_{\max}^2(V_+^{-1})\|I - V_+^2\|_F^2 = \frac{1}{4}\frac{1}{\lambda_{\min}^2(V_+)}\|I - V_+^2\|_F^2 \\ = \frac{1}{4}\frac{1}{\lambda_{\min}(V_+^2)}\|I - V_+^2\|_F^2.\end{aligned}$$

Due to Lemma 3.5, we get

$$\delta_+^2 \leqslant \frac{1}{4(1-\delta^2)}\|I - V_+^2\|_F^2.$$

Next we show that

$$\|I - V_+^2\|_F^2 \leqslant \|D_{XZ}\|_F^2.$$

From $V_+^2 = \mu^{-1}D^{-1}X_+Z_+D$, $X_+ = X(1)$, $Z_+ = Z(1)$, (3.4) and (3.5), we have

$$\begin{aligned}I - V_+^2 &= I - \frac{1}{\mu}D^{-1}X(1)Z(1)D \\ &\sim I - \frac{1}{\mu}D^{-1}[B(1) + M(1)]D \sim \Big(-D_{XZ} - \frac{1}{\mu}M(1)\Big).\end{aligned}$$

Then

$$\begin{aligned}\|I - V_+^2\|_F^2 &= \mathrm{Tr}\Big(D_{XZ} + \frac{1}{\mu}M(1)\Big)^2 \\ &= \mathrm{Tr}\Big(D_{XZ}^2 - \frac{1}{\mu^2}M(1)M(1)^\top + \frac{2}{\mu}(D_{XZ}M(1))\Big).\end{aligned}$$

As $M(1) = -M(1)^\top$ and $D_{XZ} \in \mathbb{S}^n$, then the matrix $(D_{XZ}M(1))$ is anti-symmetric, which implies that $\mathrm{Tr}(D_{XZ}M(1)) = 0$. Hence, we have

$$\|I - V_+^2\|_F^2 = \mathrm{Tr}\Big(D_{XZ}^2 - \frac{1}{\mu^2}M(1)M(1)^\top\Big) \leqslant \mathrm{Tr}(D_{XZ}^2) = \|D_{XZ}\|_F^2.$$

Therefore,

$$\|I - V_+^2\|_F^2 \leqslant \|D_{XZ}\|_F^2.$$

By (3.3) in Lemma 3.3, we deduce that

$$\delta_+^2 \leqslant \frac{1}{4(1-\delta^2)}\|D_{XZ}\|_F^2 \leqslant \frac{1}{4(1-\delta^2)}\frac{\|P_V\|_F^4}{8} \leqslant \frac{16\delta^4}{32(1-\delta^2)} = \frac{\delta^4}{2(1-\delta^2)}.$$

Finally, we get $\delta_+ = \delta(X_+, Z_+; \mu) \leqslant \delta^2/\sqrt{2(1-\delta^2)}$. This proves the lemma. □

The next lemma shows the influence of a full NT-step on the duality gap and gives an upper bound for it.

Lemma 3.7. *If $\delta(X,Z;\mu) < 1/\sqrt{2}$ after a full NT-step. Then*

$$X_+ \bullet Z_+ \leqslant \mu(n+1).$$

P r o o f. One has

$$\begin{aligned}X_+ \bullet Z_+ &= \operatorname{Tr}(V_+^2) = \operatorname{Tr}(B(1)+M(1))\\ &= \operatorname{Tr}(\mu(I+D_{XZ})+M(1))\\ &= \mu\operatorname{Tr}\Big((I+D_{XZ})+\frac{1}{\mu}M(1)\Big).\end{aligned}$$

Because $\operatorname{Tr}(M(1))=0$, since the matrix $M(1)$ is anti-symmetric, it follows that

$$X_+ \bullet Z_+ = \mu(n+\operatorname{Tr}(D_{XZ})) = \mu\Big(n+\frac{1}{2}\operatorname{Tr}(D_XD_Z+D_ZD_X)\Big) = \mu(n+D_X\bullet D_Z).$$

Due to (3.1) we deduce that

$$X_+ \bullet Z_+ \leqslant \mu(n+2\delta^2).$$

As $\delta < 1/\sqrt{2}$, it is implied that

$$X_+ \bullet Z_+ \leqslant \mu(n+1).$$

This proves the lemma. □

The next lemma investigates the effect of a full NT-step on the proximity measure followed by an update of the parameter μ. For convenience, we define the matrix U by

$$U = \frac{1}{\sqrt{\mu_+}}D^{-1}X_+D^{-1} = \frac{1}{\sqrt{\mu_+}}DZ_+D = \frac{1}{\sqrt{1-\theta}}V_+,$$

where $\mu_+ = (1-\theta)\mu$.

Lemma 3.8. *Let $\mu_+ = (1-\theta)\mu$ where $0<\theta<1$. Then*

$$\delta^2(X_+,Z_+;\mu_+) \leqslant (1-\theta)\delta_+^2 + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}$$

with $\delta_+ = \delta(X_+,Z_+;\mu)$. In addition, if $\delta = \delta(X,Z;\mu) < 1/\sqrt{2}$, $\theta = 1/(3\sqrt{n})$ and $n \geqslant 2$, then

$$\delta^2(X_+,Z_+;\mu_+) < \frac{1}{2}.$$

P r o o f. We have $U = 1/\sqrt{1-\theta}V_+$. Then

$$\begin{aligned}
4\delta^2(X_+, Z_+; \mu_+) &= 4\delta^2(U) = \|P_U\|_F^2 = \|U^{-1} - U\|_F^2 \\
&= \left\| \sqrt{1-\theta}V_+^{-1} - \frac{1}{\sqrt{1-\theta}}V_+ \right\|_F^2 \\
&= \left\| \sqrt{1-\theta}(V_+^{-1} - V_+) - \frac{\theta}{\sqrt{1-\theta}}V_+ \right\|_F^2 \\
&= \mathrm{Tr}\Big(\sqrt{1-\theta}(V_+^{-1} - V_+) - \frac{\theta}{\sqrt{1-\theta}}V_+ \Big)^2 \\
&= (1-\theta)\|V_+^{-1} - V_+\|_F^2 + \frac{\theta^2}{1-\theta}\|V_+\|_F^2 - 2\theta \mathrm{Tr}((V_+^{-1} - V_+)V_+) \\
&= (1-\theta)\|V_+^{-1} - V_+\|_F^2 + \frac{\theta^2}{1-\theta}\|V_+\|_F^2 - 2\theta \mathrm{Tr}(I - V_+^2) \\
&= (1-\theta)4\delta_+^2 + \frac{\theta^2}{1-\theta}\|V_+\|_F^2 - 2\theta n + 2\theta\|V_+\|_F^2.
\end{aligned}$$

As $V_+^2 = \mu^{-1}D^{-1}X_+Z_+D$ and according to Lemma 3.7, we get:

$$\|V_+\|_F^2 = \mathrm{Tr}(V_+^2) = \frac{1}{\mu}X_+ \bullet Z_+ \leqslant (n+1).$$

By some simplifications we obtain

$$\delta^2(X_+, Z_+; \mu_+) \leqslant (1-\theta)\delta_+^2 + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}.$$

As $\delta < 1/\sqrt{2}$, Lemma 3.6 implies that $\delta_+^2 < \delta^4 < \frac{1}{4}$. Therefore,

$$\delta^2(X_+, Z_+; \mu_+) \leqslant \frac{1}{4} + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{4}.$$

Let $\theta = 1/(3\sqrt{n})$. Then $\theta^2 = \frac{1}{9}n^{-1}$ and we have

$$\delta^2(X_+, Z_+; \mu_+) \leqslant \frac{1}{4} + \frac{\frac{1}{9}(n+1)/n}{4(1-\theta)} + \frac{\theta}{4}.$$

Since $\frac{1}{9}(n+1)/n \leqslant \frac{1}{6}$ for $n \geqslant 2$, this implies that

$$\delta^2(X_+, Z_+; \mu_+) \leqslant f(\theta) = \frac{1}{4} + \frac{1}{24(1-\theta)} + \frac{\theta}{4}.$$

For $n \geqslant 2$ we get $0 \leqslant \theta \leqslant 1/\sqrt{18}$. Because

$$f'(\theta) = \frac{1}{4} + \frac{1}{24(\theta-1)^2} > 0,$$

then $f(\theta)$ is strictly increasing on the compact interval $[0, 1/\sqrt{18}]$. This implies that

$$f(\theta) \leqslant f\Big(\frac{1}{\sqrt{18}}\Big) \simeq 0.3634 < \frac{1}{2},$$

and so $\delta(X_+, Z_+; \mu_+) < 1/\sqrt{2}$. This completes the proof. □

A consequence of Lemma 3.8 is that under our defaults the algorithm is well defined, since the conditions $(X \succ 0, y, Z \succ 0)$ and $\delta(X, Z; \mu) < 1/\sqrt{2}$ are maintained throughout the algorithm.

3.2. Iterations bound. The next lemma gives an upper bound for the total number of iterations produced by the algorithm.

Lemma 3.9. *Let $\{X_k, y_k, Z_k\}$ be the sequence generated by the algorithm and $\mu = \mu_k$ with $\mu_0 = \frac{1}{2}$. Then*

$$X_k \bullet Z_k \leqslant \varepsilon if k \geqslant \frac{1}{\theta} \log\Big(\frac{n}{\varepsilon}\Big).$$

Proof. Lemma 3.7, and because $(n + 1) \leqslant 2n$ for all $n \geqslant 1$, this implies that $X_k \bullet Z_k \leqslant 2n(1 - \theta)^k \mu_0$. Then the inequality $X_k \bullet Z_k \leqslant \varepsilon$ is satisfied if $2n(1 - \theta)^k \mu_0 \leqslant \varepsilon$. Taking the logarithms, we obtain $k \log(1 - \theta) \leqslant \log \varepsilon - \log(2n\mu_0)$. Using $\log(1 - \theta) \leqslant -\theta$ for $0 < \theta < 1$, we have

$$k\theta \geqslant \log\Big(\frac{2n\mu_0}{\varepsilon}\Big).$$

Let $\mu_0 = \frac{1}{2}$. This fulfills the required claim and thus proves the lemma. □

Theorem 3.1. *Let $\theta = 1/(3\sqrt{n})$ and $\beta = 1/\sqrt{2}$. Then the previous algorithm requires $O(\sqrt{n} \log(n/\varepsilon))$ iterations to obtain an ε-approximate solution of $\mathcal{P}$.*

Proof. Let $\theta = 1/(3\sqrt{n})$. Using Lemma 3.9, the proof is straightforward. □

4. Numerical results

To evaluate the performance of our IP algorithm, we consider some examples of SDLS problems of different sizes; each example is followed by a table containing the computational results obtained by the algorithm. We implemented the algorithm in Matlab (R2013b) and the experiments were conducted on a Pentium 4 3.0 GHz PC with 2GB of RAM. In the implementation, we use $\varepsilon = 10^{-6}$, $\theta = 1/(3\sqrt{n})$ and $\beta = 1/\sqrt{2}$, and we start with a feasible point (X_0, y_0, Z_0) such that the IPC holds and $\delta(X_0, Z_0; \mu_0) < 1/\sqrt{2}$ is satisfied. The number of iterations required and the time executed by the algorithm are denoted by "Iter" and "CPU" respectively. Also, we display the following notations in the tables: "μ-Th $= (X_0 \bullet Z_0)/n$" and "μ-Pra" mean the theoretical choice of the parameter μ and the relaxed practical choice of μ, respectively. We note here that the source of Example 4.1 is taken from [16] and the others are inspired by [5], with modifications.

Example 4.1 (Nearest correlation matrix problems (NCM)). This example of SDLS is constructed from the following nearest correlation matrix problem:

$$\min_X \frac{1}{2}\|X - C\|_F^2 \quad \text{s.t. } A_i \bullet X = b_i,\ i = 1, \ldots, m,\ X \succeq 0,$$

with

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0.5 & 1 \\ 0.5 & 1 & 0.25 \\ 1 & 0.25 & 1 \end{bmatrix}, \quad b = e.$$

In this example, the triplet starting point is taken as

$$X_0 = I, \quad Z_0 = \begin{bmatrix} 2 & -0.5 & -1 \\ -0.5 & 2 & -0.25 \\ -1 & -0.25 & 1 \end{bmatrix}, \quad y_0 = [-2, -2, -1]^\top.$$

The obtained numerical results are summarized in Table 1.

θ/μ	μ-Th $= 1.6667$		μ-Pra $= 0.5$	
	Iter	CPU	Iter	CPU
$1/(3\sqrt{n})$	73	0.1045	67	0.0909

Table 1. Numerical results for Example 4.1.

The obtained approximate primal-dual optimal solution is:

$$X^* = \begin{bmatrix} 1 & 0.4910 & 0.9684 \\ 0.4910 & 1 & 0.2582 \\ 0.9684 & 0.2582 & 1 \end{bmatrix}, \quad Z^* = \begin{bmatrix} 0.0350 & -0.0090 & -0.0316 \\ -0.0090 & 0.0023 & 0.0082 \\ -0.0316 & 0.0082 & 0.0285 \end{bmatrix},$$
$$y^* = [-0.0350, -0.0023, -0.0285]^\top.$$

The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^* = d^* = 0.0011$.

Example 4.2. Consider the SDLS problem with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & 0 & -2 & -1 \\ 0 & -1 & -1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & -2 & 0 \\ 0 & 2 & 1 & 2 \\ -2 & 1 & -2 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix}, \; A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 \\ 2 & 0 & 2 & 1 \\ -1 & 2 & 0 & 1 \\ -1 & 1 & 1 & -2 \end{bmatrix},$$
$$C = \begin{bmatrix} -2 & -3 & 3 & 1 \\ -3 & -4 & -3 & -2 \\ 3 & -3 & 4 & -1 \\ 1 & -2 & -1 & 2 \end{bmatrix}, \quad b = [-2, 2, 0]^\top.$$

For this example, the triplet starting point is taken as

$$X_0 = Z_0 = I \quad \text{and} \quad y_0 = e.$$

The obtained numerical results are summarized in Table 2.

θ/μ	μ-Th $= 1$		μ-Pra1 $= 0.5$		μ-Pra2 $= 0.05$		μ-Pra3 $= 0.005$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$1/(3\sqrt{n})$	84	0.2051	80	0.1966	67	0.1256	55	0.1042

Table 2. Numerical results for Example 4.2.

The approximated primal-dual optimal solution for this example is:

$$X^* = \begin{bmatrix} 0.0574 & -0.0368 & -0.0554 & -0.0304 \\ -0.0368 & 0.0648 & 0.0536 & 0.1540 \\ -0.0554 & 0.0536 & 0.2056 & 0.1688 \\ -0.0304 & 0.1540 & 0.1688 & 0.4996 \end{bmatrix},$$
$$Z^* = \begin{bmatrix} 0.1081 & 0.1681 & 0.0311 & -0.0557 \\ 0.1681 & 0.2615 & 0.0483 & -0.0867 \\ 0.0311 & 0.0483 & 0.0089 & -0.0160 \\ -0.0557 & -0.0867 & -0.0160 & 0.0287 \end{bmatrix},$$
$$y^* = [0.8458, 1.0559, 0.9747]^\top.$$

The optimal values for both problems: The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^* = d^* = 53.2101$.

Example 4.3. Consider the following SDLS with

$$A_1 = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ 1 & 0 & 0 & 2 & 1 & 1 & 2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 2 & 1 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 0 & 1 & 0 & 2 & 0 \end{bmatrix},$$

$$C = -\begin{bmatrix} 6 & 4 & 4 & 6 & 5 & 1 & 5 & 3 \\ 4 & 4 & 6 & 5 & 6 & 4 & 3 & 4 \\ 4 & 6 & 6 & 3 & 4 & 6 & 5 & 5 \\ 6 & 5 & 3 & 0 & 5 & 5 & 3 & 4 \\ 5 & 6 & 4 & 5 & 4 & 2 & 3 & 2 \\ 1 & 4 & 6 & 5 & 2 & 2 & 6 & 4 \\ 5 & 3 & 5 & 3 & 3 & 6 & 8 & 6 \\ 3 & 4 & 5 & 4 & 2 & 4 & 6 & 2 \end{bmatrix}, \quad b = [8, 4, 12, 8]^\top.$$

The triplet starting point is taken as

$$X_0 = Z_0 = I \quad \text{and} \quad y_0 = e.$$

The obtained numerical results are summarized in Table 3.

θ/μ	μ-Th $= 1$		μ-Pra1 $= 0.5$		μ-Pra2 $= 0.05$		μ-Pra2 $= 0.005$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$1/(3\sqrt{n})$	127	1.2560	122	1.2086	103	1.0923	85	0.8650

Table 3. Numerical results for Example 4.3.

The obtained approximate primal-dual optimal solution is given by

$$X^* = \begin{bmatrix} 0.5143 & 0.0935 & 0.2653 & -0.0822 & -0.0149 & 0.0667 & 0.1155 & -0.0981 \\ 0.0935 & 0.4440 & 0.1279 & 0.1464 & 0.1198 & 0.2706 & 0.1422 & 0.0587 \\ 0.2653 & 0.1279 & 0.4490 & -0.1267 & -0.0689 & 0.0553 & 0.0632 & 0.2285 \\ -0.0822 & 0.1464 & -0.1267 & 0.2143 & 0.0729 & 0.1745 & 0.0587 & 0.0516 \\ -0.0149 & 0.1198 & -0.0689 & 0.0729 & 0.3292 & -0.0809 & 0.3082 & 0.0623 \\ 0.0667 & 0.2706 & 0.0553 & 0.1745 & -0.0809 & 0.3255 & -0.0461 & 0.0138 \\ 0.1155 & 0.1422 & 0.0632 & 0.0587 & 0.3082 & -0.0461 & 0.3455 & 0.1314 \\ -0.0981 & 0.0587 & 0.2285 & 0.0516 & 0.0623 & 0.0138 & 0.1314 & 0.4684 \end{bmatrix},$$

$$Z^* = \begin{bmatrix} 0.0098 & -0.0085 & -0.0164 & -0.0338 & 0.0334 & 0.0304 & -0.0228 & 0.0159 \\ -0.0085 & 0.0121 & -0.0034 & 0.0081 & -0.0480 & -0.0180 & 0.0401 & -0.0069 \\ -0.0164 & -0.0034 & 0.0946 & 0.1375 & 0.0157 & -0.0829 & -0.0387 & -0.0531 \\ -0.0338 & 0.0081 & 0.1375 & 0.2143 & -0.0290 & -0.1435 & -0.0138 & -0.0868 \\ 0.0334 & -0.0480 & 0.0157 & -0.0290 & 0.1908 & 0.0695 & -0.1602 & 0.0261 \\ 0.0304 & -0.0180 & -0.0829 & -0.1435 & 0.0695 & 0.1096 & -0.0342 & 0.0620 \\ -0.0228 & 0.0401 & -0.0387 & -0.0138 & -0.1602 & -0.0342 & 0.1415 & -0.0068 \\ 0.0159 & -0.0069 & -0.0531 & -0.0868 & 0.0261 & 0.0620 & -0.0068 & 0.0363 \end{bmatrix},$$

$$y^* = [1.0363, 0.8497, 1.2161, 0.9999]^\top.$$

The optimal values for both problems: The optimal values for both problems $\mathcal{P}$ and $\mathcal{D}$ are $p^* = d^* = 678.1406$.

Example 4.4. Consider the SDLS of variable size $(n = 2m)$ and with

$$C[i;j] = \begin{cases} 1 & \text{if } i = j \text{ and } i \leqslant m, \\ 0 & \text{otherwise}, \end{cases} \qquad A_i[j;k] = \begin{cases} 1 & \text{if } j = k = i \text{ or } j = k = i + m, \\ 0 & \text{otherwise}, \end{cases}$$

$$b[i] = 2, \quad i = 1, \dots, m.$$

The triplet starting point is taken as

$$X_0[i;j] = \begin{cases} \dfrac{5}{4} & \text{if } i = j = 1, \dots, m, \\ \dfrac{3}{4} & \text{if } i = j = m + 1, \dots, n, \\ 0 & \text{otherwise}, \end{cases} \qquad Z_0[i;j] = \begin{cases} \dfrac{5}{6} & \text{if } i = j = 1, \dots, m, \\ \dfrac{4}{3} & \text{if } i = j = m + 1, \dots, n, \\ 0 & \text{otherwise}, \end{cases}$$

$$y_0[i] = -\frac{7}{12}, \quad i = 1, \dots, m.$$

The obtained approximate primal-dual optimal solution is given by:

$$X^*[i;j] = \begin{cases} 1.5 & \text{if } i=j=1,\dots,m, \\ 0.5 & \text{if } i=j=m+1,\dots,n, \\ 0 & \text{otherwise,} \end{cases}$$
$$Z^* = 0_{n\times n},\ y^*[i] = 0.5,\ i=1,\dots,m.$$

The obtained numerical results for this example are summarized in Table 4.

Size	μ	μ-Th $= 1.0208$		μ-Pra1 $= 0.5$	
(m,n)	θ	Iter	CPU	Iter	CPU
$(5,10)$	$1/(3\sqrt{n})$	145	2.6305	139	2.6019
$(10,20)$	$1/(3\sqrt{n})$	218	58.2806	209	69.0568
$(25,50)$	$1/(3\sqrt{n})$	368	$6.4202\mathrm{e}+03$	353	$5.9039\mathrm{e}+03$

Size	μ	μ-Pra2 $= 0.05$		μ-Pra3 $= 0.005$	
(m,n)	θ	Iter	CPU	Iter	CPU
$(5,10)$	$1/(3\sqrt{n})$	118	2.2160	98	1.9398
$(10,20)$	$1/(3\sqrt{n})$	179	65.4664	149	49.3058
$(25,50)$	$1/(3\sqrt{n})$	306	$3.6451\mathrm{e}+03$	258	$4.0128\mathrm{e}+03$

Table 4. Numerical results for Example 4.4.

5. Conclusion and future work

We have proposed a full NT-step feasible primal-dual short-step IP algorithm for SDLS and derived its polynomial time complexity; namely, $O(\sqrt{n}\log(n/\varepsilon))$, which is as good as the complexity obtained for the LO, SDO and CQSDO cases. The analysis of the algorithm is inspired by the one used for CQSDO [5]. The efficiency of the algorithm is proven by solving SDLS problems of different sizes. However, in a numerical implementation, getting an initial starting point for problems of the large size remains a serious problem. The analysis of the algorithm with other search directions remains a good topic of research. Also, the application of kernel functions with large-step algorithms is a good subject of research.

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