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DEGREE SUMS OF ADJACENT VERTICES FOR TRACEABILITY OF CLAW-FREE GRAPHS

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Abstract. The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. For a graph H, define $\overline{\sigma}_2(H) = \min\{d(u) + d(v)\colon uv \in E(H)\}$. Let H be a 2-connected claw-free simple graph of order n with $\delta(H) \geqslant 3$. We show that, if $\overline{\sigma}_2(H) \geqslant \frac{1}{7}(2n-5)$ and n is sufficiently large, then either H is traceable or the Ryjáček's closure $\mathrm{cl}(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph that can be contracted to one of the two graphs of order 10 which have no spanning trail. Furthermore, if $\overline{\sigma}_2(H) > \frac{1}{3}(n-6)$ and n is sufficiently large, then H is traceable. The bound $\frac{1}{3}(n-6)$ is sharp. As a byproduct, we prove that there are exactly eight graphs in the family $\mathcal{G}$ of 2-edge-connected simple graphs of order at most 11 that have no spanning trail, an improvement of the result in Z. Niu et al. (2012).

Keywords: traceable graph; line graph; spanning trail; closure

MSC 2020: 05C07, 05C38, 05C45

1. Introduction

We follow Bondy and Murty (see [1]) for undefined terms and notation. We consider finite, undirected and loopless graphs only, but we allow multiple edges. For a vertex x of G, $N_G(x)$ is the neighborhood of x in G, and $d_G(x)$ or $d(x)$ is the degree of x in G. For a vertex set $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{x\in S} N_G(x)$ and $N_G[S] = \bigcup_{x\in S} N_G(x) \cup S$. By $\kappa(G)$, we denote the connectivity of G. A graph is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. A graph G is *hamiltonian*

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(or *traceable*) if it has a Hamilton cycle (or Hamilton path), i.e., a spanning cycle (or a spanning path). The circumference of G, denoted by $c(G)$, is the length of the longest cycle of G.

Degree conditions are by now known as the classical approach to hamiltonian problems. In [7], Dirac proved that if the degree of each vertex of a graph is at least half of the *order*, i.e., the number of vertices, (at least three), of the graph (the Dirac condition), then it contains a Hamilton cycle. In addition to Dirac's minimum degree condition, various degree conditions such as the minimum degree sum of an independent set (the Ore condition—on two independent vertices, the Bondy and Chvátal condition—on at least two independent vertices) and the maximum degree of pairs of vertices with distance two (the Fan condition) have been studied for the hamiltonicity of graphs, circumferences of graphs or other structural properties of graphs (see the surveys [8], [9]). Here, we study a particular type of conditions, inspired by the early work of Brualdi and Shanny from the 1980s. In [3], they considered a degree sum condition on adjacent pairs of vertices of graphs guaranteeing that their line graphs are hamiltonian. The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common.

For a graph G, let

$$\overline{\sigma}_2(G) = \min\{d(u) + d(v)\colon\ uv \in E(G)\}.$$

It is easy to obtain a corollary of Dirac's theorem that every connected graph G of order $n \geqslant 3$ with $\overline{\sigma}_2(G) \geqslant \frac{1}{2}(3n-2)$ (or $\overline{\sigma}_2(G) \geqslant \frac{1}{2}(3n-3)$) is hamiltonian (or traceable). The bounds $\frac{1}{2}(3n-2)$ and $\frac{1}{2}(3n-3)$ are sharp. The counterexamples hamiltonian (or traceable) graphs can be seen from the graphs $G_m = (m+1)K_1 \vee K_m$ (or $G_m^1 = (m+2)K_1 \vee K_m$). One easily checks that G_m with $n = |V(G_m)| = 2m+1$, $\delta(G_m) = \frac{1}{2}(n-1)$, and $\overline{\sigma}_2(G_m) = \frac{1}{2}(3n-3)$, while G_m is not hamiltonian since the number of the components of $G_m - V(K_m)$ is $m+1$. Similarly, the nontraceable graphs G_m^1 with $n = |V(G_m^1)| = 2m+2$, $\delta(G_m^1) = \frac{1}{2}(n-2)$, and $\overline{\sigma}_2(G_m^1) = \frac{1}{2}(3n-4)$.

The above discussion reveals that considering degree sum conditions on adjacent pairs of vertices for general graphs does not provide anything relevant in the sense of essentially new and more general results. However, if we consider claw-free graphs, this picture changes. This was first observed by Chen (see [6]) who considered the Brualdi-Shanny condition for guaranteeing the hamiltonicity of claw-free graphs.

Before stating the results, we need the following terminology and notation. A graph is *triangle-free* if it has no K_3. As in [1], $\kappa'(G)$ denotes the edge-connectivity of G. An edge cut X of G is *essential* if $G-X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge-cut X with $|X| < k$. Note that a graph G is essentially k-edge-connected if and only if $L(G)$ is k-connected or complete.

Next, we review some key concepts that we use throughout the paper. The first concept yields a way to shift attention and considerations from a claw-free graph H to a closely related line graph $L(G)$ of a triangle-free graph G. This will enable us to show the validity of statements about the hamiltonicity and traceability of H by proving equivalent statements about G. Since we will mainly deal with the latter, we find it convenient to use H for the original claw-free graph for which we establish hamiltonicity and traceability results.

Ryjáček in [13] introduced the *closure* operation of a claw-free graph H, which becomes a very useful tool in investigating the hamiltonicity or traceability in claw-free graphs. A vertex $v \in V(H)$ is *locally connected* if the neighborhood of v induces a connected subgraph in H. The closure of a claw-free graph H, denoted by $\mathrm{cl}(H)$, is obtained from H by recursively joining all pairs of nonadjacent vertices in the neighborhood of each locally connected vertex as long as it is possible. The closure $\mathrm{cl}(H)$ remains a claw-free graph and its connectivity is no less than the connectivity of H. A claw-free graph H is said to be *closed* if $H = \mathrm{cl}(H)$. The following theorem summarizes the basic properties of $\mathrm{cl}(H)$.

Theorem 1.1 ([13]). *Let H be a claw-free graph and $\mathrm{cl}(H)$ its closure. Then*
(i) *$\mathrm{cl}(H)$ is well-defined and $\kappa(\mathrm{cl}(H)) \geqslant \kappa(H)$;*
(ii) *there is a triangle-free graph G such that $\mathrm{cl}(H) = L(G)$;*
(iii) *both the graphs H and $\mathrm{cl}(H)$ have the same circumference.*

Let G be a connected graph. For $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge $e \in X$ and deleting the resulting loops. Even when G is simple, G/X may not be simple. If Γ is a connected subgraph of G, then we write G/Γ for $G/E(\Gamma)$ and use v_Γ for the vertex in G/Γ to which Γ is contracted, and v_Γ is called a *contracted vertex* if $\Gamma \neq K_1$.

A (closed) trail Ψ is called a *spanning* (*closed*) *trail* in G if $V(G) = V(\Psi)$, and is called a *dominating* (*closed*) *trail* if $E(G - V(\Psi)) = \emptyset$. Let $\mathcal{Q}_0(r,k)$ be the family of k-edge-connected graphs of order at most r that do not admit a spanning closed trail. For a given integer $p > 0$ and a given real number ε, we use "$n \gg L(p,\varepsilon)$" for "n is sufficiently large related to p and ε". In [6], Chen proved the following result.

Theorem 1.2 ([6]). *Let $p > 0$ be a given integer and ε be a given number, and $k \in \{2,3\}$. Suppose H is a k-connected claw-free simple graph of order n with $\delta(H) \geqslant 3$. If $\overline{\sigma}_2(H) \geqslant (2n+\varepsilon)/p$ and $n \gg L(p,\varepsilon)$, then either H is hamiltonian or $\mathrm{cl}(H) = L(G)$, where G is an essentially k-edge-connected triangle-free graph that can be contracted to a graph in $\mathcal{Q}_0(5p-10,k)$ and $p \geqslant 3$.*

As a special case of Theorem 1.2 with fixed given values of p and ε, Chen in [6] has shown that if $\overline{\sigma}_2(H) \geqslant \frac{1}{4}(2n-4)$, then either H is hamiltonian or H is a member of a well-defined class of exceptional graphs. In [19], Tian and Xiong extended Chen's result and proved the case $\overline{\sigma}_2(H) \geqslant \frac{2}{5}n-1$.

Motivated by the results above, in this paper, we give best possible degree sum conditions of adjacent vertices for claw-free graphs H with $\delta(H) \geqslant 3$ to be traceable.

First, we obtain the following analogue of Theorem 1.2 for traceability.

Corollary 1.3. *Let $p > 0$ be a given integer and ε be a given number, and $k \in \{2,3\}$. Suppose H is a k-connected claw-free simple graph of order n with $\delta(H) \geqslant 3$. If $\overline{\sigma}_2(H) \geqslant (2n+\varepsilon)/p$ and $n \gg L(p,\varepsilon)$, then either H is traceable or $\mathrm{cl}(H) = L(G)$, where G is an essentially k-edge-connected triangle-free graph that can be contracted to a graph in $\mathcal{R}_0(5p-10,k)$ and $p \geqslant 4$.*

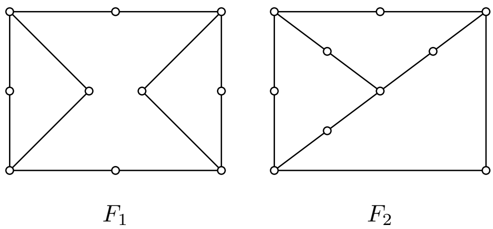

Figure 1. Two graphs of order 10 that have no spanning trail.

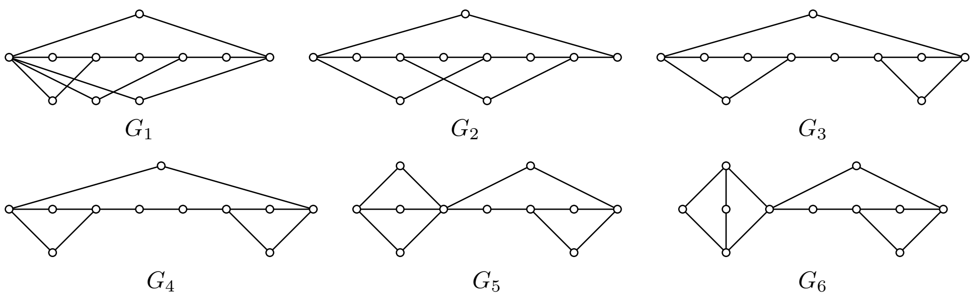

Figure 2. Six graphs of order 11 that have no spanning trail.

Here $\mathcal{R}_0(r,k)$ denotes the family of k-edge-connected graphs of order at most r that do not admit a spanning trail, which will be used for describing the exceptional classes for the traceability results that will follow. Since some graphs in $\mathcal{Q}_0(r,k)$ have spanning trails, such as $K_{2,3}$ for $k=2$ and the Petersen graph for $k=3$, $\mathcal{R}_0(r,k) \subseteq \mathcal{Q}_0(r,k)$. Let F_1 and F_2 be the graphs depicted in Figure 1, and let $G_1, G_2, \ldots, G_6$ be the graphs that are depicted in Figure 2. By Theorem 2.4 of Section 2, we know that $\mathcal{R}_0(11,2) = \{F_1, F_2, G_1, G_2, \ldots, G_6\}$. As an application of Corollary 1.3, we get the following result.

Theorem 1.4. *Let H be a 2-connected claw-free simple graph of order n and $\delta(H) \geqslant 3$. If $\overline{\sigma}_2(H) \geqslant \frac{1}{7}(2n-5)$ and n is sufficiently large, then either H is traceable or $\mathrm{cl}(H) = L(G)$, where G is an essentially 2-edge-connected triangle-free graph that can be contracted to either F_1 or F_2 such that all vertices of degree two are contracted vertices.*

For a graph G, we put $D_i(G) = \{v \in V(G) \colon d_G(v) = i\}$. For $F \in \{F_1, F_2\}$, let $D_2(F) = \{v_1, v_2, \ldots, v_6\}$. Let $\mathcal{F}(n,s)$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from those F by replacing each $v_i \in D_2(F)$ by a triangle-free subgraph of size $s_i \geqslant s$ such that $n = 12 + \sum\limits_{i=1}^{6} s_i$. Note that each graph in $\mathcal{F}(n,s)$ may be contractible to F_1 or F_2.

Let $\mathcal{R}_{\mathcal{F}}(n,s)$ be the set of 2-connected claw-free graphs H whose Ryjáček's closure is the line graph of a graph G in $\mathcal{F}(n,s)$, i.e., $\mathrm{cl}(H) = L(G)$.

Theorem 1.4 in fact can be deduced from the following result.

Theorem 1.5. *Let H be a 2-connected claw-free simple graph of order n with $\delta(H) \geqslant 3$. If $\overline{\sigma}_2(H) \geqslant \frac{1}{7}(2n-5)$ and n is sufficiently large, then either H is traceable or $\overline{\sigma}_2(H) \leqslant \frac{1}{3}(n-6)$ and $H \in \mathcal{R}_{\mathcal{F}}(n, \frac{1}{14}(2n-19))$.*

Theorem 1.5 implies the following result immediately.

Corollary 1.6. *Let H be a 2-connected claw-free simple graph of order n. If $\delta(H) \geqslant \frac{1}{14}(2n-5)$ and n is sufficiently large, then either H is traceable or $\delta(H) \leqslant \frac{1}{6}(n-6)$ and $H \in \mathcal{R}_{\mathcal{F}}(n, \frac{1}{14}(2n-19))$.*

An edge $e = uv \in E(G)$ is called a *pendant edge* of G if $\min\{d(u), d(v)\} = 1$. For $\mathcal{F}(n,s)$, if $s = \frac{1}{6}(n-12)$, then let $\mathcal{F}(n, \frac{1}{6}(n-12))$ be the family of essentially 2-edge-connected graphs in which each graph is obtained from those $F \in \{F_1, F_2\}$ by adding $\frac{1}{6}(n-12)$ pendant edges to each vertex of degree two in F. From our proof of Theorem 1.5 (which is given in Section 4), we also obtain the following results.

Theorem 1.7. *Let H be a 2-connected claw-free simple graph of order n with $\delta(H) \geqslant 3$. If $\overline{\sigma}_2(H) \geqslant \frac{1}{3}(n-6)$ and n is sufficiently large, then either H is traceable or $\overline{\sigma}_2(H) = \frac{1}{3}(n-6)$ and $H \in \mathcal{R}_{\mathcal{F}}(n, \frac{1}{6}(n-12))$.*

From Theorem 1.7, we immediately get the following corollary.

Corollary 1.8. *Let H be a 2-connected claw-free simple graph of order n. If $\delta(H) \geqslant \frac{1}{6}(n-6)$ and n is sufficiently large, then either H is traceable or $\delta(H) = \frac{1}{6}(n-6)$ and $H \in \mathcal{R}_{\mathcal{F}}(n, \frac{1}{6}(n-12))$.*

Remark 1.9. Let G^* be a graph obtained from the graph G_1 of Figure 2 by adding $\frac{1}{7}(n-14) \geqslant 2$ pendant edges (for a suitable choice of n) at each vertex of degree two in G_1. Then $\overline{\sigma}_2(L(G^*)) = \frac{1}{7}(2n-14) < \frac{1}{7}(2n-5)$. Clearly, $L(G^*) \notin \mathcal{R}_{\mathcal{F}}(n, \frac{1}{14}(2n-19))$. Note that G^* cannot be contracted to a graph in $\{F_1, F_2\}$. This example shows that the bound $\frac{1}{7}(2n-5)$ in Theorems 1.4 and 1.5 is asymptotically sharp.

Corollary 1.6 is a substantial improvement of the "$\delta(H) \geqslant \frac{1}{3}(n-2)$" theorem obtained by Matthews and Sumner in [11] if H is a 2-connected claw-free graph of sufficiently large order n. Corollary 1.6 is also an improvement of the "$\delta(H) > \frac{1}{7}n+4$" theorem obtained by Wang and Xiong in [21].

The remainder of this paper is organized as follows. In Section 2, we present some auxiliary results and give a brief discussion of Catlin's reduction. In Section 3, we give a brief discussion of the core of essentially 2-edge-connected graphs and present some useful results. In Section 4, the proofs of Corollary 1.3 and Theorem 1.5 are given.

2. Preliminaries and auxiliary results

Niu, Xiong and Zhang in [12] defined the *smallest graph* in a collection of graphs as a graph that has the least order and subject to that it has the least size amongst all graphs of that order in the collection. In particular, they considered the smallest order and size of 2-edge-connected graphs without spanning trails as follows.

Theorem 2.1 ([12]). *If G is a 2-edge-connected simple graph of order at most 10, then either G has a spanning trail or $G \in \{F_1, F_2\}$.*

In [21], Wang and Xiong proved the following two useful results.

Theorem 2.2 ([21]). *Let G be a 2-connected graph with circumference $c(G)$.*
(a) *If $c(G) \leqslant 5$, then G has a spanning trail that starts from any given vertex.*
(b) *If $c(G) \leqslant 7$, then G has a spanning trail.*

The following result is needed in our proof of Theorem 1.5.

Theorem 2.3 ([21]). *Let G be a 2-edge-connected simple graph. Then for any subset $S \subseteq V(G)$ with $|S| \leqslant 6$ and $E(G-S) = \emptyset$, either G has a trail passing through all vertices of S or $G \in \{F_1, F_2\}$.*

Using Theorem 2.2, Theorem 2.1 can be extended as follows.

Theorem 2.4. *If G is a 2-edge-connected simple graph of order at most 11, then either G has a spanning trail or $G \in \{F_1, F_2, G_1, G_2, \ldots, G_6\}$.*

Since all graphs depicted in Figures 1 and 2 are not 3-edge-connected, Theorem 2.4 implies the following result.

Corollary 2.5. *If G is a 3-edge-connected simple graph of order at most 11, then G has a spanning trail.*

The following theorem shows the relationship between a graph and its line graph.

Theorem 2.6 ([10]). *Let G be a graph with $|E(G)| \geqslant 1$. Then the line graph $L(G)$ of G is traceable if and only if G has a dominating trail.*

Theorem 2.7 ([2]). *Let H be a claw-free graph. Then H is traceable if and only if $\mathrm{cl}(H)$ is traceable.*

2.1. Catlin's reduction method. Let $O(G)$ be the set of vertices of odd degree in G. A graph in which each vertex has even degree is called an *even graph*. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph Γ_R of G with $O(\Gamma_R) = R$.

In [4], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_c$. The *reduction* of G is $G' = G/\Bigl(\bigcup\limits_{i=1}^{c} \Gamma_i\Bigr)$, the graph obtained from G by contracting each Γ_i into a single vertex v_i $(1 \leqslant i \leqslant c)$. So each Γ_i is the *preimage* of a vertex v_i in G. A graph G is *reduced* if $G' = G$.

A graph is *supereulerian* if it contains a spanning closed trail. The family of supereulerian graphs is denoted by $\mathcal{SL}$. The graph K_1 is regarded as a collapsible and supereulerian graph.

Theorem 2.8 ([4], [5]). *Let G be a connected graph and let G' be the reduction of G.*

(a) *G is collapsible if and only if $G' = K_1$, and $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.*

(b) *G has a dominating closed trail if and only if G' has a dominating closed trail containing all the contracted vertices of G'.*

(c) *If G is a reduced graph, then G is simple and triangle-free with $\delta(G) \leqslant 3$. For any subgraph Ψ of G, Ψ is reduced and either $\Psi \in \{K_1, K_2, K_{2,t}\ (t \geqslant 2)\}$ or $|E(\Psi)| \leqslant 2|V(\Psi)| - 5$.*

Theorem 2.9 ([22]). *Let G be a connected graph of order n and let G' be the reduction of G. Then G has a spanning trail if and only if G' has a spanning trail.*

2.2. Proof of Theorem 2.4. In a graph G, let $C = v_0v_1v_2\dots v_{c(G)-1}v_0$ denote the longest cycle containing the vertices $v_0, v_1, \dots, v_{c(G)-1}$ of G. For convenience, in the following, the subscripts are taken modulo $c(G)$. For any $v_i, v_j \in V(C)$ (with $v_i \neq v_j$), without loss of generality, we assume that $i < j$. We use $v_i\overrightarrow{C}v_j$ to denote the segment $v_iv_{i+1}\dots v_{j-1}v_j$ of C, i.e., $v_i\overrightarrow{C}v_j$ is a trail (path) along the edges of C starting from the vertex v_i and terminating at the vertex v_j. Note that $v_i\overrightarrow{C}v_j$ contains the vertices v_i and v_j exactly once.

Proof of Theorem 2.4. Let G be a 2-edge-connected simple graph of order at most 11. If G has a spanning trail, then we are done. In the following, we assume that G has no spanning trail. Assume first that G has a triangle. Then we let G' be the reduction of G. By Theorem 2.8 (c), G' is triangle-free. Then, since $|V(G)| \leqslant 11$, we obtain that $|V(G')| \leqslant 9$. Now, since G is 2-edge-connected, G' is also 2-edge-connected. By Theorem 2.1, G' has a spanning trail. Then by Theorem 2.9, G has a spanning trail, a contradiction. Therefore, we next assume that G is triangle-free. If $|V(G)| \leqslant 10$, then by Theorem 2.1, G is isomorphic to one of the graphs F_1 and F_2 depicted in Figure 1. Hence, in the remainder of the proof, we only need to consider the case when $|V(G)| = 11$. We distinguish two cases based on the connectivity $\kappa(G)$ of G.

Case 1: $\kappa(G) \geqslant 2$. Since G has no spanning trail then by Theorem 2.2, $c(G) \geqslant 8$. Therefore, $8 \leqslant c(G) \leqslant 9$; otherwise $G - C$ has at most one vertex and we can find a spanning trail of G, a contradiction. Here, $C = v_0v_1v_2\dots v_{c(G)-1}v_0$ denotes a longest cycle of G ($c(G) = 8$ or 9). By deleting all the chords of C, the resulting 2-connected graph G_0 is a spanning subgraph of G. Thus, G_0 has no spanning trail; otherwise G has a spanning trail, a contradiction.

Claim 2.10. *$V(G_0 - C)$ is an independent set.*

Proof. It suffices to prove $|V(D)| = 1$ for each component D of $G_0 - C$. Let D be a component of $G_0 - C$ with the most vertices. Suppose that $|V(D)| > 1$. Let $u_1u_2\dots u_k$ ($2 \leqslant k \leqslant 3$) be the maximal path in D. Because G_0 is triangle-free and 2-connected, $u_1v_i \in E(G_0)$ for some $v_i \in V(C)$ and G_0 has a spanning trail unless $c(G_0) = 8$ and $|V(D)| = 2$. Then $c(G_0) = 8$ and $D = K_2$. Since G_0 is 2-connected, we assume that xy is an edge of D with $v_i \in N_{G_0}(x) \cap V(C)$, $v_j \in N_{G_0}(y) \cap V(C)$ (and $v_i \neq v_j$). Put $G^* = G_0[E(G_0 - x - y) \cup \{xv_i, xy, yv_j\}]$. Then G^* is a 2-connected spanning subgraph of G_0 and v_ixyv_j is an induced path of length 3 of G^*. Let $\widetilde{G} = G^*/\{xy\}$. Then by Theorem 2.1, either $\widetilde{G}$ has a spanning trail or $\widetilde{G} \in \{F_1, F_2\}$. In the first case, G^* has a spanning trail, thus G_0 has a spanning trail as well.

Then G has a spanning trail, a contradiction. In the second case, so if $\widetilde{G} \in \{F_1, F_2\}$, then by the construction of $\widetilde{G}$, G^* has a cycle of length 9, a contradiction. Hence, $|V(D)| = 1$, as required. $\square$

Using Claim 2.10, let $V(G_0 - C) = \{u_1, u_2, \ldots, u_t\}$. Then since $|V(G_0)| = 11$ and $8 \leqslant c(G_0) \leqslant 9$, $2 \leqslant t \leqslant 3$. We prove another claim.

Claim 2.11. *For any two vertices $x, y \in V(G_0 - C)$, $|N_{G_0}(x) \cap N_{G_0}(y)| \leqslant 1$.*

P r o o f. We establish the claim by contradiction. We assume that $v_i, v_j \in N_G(x) \cap N_G(y)$ (with $v_i \neq v_j$). Then $G^\tau = G_0[E(C) \cup \{xv_i, xv_j, yv_i, yv_j\}]$ is a spanning even subgraph of $G_0[V(C) \cup \{x, y\}]$. Since $8 \leqslant |V(C)| \leqslant 9$, $G_0 - C - x - y$ has at most one vertex. Then G_0 has a spanning trail containing all edges of G^τ, a contradiction. $\square$

Since $\kappa(G) \geqslant 2$, for any $x \in V(G_0 - C)$, $|N_{G_0}(x) \cap V(C)| \geqslant 2$, and we consider exactly two edges e_x, e'_x that are incident with x. Let $E_1 = \{e_x, e'_x\colon\ x \in V(G_0 - C)\}$ and $G^\star = G_0\Bigl[E\Bigl(G_0 - \bigcup\limits_{i=1}^{t}\{u_i\}\Bigr) \cup E_1\Bigr]$. Then $G^\star$ is a 2-connected spanning subgraph of G_0 and $G^\star$ has no spanning trail; otherwise, G_0 has a spanning trail and thus G has a spanning trail as well, a contradiction. Let V_1 be the set of all vertices of odd degree in $G^\star$. Then $V_1 \subseteq V(C)$. Since $|V_1| \leqslant 6$, $|V_1| \in \{0, 2, 4, 6\}$, and it suffices to consider the cases when $|V_1| = 4$ or 6 (since, if $|V_1| = 0$ or 2, it is immediate that $G^\star$ has a spanning trail, a contradiction).

We distinguish the two remaining subcases for Case 1.

Subcase 1.1: $|V_1| = 6$. Then $c(G^\star) = 8$ and $|V(G^\star - C)| = 3$. Moreover, $N_{G^\star}(x) \cap N_{G^\star}(y) = \emptyset$ for any $x, y \in V(G^\star - C)$ with $x \neq y$. Since $|V(C)| = 8$ and $|V_1| = 6$, there exist at least three consecutive vertices of V_1 on C. Without loss of generality, we assume that $v_i, v_{i+1}, \ldots, v_{i+l} \in V_1 \cap V(C)$ with $2 \leqslant l \leqslant 5$.

First suppose that V_1 has exactly three consecutive vertices on C. Then $l = 2$ and $V_1 = \{v_i, v_{i+1}, v_{i+2}, v_{i+4}, v_{i+5}, v_{i+6}\}$. Then, since $G^\star - \{v_iv_{i+1}, v_{i+4}v_{i+5}\}$ is connected and has exactly two vertices of odd degree, $G^\star$ has a spanning trail, a contradiction.

Next suppose that V_1 has at least four consecutive vertices on C. Then $3 \leqslant l \leqslant 5$. Since $G^\star$ is triangle-free, $G^\star - \{v_iv_{i+1}, v_{i+2}v_{i+3}\}$ is connected and has exactly two vertices of odd degree. Then $G^\star$ has a spanning trail, a contradiction.

Subcase 1.2: $|V_1| = 4$. We prove another claim.

Claim 2.12. *For any pair of distinct vertices $v_i, v_j \in V_1$, v_i, v_j are nonadjacent on C.*

P r o o f. We establish the claim by contradiction. We assume that $v_i, v_{i+1} \in V_1$. Then $G^\star - \{v_i v_{i+1}\}$ has exactly two vertices of odd degree. Then $G^\star$ has a spanning trail, a contradiction. □

Using Claim 2.12 and by $8 \leqslant c(G^\star) \leqslant 9$, without loss of generality, we assume that $V_1 = \{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$. Note that $|V(G^\star - C)| \leqslant 3$ and $|N_{G^\star}(x) \cap V(C)| = 2$ for any $x \in V(G^\star - C)$. Then by Claim 2.11 and by $V_1 = \{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$, it is easy to check that $G^\star$ is isomorphic to one of the graphs in $\{G_1, G_2, G_3, G_4\}$ as depicted in Figure 2.

Since joining any two nonadjacent vertices of a graph in $\{G_1, G_2, G_3, G_4\}$ by an edge results in a triangle or a spanning trail of the new graph, $G = G_0 = G^\star$. Hence, in this situation $G \in \{G_1, G_2, G_3, G_4\}$. This completes the proof for Case 1.

Case 2: $\kappa(G) = 1$. Let $B_1, B_2, \ldots, B_t (t \geqslant 2)$ be the blocks of G. Since G is triangle-free, $|V(B_i)| \geqslant 4$ for $1 \leqslant i \leqslant t$. We first prove two claims.

Claim 2.13. *Each end-block of* G *has at least* 5 *vertices.*

P r o o f. If there exists an end-block B_i of G with 4 vertices, then $G[V(B_i)]$ is a cycle of length 4. Obviously, G/B_i is a 2-edge-connected triangle-free simple graph of order 8. By Theorem 2.1, G/B_i has a spanning trail. Since B_i and G/B_i have a vertex in common, the spanning trail of G/B_i can be extended to be a spanning trail of G, a contradiction. □

Claim 2.14. $t = 2$.

P r o o f. We establish the claim by contradiction. Without loss of generality, we assume that B_1 and B_t are two end-blocks of G and B_k is a third distinct block of G. By Claim 2.13, $|V(B_1)| \geqslant 5$ and $|V(B_t)| \geqslant 5$. Since both B_1 and B_t have at most one vertex in common with B_k, $11 = |V(G)| \geqslant |V(B_1)| + |V(B_t)| + |V(B_k)| - 2 \geqslant 5 + 5 + 4 - 2 = 12$, a contradiction. □

Since $|V(G)| = 11$ and $t = 2$, either $|V(B_1)| = |V(B_2)| = 6$ or $|V(B_1)| = 5$ and $|V(B_2)| = 7$. Then $B_i \notin \mathcal{SL}$; otherwise, the spanning trail of G/B_i can be extended to be a spanning trail of G, a contradiction.

First suppose that $|V(B_1)| = |V(B_2)| = 6$. Since $B_i \notin \mathcal{SL}$, $c(B_i) \leqslant 5$. By Theorem 2.2 (a), both B_1 and B_2 have a spanning trail that starts from any given vertex. Since B_1 and B_2 have a vertex in common, there exists a spanning trail of G, a contradiction.

Next suppose that $|V(B_1)| = 5$ and $|V(B_2)| = 7$. Since B_1 is 2-connected, triangle-free and $B_1 \notin \mathcal{SL}$, $B_1 = K_{2,3}$. Since $B_2 \notin \mathcal{SL}$, $c(B_2) = 6$; otherwise, by Theorem 2.2 (a), both B_1 and B_2 have a spanning trail that starts from any given vertex, there exists a spanning trail of G, a contradiction. We assume that $C = v_0v_1v_2v_3v_4v_5v_0$ is the longest cycle of B_2.

Then $V(B_1) \cap V(C) \neq \emptyset$; otherwise, there exists a vertex $u \in V(B_2) \setminus V(C)$ such that $V(B_1) \cap V(B_2) = \{u\}$. Since B_2 is 2-connected, there exists a vertex $v_i \in N_G(u) \cap V(C)$. Then $T_2 = uv_i\overrightarrow{C}v_{i+5}$ is a spanning trail of B_2. Since B_1 has a spanning trail T_1 starting from vertex u, by combining T_1 and T_2, we can get a spanning trail of G, a contradiction.

Without loss of generality, we assume that $V(B_1) \cap V(C) = \{v_0\}$ and $V(B_2) \setminus V(C) = \{u\}$. Then $v_0, v_1, v_5 \notin N_G(u)$; otherwise, we denote by T_1 a spanning trail of B_1 starting from the vertex v_0 and by $T_2 = v_0\overrightarrow{C}v_5v_0u$ or $v_0v_5v_4v_3v_2v_1u$ or $v_0\overrightarrow{C}v_5u$ a spanning trail of B_2 starting from vertex v_0. By combining T_1 with T_2, we can get a spanning trail of G, a contradiction. Since G is 2-edge-connected and triangle-free, $N_G(u) = \{v_2, v_4\}$.

Then G has a spanning subgraph isomorphic to the graph G_5 or G_6 as depicted in Figure 2. Furthermore, by joining any two nonadjacent vertices of G_5 or G_6 by an edge, the new graph contains a triangle or a spanning trail of the new graph. Hence, $G \in \{G_5, G_6\}$. The proof of Theorem 2.4 is completed. $\square$

3. The reduction of the core of a graph and a technical lemma

3.1. The reduction of the core of a graph. Let G be an essentially 2-edge-connected simple graph with $\overline{\sigma}_2(G) \geqslant 5$. Then $D_1(G) \cup D_2(G)$ is an independent set. Let E_1 be the set of pendant edges in G. For each $x \in D_2(G)$, there are two edges e_x^1 and e_x^2 incident with x. Let $X_2(G) = \{e_x^1 \colon x \in D_2(G)\}$. Put

$$G_0 = G/(E_1 \cup X_2(G)).$$

In other words, G_0 is obtained from G by deleting the vertices in $D_1(G)$ and replacing each path of length 2, whose internal vertex is a vertex in $D_2(G)$, by an edge. Note that G_0 may not be simple.

The vertex set $V(G_0)$ is regarded as a subset of $V(G)$. A vertex in G_0 is *nontrivial* if it is obtained by contracting some edge(s) in $E_1 \cup X_2(G)$ or it is adjacent to a vertex in $D_2(G)$ in G. For instance, if $v \in D_2(G)$ and $N_G(v) = \{x, y\}$, and if x_v is a vertex in G_0 obtained by contracting the edge xv, then both x_v and y are nontrivial in G_0 (although x_v is a contracted vertex and y is not a contracted vertex in G_0). Since $\overline{\sigma}_2(G) \geqslant 5$, all vertices in $D_2(G_0)$ are nontrivial.

Let $X = D_1(G) \cup D_2(G)$. In [20], G_0 is denoted by $I_X(G)$. Following [14], we call G_0 the *core* of G.

Let G_0' be the reduction of G_0. For a vertex $v \in V(G_0')$, let $\Gamma_0(v)$ be the maximum collapsible preimage of v in G_0 and let $\Gamma(v)$ be the preimage of v in G. Note that $\Gamma(v)$ is the graph induced by edge(s) composing of $E(\Gamma_0(v))$ and some edge(s) in $E_1 \cup X_2(G)$, for an example, see Figure 3. A vertex v in G_0' is a *nontrivial vertex* if v is a contracted vertex (i.e., $|V(\Gamma(v))| > 1$) or v is adjacent to a vertex in $D_2(G)$.

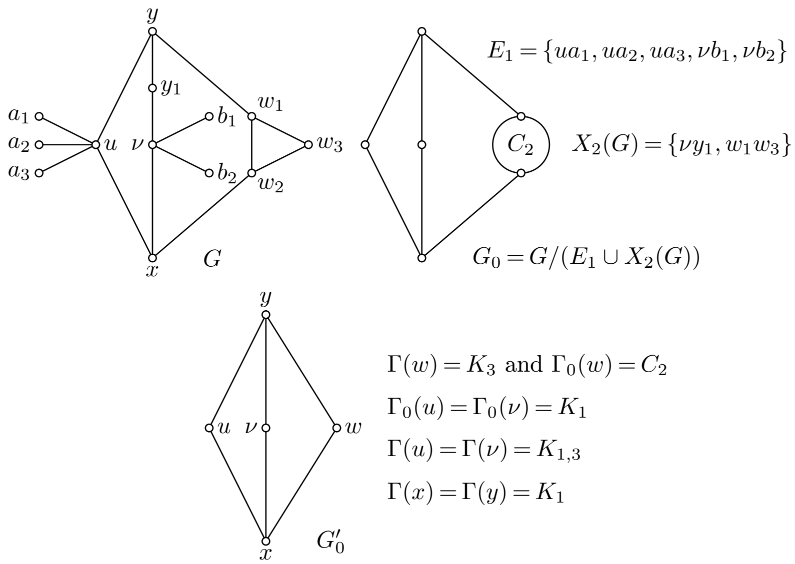

Figure 3. The reduction G_0' of the core G_0 of a graph G.

Using Theorem 2.8, Veldman in [20] and Shao in [14] proved the following theorem.

Theorem 3.1. *Let G be a connected and essentially k-edge-connected graph with $\overline{\sigma}_2(G) \geqslant 5$, where $k \in \{2,3\}$ and $L(G)$ is not complete. Let G_0' be the reduction of the core G_0 of G. Then each of the following holds:*

(a) *G_0 is well-defined, nontrivial, $\delta(G_0) \geqslant \kappa'(G_0) \geqslant k$, and $\kappa'(G_0') \geqslant \kappa'(G_0) \geqslant k$.*

(b) *G has a dominating closed trail if and only if G_0' has a dominating closed trail containing all the nontrivial vertices, see* [20], *Lemma* 5.

We have the following similar result.

Theorem 3.2. *Under the conditions of Theorem* 3.1, *G has a dominating trail if and only if G_0' has a dominating trail containing all the nontrivial vertices.*

P r o o f. Clearly, if G has a dominating trail, then G_0' has a dominating trail containing all the nontrivial vertices of G_0'. Conversely, we assume that G_0' has a dominating trail T' containing all the nontrivial vertices of G_0'. Set $G_s' = G_0'[V(T')]$

and $U = V(G'_0) - V(T')$. Then U is an independent subset of both $V(G'_0)$ and $V(G)$, $U \cap N_G[D_1(G) \cup D_2(G)] = \emptyset$ and T' is a spanning trail of G'_s. Set $G_s = G_0 - U$ and $G_t = G - (U \cup D_1(G))$. By our definitions, G_t is a subdivision of G_s and G'_s is the reduction of G_s. Since G'_s has a spanning trail, by Theorem 2.9, G_s has a spanning trail. Since $\overline{\sigma}_2(G) \geqslant 5$, G_t is a subdivision of G_s with each edge of G_s subdivided at most once. It follows that G_t has a dominating trail T such that $V(G_t) - V(T) \subseteq D_2(G)$. Then $V(G) - V(T) \subseteq U \cup D_1(G) \cup D_2(G)$. Since $U \cup D_1(G) \cup D_2(G)$ is an independent subset of $V(G)$, T is a dominating trail of G. This completes the proof. □

In the following, let $H = L(G)$ and assume that H is not complete. Then $|V(H)| = |E(G)|$ and $\overline{\sigma}_2(G) = \delta(H) + 2$. If $H = L(G)$ is k-connected with $\delta(H) \geqslant 3$, then G is essentially k-edge-connected with $\overline{\sigma}_2(G) \geqslant 5$. For each $v \in V(H)$, there is an edge xy in G corresponding to v and $d_H(v) = d_G(x) + d_G(y) - 2$. We call a path of length k a k-path. For each edge uv in H, there is a 2-path, $P_2 = xyz$ in G such that xy corresponds to the vertex u and the edge yz corresponds to the vertex v in H. Then $d_H(u) + d_H(v) = d_G(x) + 2d_G(y) + d_G(z) - 4$.

For any 2-path $P_2 = xyz$ in G, put $d_G(P_2) = d_G(x) + 2d_G(y) + d_G(z)$. Set

$$\delta_2(G) = \min\{d_G(P_2)\colon\ P_2 \text{ is a 2-path in } G\}.$$

Thus, for a graph $H = L(G)$,

$$\delta_2(G) = \overline{\sigma}_2(H) + 4. \tag{3.1}$$

For a given integer $p > 0$ and a given real number ε, if $\overline{\sigma}_2(H) \geqslant (2n + \varepsilon)/p$, then the preimage G of $H = L(G)$ has

$$\delta_2(G) \geqslant \frac{2n + \varepsilon}{p} + 4. \tag{3.2}$$

3.2. Notation and a technical lemma. Let G, G_0 and G'_0 be the graphs defined above. For $v \in V(G'_0)$, let $\Gamma_0(v)$ be the collapsible preimage of v in G_0 and let $\Gamma(v)$ be the preimage of v in G. For convenience, we use the following notation.

- ▷ $V^* = \{v \in V(G'_0)\colon\ |V(\Gamma(v))| \geqslant 3\}$;
- ▷ $V_1 = \{v \in V(G'_0)\colon\ |V(\Gamma(v))| = 1$ and v is not adjacent to any vertices in $D_1(G) \cup D_2(G)\}$;
- ▷ $V_2 = \{v \in V(G'_0)\colon\ |V(\Gamma(v))| = 2$ or $|V(\Gamma(v))| = 1$ and v is adjacent to a vertex in $D_2(G)\}$;

(Note that $V^* \cup V_2$ is the set of all nontrivial vertices in G'_0.)

▷ $\Phi = G_0'[V_1]$, the subgraph induced by V_1 in G_0' if $V_1 \neq \emptyset$;
▷ $E_\Phi = E(\Phi)$, which is a matching under the conditions of Lemma 3.3 (see below);
▷ $V_\Phi = \{v \in V_1 \colon v$ is incident with an edge in $E_\Phi\}$;
▷ $V_\Phi^0 = V_1 - V_\Phi$;
▷ $N_{\Phi,2} = \bigcup_{v \in V_\Phi \cup V_2} (N_{G_0'}(v) \cap V^*)$ if $V_\Phi \cup V_2 \neq \emptyset$ (otherwise, $N_{\Phi,2} = \emptyset$).

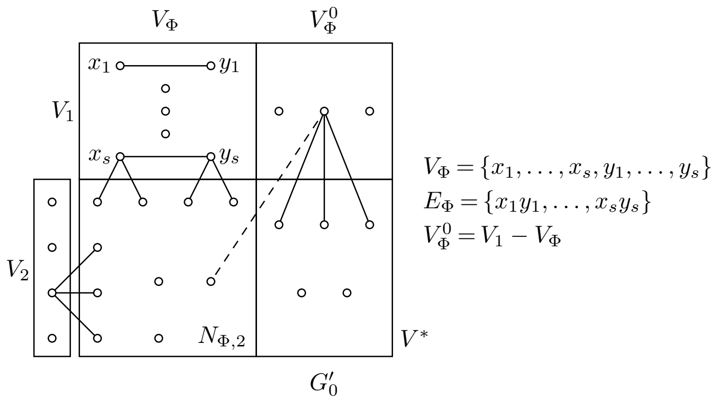

Figure 4. $V(G_0') = V^* \cup V_1 \cup V_2 = V^* \cup (V_\Phi \cup V_\Phi^0) \cup V_2$.

In [6], Chen proved a technical lemma that follows.

Lemma 3.3 ([6]). *Let G be an essentially 2-edge-connected triangle-free graph such that $G \neq K_{1,t}$ with size n and $\overline{\sigma}_2(G) \geqslant 5$, and satisfying (3.2) and $n \gg L(p, \varepsilon)$. Assume that $G_0' \notin \mathcal{SL}$. For V^*, $N_{\Phi,2}$, V_1, V_2, Φ, E_Φ, V_Φ, and V_Φ^0 defined above, we have:*

(a) *For each $v \in V^*$, $|V(\Gamma(v))| \geqslant \frac{1}{2}\delta_2(G) - d_{G_0'}(v)$ and $|E(\Gamma(v))| \geqslant \frac{1}{2}\delta_2(G) - d_{G_0'}(v) - 1$.*
(b) *$D_2(G_0') \subseteq V^*$ and so $d_{G_0'}(v) \geqslant 3$ for $v \in V_1 \cup V_2$.*
(c) *If $E_\Phi \neq \emptyset$ for each $xy \in E_\Phi$, $(N_{G_0'}(x) - \{y\}) \cup (N_{G_0'}(y) - \{x\}) \subseteq N_{\Phi,2}$ and so E_Φ is a matching.*
(d) *For each vertex v in $V_\Phi^0 \cup V_2, N_{G_0'}(v) \subseteq V^*$, and so $V_\Phi^0 \cup V_2$ is an independent set.*
(e) *If $|V_1 \cup V_2| \geqslant 3$, then $|V_\Phi^0 \cup V_2| + \frac{1}{2}|V_\Phi| \leqslant 2|V^*| - 5$. If $|V_2| \geqslant 3$ or $V_\Phi \neq \emptyset$, then $|V_2| + \frac{1}{2}|V_\Phi| \leqslant 2|N_{\Phi,2}| - 5$.*
(f) *$|V^*| \leqslant p$. Furthermore, if $|V^*| = p$ and $G_0' \neq K_{2,t}$ for $t \geqslant 2$, then $|V(G_0')| \leqslant 2p - 5 - \frac{1}{2}\varepsilon$.*
(g) *For $v \in N_{\Phi,2}$, $|E(\Gamma(v))| \geqslant \delta_2(G) - 5p - 3$ and $|V^*| + |N_{\Phi,2}| \leqslant p$.*
(h) *If $V_2 \neq \emptyset$, then $|N_{\Phi,2}| \geqslant 3$. If $V_\Phi \neq \emptyset$, then $|N_{\Phi,2}| \geqslant 4$. Thus, $|N_{\Phi,2}| \geqslant 3$ if $|V_2 \cup V_\Phi| \neq 0$.*

4. Proofs of Corollary 1.3 and Theorem 1.5

In this section, we present the proofs of Corollary 1.3 and Theorem 1.5.

P r o o f of Corollary 1.3. If H is traceable, then we are done. Thus, in the following, we assume that H is not traceable, and so H is not hamiltonian and H is not complete. By Theorem 1.1, $\mathrm{cl}(H)$ is not complete and there exists an essentially k-edge-connected triangle-free graph G such that $\mathrm{cl}(H) = L(G)$ and $|E(G)| = |V(H)|$. Let G_0' be the reduction of the core G_0 of G. By Theorem 3.1, $\kappa'(G_0') \geqslant \kappa'(G_0) \geqslant k$. Since H is not traceable, by Theorem 2.6, G has no dominating trail. By Theorem 3.2, G_0' has no dominating trail containing all the nontrivial vertices. Then G_0' has no dominating closed trail containing all the nontrivial vertices. Then by Theorem 1.2, $G_0' \in \mathcal{Q}_0(5p-10, k)$. Note that $\mathcal{R}_0(r,k) \subseteq \mathcal{Q}_0(r,k)$. Since G_0' has no spanning trail and by Theorem 2.4, $G_0' \in \mathcal{R}_0(5p-10, k)$ and $|V(G_0')| \geqslant 10$. Then $5p - 10 \geqslant 10$. We conclude that $p \geqslant 4$. This completes the proof. □

P r o o f of Theorem 1.5. This is a special case of Corollary 1.3 with $p = 7$, $\varepsilon = -5$ and $k = 2$. By Theorem 1.1, there is an essentially 2-edge-connected triangle-free graph G such that the closure $\mathrm{cl}(H) = L(G)$ and $|E(G)| = |V(H)| = n$. Since $\delta(H) \geqslant 3$ and $\overline{\sigma}_2(H) \geqslant \frac{1}{7}(2n-5)$, $\overline{\sigma}_2(G) \geqslant 5$ and $\delta_2(G) \geqslant \frac{1}{7}(2n-5)+4 = \frac{1}{7}(2n+23)$ by (3.2).

Suppose that H is not traceable. Then $G \neq K_{1,t}$; otherwise, by Theorems 2.6 and 2.7, H is traceable, a contradiction. By Corollary 1.3 and Theorem 2.4, G_0' has no spanning trail and $|V(G_0')| \geqslant 10$. Therefore, $G_0' \notin \mathcal{SL}$.

Let V^*, V_2, V_Φ, V_Φ^0 and $N_{\Phi,2}$ be the sets relating to G_0' as defined in Section 3. If $V_\Phi \cup V_2 \neq \emptyset$, then by the definition, $N_{\Phi,2} \neq \emptyset$. By Lemma 3.3 (h), $|N_{\Phi,2}| \geqslant 3$. Since $N_{\Phi,2} \subseteq V^*$, by Lemma 3.3 (g), $|N_{\Phi,2}| \leqslant 3$. So, $|N_{\Phi,2}| = 3$. Then $V_\Phi = \emptyset$, otherwise, by Lemma 3.3 (h), $|N_{\Phi,2}| \geqslant 4$, a contradiction. Since $N_{\Phi,2} \subseteq V^*$, by Lemma 3.3 (g), $3 \leqslant |V^*| \leqslant 4$. Then $|V_\Phi^0 \cup V_2| \leqslant 3$; otherwise, by Lemma 3.3 (e), $|V^*| \geqslant 5$, a contradiction. Therefore, $|V(G_0')| = |V_\Phi^0 \cup V_2| + |V^*| \leqslant 3 + 4 = 7$, a contradiction. Hence, $V_\Phi = V_2 = \emptyset$ and $V(G_0') = V_\Phi^0 \cup V^*$. Then V^* is the set of all nontrivial vertices of G_0'. By Lemma 3.3 (f), $|V^*| \leqslant 7$. We distinguish the two cases that $|V^*| \leqslant 6$ and $|V^*| = 7$.

Case 1: $|V^*| \leqslant 6$. By Lemma 3.3 (d), $E(G_0' - V^*) = \emptyset$. Note that G_0' is 2-edge-connected. Then by Theorem 2.3, either G_0' has a trail passing through all vertices of V^* or $G_0' \in \{F_1, F_2\}$. For the first case, G_0' has a dominating trail containing all vertices of V^*. Then by Theorems 2.6, 2.7 and 3.2, H is traceable, a contradiction.

Hence, $G_0' \in \{F_1, F_2\}$. By Lemma 3.3 (b), $D_2(G_0') \subseteq V^*$. Then, since $|D_2(G_0')| = 6$, $|V^*| = 6$. Let $V^* = D_2(G_0') = \{v_1, v_2, \ldots, v_6\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 3.3 (a) and since $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$, $s_i = |E(\Gamma(v_i))| \geqslant \frac{1}{2}\delta_2(G) - d_{G_0'}(v_i) - 1 = \frac{1}{2}\delta_2(G) - 3 \geqslant \frac{1}{14}(2n-19)$. Since $n = |E(G)| = |E(G_0')| + \sum_{i=1}^{6} s_i \geqslant 12 + 6(\frac{1}{2}\delta_2(G) - 3) = 3\delta_2(G) - 6$, $\delta_2(G) \leqslant \frac{1}{3}(n+6)$. Then by (3.1), $\overline{\sigma}_2(H) = \delta_2(G) - 4 \leqslant \frac{1}{3}(n-6)$. Thus, $G \in \mathcal{F}(n, \frac{1}{14}(2n-19))$, and so $\overline{\sigma}_2(H) \leqslant \frac{1}{3}(n-6)$ and $H \in \mathcal{R}_{\mathcal{F}}(n, \frac{1}{14}(2n-19))$.

In particular, if $\overline{\sigma}_2(H) = \frac{1}{3}(n-6)$, then by (3.1), $\delta_2(G) = \overline{\sigma}_2(H) + 4 = \frac{1}{3}(n+6)$. By Lemma 3.3 (a) and since $\delta_2(G) \geqslant \frac{1}{3}(n+6)$, $s_i = |E(\Gamma(v_i))| \geqslant \frac{1}{2}\delta_2(G) - d_{G_0'}(v_i) - 1 = \frac{1}{2}\delta_2(G) - 3 \geqslant \frac{1}{6}(n-12)$. Since $n = |E(G)| = |E(G_0')| + \sum_{i=1}^{6} s_i$, $s_i = \frac{1}{6}(n-12)$ for $v_i \in V^*$. By Lemma 3.3 (a), $|V(\Gamma(v_i))| \geqslant \frac{1}{2}\delta_2(G) - d_{G_0'}(v_i) = |E(\Gamma(v_i))| + 1$. Thus, $|V(\Gamma(v_i))| = |E(\Gamma(v_i))| + 1$ and so $\Gamma(v_i)$ is a tree. Since G is essentially 2-edge-connected, $\Gamma(v_i) = K_{1,s}$, where $s = \frac{1}{6}(n-12)$. Because $G \in \mathcal{F}(n, \frac{1}{6}(n-12))$, $H \in \mathcal{R}_{\mathcal{F}}(n, \frac{1}{6}(n-12))$, this settles Case 1.

In the following, we show that the case $|V^*| = 7$ is impossible.

Case 2: $|V^*| = 7$. Then G_0' cannot be isomorphic to a $K_{2,t}$ for any $t \geqslant 2$; otherwise G_0' has a spanning trail, a contradiction. By Lemma 3.3 (f), $|V(G_0')| \leqslant 2p - 5 - \frac{1}{2}\varepsilon = 2 \times 7 - 5 - \frac{(-5)}{2} = \frac{23}{2}$. Then $|V(G_0')| \leqslant 11$. Then by Theorem 2.4, $G_0' \in \{F_1, F_2, G_1, G_2, \dots, G_6\}$. By Lemma 3.3 (b), $D_2(G_0') \subseteq V^*$. We distinguish the following two subcases for Case 2.

Subcase 2.1: $G_0' \in \{F_1, F_2\}$. Let $V^* = D_2(G_0') \cup \{v\} = \{v_1, v_2, \dots, v_6, v\}$, where $v \in D_3(G_0')$. Then $d_{G_0'}(v_i) = 2$ and $d_{G_0'}(v) = 3$. By Lemma 3.3 (a) and since $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$, $s_i = |E(\Gamma(v_i))| \geqslant \frac{1}{2}\delta_2(G) - 3 \geqslant \frac{1}{14}(2n-19)$, $s = |E(\Gamma(v))| \geqslant \frac{1}{2}\delta_2(G) - 4 \geqslant \frac{1}{14}(2n-33)$. Furthermore, since $n = 12 + s + \sum_{i=1}^{6} s_i \geqslant 12 + (\frac{1}{2}\delta_2(G) - 4) + 6(\frac{1}{2}\delta_2(G) - 3) = \frac{7}{2}\delta_2(G) - 10$, $\delta_2(G) \leqslant \frac{1}{7}(2n+20)$, contradicting that $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$.

Subcase 2.2: $G_0' \in \{G_1, G_2, \dots, G_6\}$. First suppose that $G_0' \in \{G_1, G_2, \dots, G_5\}$. Then $V^* = D_2(G_0')$. Let $V^* = D_2(G_0') = \{v_1, v_2, \dots, v_7\}$. Then $d_{G_0'}(v_i) = 2$. By Lemma 3.3 (a) and since $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$, $s_i = |E(\Gamma(v_i))| \geqslant \frac{1}{2}\delta_2(G) - 3 \geqslant \frac{1}{14}(2n-19)$. Furthermore, since $n \geqslant 13 + \sum_{i=1}^{7} s_i \geqslant 13 + 7(\frac{1}{2}\delta_2(G) - 3) = \frac{7}{2}\delta_2(G) - 8$, $\delta_2(G) \leqslant \frac{1}{7}(2n+16)$, contradicting that $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$.

Next suppose that $G_0' = G_6$. Since $|D_2(G_0')| = 6$ and $|V^*| = 7$, there exists one vertex $v \in V(G_0') \setminus D_2(G_0')$ such that $v \in V^*$. By Lemma 3.3 (d), V_Φ^0 is an independent set. Then $v \in D_4(G_0')$. Let $V^* = D_2(G_0') \cup \{v\} = \{v_1, v_2, \dots, v_6, v\}$. Then $d_{G_0'}(v_i) = 2$ and $d_{G_0'}(v) = 4$. By Lemma 3.3 (a) and since $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$, $s_i = |E(\Gamma(v_i))| \geqslant \frac{1}{2}\delta_2(G) - 3 \geqslant \frac{1}{14}(2n-19)$, $s = |E(\Gamma(v))| \geqslant \frac{1}{2}\delta_2(G) - 5 \geqslant \frac{1}{14}(2n-47)$. Furthermore, since $n = 14 + s + \sum_{i=1}^{6} s_i \geqslant 14 + (\frac{1}{2}\delta_2(G) - 5) + 6(\frac{1}{2}\delta_2(G) - 3) = \frac{7}{2}\delta_2(G) - 9$, $\delta_2(G) \leqslant \frac{1}{7}(2n+18)$, contradicting that $\delta_2(G) \geqslant \frac{1}{7}(2n+23)$.

This settles Case 2 and completes the proof. $\square$

5. Concluding remark

In this paper, we have been mainly discussing the traceability of 2-connected claw-free graphs. In order to prove our main results, one of the essential elements we needed was a characterization of all the 2-edge-connected graphs of order at most 11 that have no spanning trail. This has resulted in a more or less explicit description of the obstructions that prevent graphs satisfying the degree conditions from being traceable for given values of $p \leqslant 7$. For given values of $p \geqslant 8$, it is much harder to obtain (and write down) such an explicit description, but our main result still implies that there are only a finite number of these obstructions. In principle, for a given p, this finite set of obstructions can be found with the help of a computer, but the numbers grow fast with increasing values of p.

By using Theorems 2.4, 3.2 and a more detailed discussion than in Theorem 1.5, Tian et al. characterized the traceability of 2-connected claw-free graphs with minimum degree sum of t independent vertices (see [17]) and generalized Dirac conditions (see [16]), as well as the traceability of 2-connected line graphs among graphs with the minimum degree sum of a pair of adjacent vertices, see [18].

As far as we know, the smallest 3-edge-connected graph without a spanning trail is still unknown, but a likely candidate is the cubic (i.e., 3-regular) graph on 28 vertices that is shown in Figure 5. In [15], the author proved that the cubic graph has no spanning path. Since this cubic graph is 3-regular, it is easy to prove that it has no spanning trail. If one would be able to characterize the smallest 3-edge-connected graphs without spanning trails, then, using a similar approach, one can deduce a best possible adjacent degree sum condition for the traceability of 3-connected claw-free graphs.

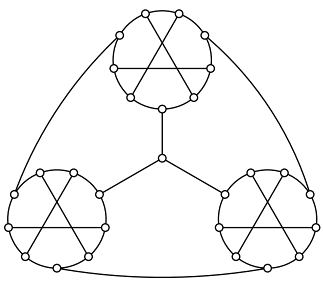

Figure 5. A 3-edge-connected cubic graph which has no spanning trail.

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