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Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 2, 391–409

Persistent URL: http://dml.cz/dmlcz/150408

Czechoslovak Mathematical Journal, 72 (147) (2022), 391–409

ON THE SYMMETRIC ALGEBRA OF CERTAIN FIRST SYZYGY MODULES

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Received November 22, 2020. Published online October 5, 2021.

Abstract. Let $(R,\mathfrak{m})$ be a standard graded K-algebra over a field K. Then R can be written as S/I, where $I\subseteq (x_1,\dots,x_n)^2$ is a graded ideal of a polynomial ring $S=K[x_1,\dots,x_n]$. Assume that $n\geqslant 3$ and I is a strongly stable monomial ideal. We study the symmetric algebra $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$ of the first syzygy module $\mathrm{Syz}_1(\mathfrak{m})$ of $\mathfrak{m}$. When the minimal generators of I are all of degree 2, the dimension of $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$ is calculated and a lower bound for its depth is obtained. Under suitable conditions, this lower bound is reached.

Keywords: symmetric algebra; syzygy; dimension; depth

MSC 2020: 13D02, 13C15

1. Introduction

Symmetric algebras are important topics in commutative algebra and algebraic geometry. For instance, let W be a closed subscheme of a scheme X, which is defined by a quasi-coherent sheaf of ideals I. Then the normal bundle to W in X is defined by the symmetric algebra of I/I^2. On the other hand, from the normal cone to the normal bundle, there is a closed immersion, which is isomorphic if and only if the symmetric and Rees algebra of I are isomorphic.

Let M be a finitely generated module over a commutative Noetherian ring R with identity. There is an effective method to study the invariants of the symmetric algebra $\mathrm{Sym}_R(M)$ in [5], where the authors introduced the notion of s-sequences. If M

This work was supported by the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA - INDAM, Istituto Nazionale di Alta Matematica "F. Severi", Roma, Italy). The second author would like to thank the Natural Science Foundation of Jiangsu Province (No. BK20181427) for financial support.

DOI: 10.21136/CMJ.2021.0508-20

is generated by an s-sequence, one can obtain an exact value for $\dim_R(\mathrm{Sym}(M))$, $e(\mathrm{Sym}(M))$ and a bound for $\mathrm{depth}(\mathrm{Sym}(M))$ and the Castelnuovo-Mumford regularity $\mathrm{reg}(\mathrm{Sym}(M))$ by the computation of the same invariants of some special quotients of the base ring R by the annihilator ideals.

Let M be an R-module generated by $f_1, \ldots, f_n$. Then M has a presentation

$$R^m \to R^n \to M \to 0$$

with $m \times n$ relation matrix $A = (a_{ij})$. The symmetric algebra $\mathrm{Sym}(M)$ has the presentation

$$R[y_1, \ldots, y_n]/J,$$

where $J = (g_1, \ldots, g_m)$ and $g_i = \sum_{j=1}^{n} a_{ij} y_j$ with $i = 1, \ldots, m$. Consider $P = R[y_1, \ldots, y_n]$ as a graded R-algebra assigning degree one to each variable y_i and degree zero to the elements of R. Then J is a graded ideal and $\mathrm{Sym}(M)$ is a graded R-algebra. Let $<$ be a monomial order induced by $y_1 < \ldots < y_n$. For $f \in P$, $f = \sum_{\alpha} a_\alpha y^\alpha$ we put $\mathrm{in}(f) = a_\alpha y^\alpha$, where y^α is the largest monomial with respect to the given order such that $a_\alpha \neq 0$. We call $\mathrm{in}(f)$ the *initial term* of f. Note that in contrast to the ordinary Gröbner basis theory, the base ring R is not a field. Nevertheless, we may define the ideal

$$\mathrm{in}(J) = (\mathrm{in}(f)\colon\ f \in J).$$

The ideal is generated by terms which are monomials in $y_1, \ldots, y_n$ with coefficients in R and is finitely generated since P is Noetherian. For $i = 1, \ldots, n$ we set $M_i = \sum_{j=1}^{i} R f_j$ and let $I_i = M_{i-1} :_R f_i = \{a \in R\colon\ a f_i \in M_{i-1}\}$. We also set $I_0 = 0$. Note that I_i is the annihilator ideal of the cyclic module $M_i/M_{i-1} \cong R/I_i$.

It is clear that

$$(I_1 y_1, \ldots, I_n y_n) \subseteq \mathrm{in}(J),$$

and the two ideals coincide in degree one. If $(I_1 y_1, \ldots, I_n y_n) = \mathrm{in}(J)$, the generators $f_1, \ldots, f_n$ of M are called an *s-sequence* (with respect to $<$). If, in addition, $I_1 \subseteq \ldots \subseteq I_n$, then $f_1, \ldots, f_n$ is called a *strong s-sequence.*

If $f_1, \ldots, f_n$ forms a strong s-sequence, then Propositions 2.4 and 2.6 in [5] shows that

$$\dim(\mathrm{Sym}_R(M)) = \max\{\dim(R/I_r) + r\colon\ r = 0, 1, \ldots, n\},$$
$$\mathrm{depth}(\mathrm{Sym}_R(M)) \geqslant \min\{\mathrm{depth}(R/I_r) + r\colon\ r = 0, 1, \ldots, n\}.$$

Using s-sequences, some new results for symmetric algebras are obtained (cf. [5], [6], [7], [8], [9]).

Let $\mathrm{Syz}_1(\mathfrak{M})$ be the first syzygy module of the graded maximal ideal $\mathfrak{M} = (x_1, \ldots, x_n)$ of a polynomial ring $K[x_1, \ldots, x_n]$ over a field K. Although the generators of $\mathrm{Syz}_1(\mathfrak{M})$ do not form an s-sequence, in virtue of Jacobian dual, some invariants of $\mathrm{Sym}(\mathrm{Syz}_1(\mathfrak{M}))$ are evaluated in [7] by the theory of s-sequences.

On the other hand, when R is a standard graded K-algebra whose defining ideal is componentwise linear and M is the graded maximal ideal of R, the depth and regularity of $\mathrm{Sym}_R(M)$ are bounded in [4]. Using Gröbner bases, in order to get certain invariants of $\mathrm{Sym}_R(M)$, it suffices to study standard graded K-algebras with monomial relations. Stable and strongly stable monomial ideals are suitable candidates.

Combining the above two situations, we consider the case $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$, where R is a standard graded algebra over a field K with the graded maximal ideal $\mathfrak{m} = (a_1, \ldots, a_n)$. Then the algebra R can be written as S/I and $\mathfrak{m} = \mathfrak{M}/I$, where $S = K[x_1, \ldots, x_n]$ is a polynomial ring, $\mathfrak{M} = (x_1, \ldots, x_n)$ and $I \subseteq \mathfrak{M}^2$ is a graded ideal of S. We are interested in the dimension and depth of $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$.

In the case M is generated by a strong s-sequence, the dimension and depth of $\mathrm{Sym}_R(M)$ are estimated by that of $R[y_1, \ldots, y_n]/(I_1y_1, \ldots, I_ny_n)$, where $I_1 \subseteq \ldots \subseteq I_n$. In our case, we have to treat a ring $R[y_1, \ldots, y_n]/(I_1y_1, \ldots, I_ny_n, I)$, where $I_1 \supseteq \ldots \supseteq I_s \subseteq I_{s+1} \subseteq \ldots \subseteq I_n$ with some $s \geqslant 1$, and I is generated by some monomials in $y_1, \ldots, y_n$. In Section 2, we will compute the dimension and depth of $R[y_1, \ldots, y_n]/(I_1y_1, \ldots, I_ny_n, I)$.

Write $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$ as $S[y_{ij}\colon 1 \leqslant i < j \leqslant n]/J$. In order to get the initial ideal $\mathrm{in}(J)$, we find one Gröbner basis of J in Section 3. Section 4 is devoted to calculate the dimension of $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$ and obtain one lower bound for its depth.

2. Preliminaries

Let R be a Noetherian ring and

$$0 \to L \to M \to N \to 0$$

be an exact sequence of R-modules. Then there is an exact sequence of symmetric algebras:

$$L \otimes_R \mathrm{Sym}_R(M) \to \mathrm{Sym}_R(M) \to \mathrm{Sym}_R(N) \to 0.$$

When L is a submodule of M, one has an isomorphism

$$\mathrm{Sym}_R(N) \cong \mathrm{Sym}_R(M)/(\widetilde{L}),$$

where $\widetilde{L}$ is the set of 1-forms of elements of L, cf. [1], Proposition A2.2.

Furthermore, suppose that M is an R-module generated by $f_1, \dots, f_n$. Then M has a presentation

$$R^m \to R^n \to M \to 0$$

with $m \times n$ relation matrix $A = (a_{ij})$. The symmetric algebra $\mathrm{Sym}(M)$ has the presentation

$$R[y_1, \dots, y_n]/J,$$

where $J = (g_1, \dots, g_m)$ and $g_i = \sum_{j=1}^{n} a_{ij} y_j$ with $i = 1, \dots, m$. Under this presentation, we get

$$\mathrm{Sym}_R(N) \cong R[y_1, \dots, y_n]/(J, \widetilde{L}),$$

where $\widetilde{L} = \{r_1 y_1 + \ldots + r_n y_n \colon\ r_1 f_1 + \ldots + r_n f_n \in L\}$. We will use this presentation in the next section.

Let K be a field, $S = K[x_1, \dots, x_n]$ and I be a monomial ideal of S. Denote the minimal generating set of I by $G(I)$. For any monomial u of S, set $\max(u) = \max\{i\colon\ x_i \mid u\}$ and $\min(u) = \min\{i\colon\ x_i \mid u\}$. Put $m(I) = \max\{\min(u)\colon\ u \in G(I)\}$ and $M(I) = \max\{\max(u)\colon\ u \in G(I)\}$. For a monomial ideal W of $K[x_r, \dots, x_t]$, $M(W)$ and $m(W)$ are defined exactly as in $K[x_1, \dots, x_n]$.

Definition 2.1. If for any monomial $u \in I$, $x_i u / x_{\max(u)} \in I$ holds for any $i < \max(u)$, we say that I is *stable*. Furthermore, if for any monomial $u \in I$ and any integer j such that $x_j \mid u$, one has that $x_i u / x_j \in I$ for any $i < j$, then we say that I is *strongly stable*.

When I is stable, it is shown in [2] that

$$\mathrm{Proj.dim}(S/I) = \max\{\max(u)\colon\ u \in G(I)\}.$$

Then, by Auslander-Buchsbaum formula, one gets

$$\mathrm{depth}(S/I) = n - \max\{\max(u)\colon\ u \in G(I)\}.$$

On the other hand, for the dimension we have

$$\dim(S/I) = n - \max\{\min(u)\colon\ u \in G(I)\},$$

which follows from the equality $\mathrm{height}(I) = \max\{\min(u)\colon\ u \in G(I)\}$ (cf. [3], Exercise 8.9). Then we have the following lemma.

Lemma 2.2. *Let I be a stable monomial ideal of $S = K[x_1, \dots, x_n]$. Then $\dim(S/I) = n - M(I)$ and $\mathrm{depth}(S/I) = n - m(I)$.*

In order to estimate the dimension and depth of a factor ring, we need to express an ideal as an intersection of some satisfied ideals. By using the same arguments as in the proof of Lemma 2.3 of [5], we get the following two lemmas.

Lemma 2.3. *Let R be a Noetherian ring, $I_1, \ldots, I_n$ be ideals of R and $u_1, \ldots, u_t$ be monomials in $y_1, \ldots, y_n$. Then in $R[y_1, \ldots, y_n]$,*

$$\begin{aligned}(I_1y_1, \ldots, I_ny_n, u_1, \ldots, u_t) \\ = \bigcap_{\substack{0\leqslant r\leqslant n \\ 1\leqslant i_1<\ldots<i_r\leqslant n}} (I_{i_1} + \ldots + I_{i_r}, y_1, \ldots, \widehat{y}_{i_1}, \ldots, \widehat{y}_{i_r}, \ldots, y_n, u_1, \ldots, u_t),\end{aligned}$$

where $I_0 = 0$ by convention.

Lemma 2.4. *Let R be a Noetherian ring, $I_1, \ldots, I_n$ be ideals of R and $u_1, \ldots, u_t$ be monomials in $y_1, \ldots, y_n$. Suppose that there is an $1 \leqslant s \leqslant n$ such that $I_1 \supseteq \ldots \supseteq I_s \subseteq I_{s+1} \subseteq \ldots \subseteq I_n$. Then in $R[y_1, \ldots, y_n]$,*

$$\begin{aligned}(I_1y_1, \ldots, I_ny_n, u_1, \ldots, u_t) \\ = (y_1, \ldots, y_n) \bigcap \Big(\bigcap_{r=1}^{s} \bigcap_{t=s}^{n} (I_r + I_t, y_1, \ldots, y_{r-1}, y_{t+1}, \ldots, y_n, u_1, \ldots, u_t) \Big).\end{aligned}$$

In particular, when $s = 1$, i.e. $I_1 \subseteq \ldots \subseteq I_n$,

$$(I_1y_1, \ldots, I_ny_n, u_1, \ldots, u_t) = \bigcap_{r=0}^{n} (I_r, y_{r+1}, \ldots, y_n, u_1, \ldots, u_t).$$

Let I and J be two ideals of a Noetherian ring R. It is well-known that

$$\dim(R/(I \cap J)) = \max\{\dim(R/I), \dim(R/J)\}.$$

On the other hand, from the short exact sequence

$$0 \to R/(I \cap J) \to R/I \oplus R/J \to R/(I + J) \to 0$$

we have

$$\operatorname{depth}(R/(I \cap J)) \geqslant \min\{\operatorname{depth}(R/I), \operatorname{depth}(R/J), \operatorname{depth}(R/(I + J)) + 1\}.$$

The following result generalizes Proposition 2.4 of [5].

Proposition 2.5. *Let K be a field, $R = K[x_1, \ldots, x_m]$ and $I_1 \supseteq \ldots \supseteq I_s \subseteq I_{s+1} \subseteq \ldots \subseteq I_n$ be ideals of R. Then for any monomial ideal I of $K[y_1, \ldots, y_n]$,*

$$\begin{aligned}\dim(R[y_1, \ldots, y_n]/(I_1y_1, \ldots, I_ny_n, I)) \\ = \max_{\substack{1\leqslant r\leqslant s \\ s\leqslant t\leqslant n}} \{\dim(R), \dim(R/(I_r + I_t)) + \dim(K[y_r, \ldots, y_t]/I \cap K[y_r, \ldots, y_t])\}.\end{aligned}$$

P r o o f. By Lemma 2.4 we have

$$\begin{aligned}
&\dim(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\
&\quad=\dim\biggl(R[y_1,\dots,y_n]/(y_1,\dots,y_n)\\
&\qquad\bigcap\biggl(\bigcap_{r=1}^{s}\bigcap_{t=s}^{n}(I_r+I_t,y_1,\dots,y_{r-1},y_{t+1},\dots,y_n,I)\biggr)\biggr)\\
&\quad=\max_{\substack{1\leqslant r\leqslant s\\ s\leqslant t\leqslant n}}\{\dim(R),\dim(R[y_1,\dots,y_n]/(I_r+I_t,y_1,\dots,y_{r-1},y_{t+1},\dots,y_n,I))\}\\
&\quad=\max_{\substack{1\leqslant r\leqslant s\\ s\leqslant t\leqslant n}}\{\dim(R),\dim(R[y_r,\dots,y_t]/(I_r+I_t,I\cap K[y_r,\dots,y_t]))\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&R[y_r,\dots,y_t]/(I_r+I_t,I\cap K[y_r,\dots,y_t])\\
&\qquad\cong R/(I_r+I_t)\otimes_K K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t]),
\end{aligned}$$

by Proposition 2.2.20 of [10]. Then by [10], Exercise 2.1.14

$$\begin{aligned}
&\dim(R[y_r,\dots,y_t]/(I_r+I_t,I\cap K[y_r,\dots,y_t]))\\
&\qquad=\dim(R/(I_r+I_t))+\dim(K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t]).
\end{aligned}$$

Thus, the result follows. □

By [10], Theorem 2.2.21

$$\begin{aligned}
&\operatorname{depth}(R[y_r,\dots,y_t]/(I_r+I_t,I\cap K[y_r,\dots,y_t])))\\
&\qquad=\operatorname{depth}(R/(I_r+I_t))+\operatorname{depth}(K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t])),
\end{aligned}$$

which will be used in the following arguments for depth.

Lemma 2.6. *Let K be a field, $R=K[x_1,\dots,x_m]$ and $I_1\subseteq\dots\subseteq I_n$ be ideals of R. Then for any monomial ideal I of $K[y_1,\dots,y_n]$,*

$$\begin{aligned}
&\operatorname{depth}(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\
&\qquad\geqslant\min_{0\leqslant r\leqslant n}\{\operatorname{depth}(R/I_r)+\operatorname{depth}(K[y_1,\dots,y_r]/(I\cap K[y_1,\dots,y_r])),\\
&\qquad\quad\operatorname{depth}(R/I_r)+\operatorname{depth}(K[y_1,\dots,y_{r-1}]/(I\cap K[y_1,\dots,y_{r-1}]))+1\}.
\end{aligned}$$

P r o o f. We use induction on n. When $n=1$, one has

$$\begin{aligned}
&\operatorname{depth}(R[y_1]/(I_1y_1, I))\\
&\quad=\operatorname{depth}(R[y_1]/((y_1)\cap(I_1,I)))\\
&\quad\geqslant \min\{\operatorname{depth}(R[y_1]/(y_1)), \operatorname{depth}(R[y_1]/(I_1,I)), \operatorname{depth}(R[y_1]/(y_1,I_1))+1\}\\
&\quad= \min\{\operatorname{depth}(R), \operatorname{depth}(R/I_1)+\operatorname{depth}(K[y_1]/I), \operatorname{depth}(R/I_1)+1\}.
\end{aligned}$$

Now assume that $n>1$. Notice that by Lemma 2.4, $\bigcap\limits_{r=0}^{n-1}(I_r, y_{r+1},\dots,y_n,I) = \Big(\bigcap\limits_{r=0}^{n-1}(I_r,y_{r+1},\dots,y_n,I)\Big)\cap(R,I) = (I_1y_1,\dots,I_{n-1}y_{n-1},y_n,I)$, hence

$$\Big(\bigcap_{r=0}^{n-1}(I_r,y_{r+1},\dots,y_n,I)\Big)+(I_n,I)=(I_n,y_n,I).$$

Then

$$\begin{aligned}
&\operatorname{depth}(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\
&\quad=\operatorname{depth}\Big(R[y_1,\dots,y_n]/\bigcap_{r=0}^{n}(I_r,y_{r+1},\dots,y_n,I)\Big)\\
&\quad=\operatorname{depth}\Big(R[y_1,\dots,y_n]/\Big(\Big(\bigcap_{r=0}^{n-1}(I_r,y_{r+1},\dots,y_n,I)\Big)\cap(I_n,I)\Big)\Big)\\
&\quad\geqslant\min\Big\{\operatorname{depth}\Big(R[y_1,\dots,y_n]/\bigcap_{r=0}^{n-1}(I_r,y_{r+1},\dots,y_n,I)\Big),\\
&\qquad\operatorname{depth}(R[y_1,\dots,y_n]/(I_n,I)),\\
&\qquad\operatorname{depth}\Big(R[y_1,\dots,y_n]/\Big(\Big(\bigcap_{r=0}^{n-1}(I_r,y_{r+1},\dots,y_n,I)\Big)+(I_n,I)\Big)\Big)+1\Big\}\\
&\quad=\min\{\operatorname{depth}(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_{n-1}y_{n-1},y_n,I),\\
&\qquad\operatorname{depth}(R[y_1,\dots,y_n]/(I_n,I)), \operatorname{depth}(R[y_1,\dots,y_n]/(I_n,y_n,I))+1\}\\
&\quad=\min\{\operatorname{depth}(R[y_1,\dots,y_{n-1}]/(I_1y_1,\dots,I_{n-1}y_{n-1},I\cap K[y_1,\dots,y_{n-1}])),\\
&\qquad\operatorname{depth}(R/I_n)+\operatorname{depth}(K[y_1,\dots,y_n]/I),\\
&\qquad\operatorname{depth}(R/I_n)+\operatorname{depth}(K[y_1,\dots,y_{n-1}]/(I\cap K[y_1,\dots,y_{n-1}]))+1\}.
\end{aligned}$$

The results follow by the induction hypothesis. □

The following proposition reduces the general case to the case above.

Proposition 2.7. *Let K be a field, $R = K[x_1,\dots,x_m]$ and $I_1 \supseteq \dots \supseteq I_s \subseteq I_{s+1} \subseteq \dots \subseteq I_n$ be ideals of R. Then for any monomial ideal I of $K[y_1,\dots,y_n]$,*

$$\begin{aligned}
&\operatorname{depth}(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\
&\qquad \geqslant \min_{1\leqslant r\leqslant s-1}\{\operatorname{depth}(R[y_s,\dots,y_n]/(I_sy_s,\dots,I_ny_n,I\cap K[y_s,\dots,y_n])),\\
&\qquad\quad \operatorname{depth}(R[y_r,\dots,y_n]/((I_r+I_r)y_r,\dots,(I_r+I_n)y_n,I\cap K[y_r,\dots,y_n])),\\
&\qquad\quad \operatorname{depth}(R[y_{r+1},\dots,y_n]\\
&\qquad\quad /((I_r+I_{r+1})y_{r+1},\dots,(I_r+I_n)y_n,I\cap K[y_{r+1},\dots,y_n]))+1\}.
\end{aligned}$$

P r o o f. It is enough to show that

$$\begin{aligned}
&\operatorname{depth}(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\
&\quad\geqslant \min\{\operatorname{depth}(R[y_2,\dots,y_n]/(I_2y_2,\dots,I_ny_n,I\cap K[y_2,\dots,y_n])),\\
&\qquad \operatorname{depth}(R[y_1,\dots,y_n]/((I_1+I_1)y_1,\dots,(I_1+I_n)y_n,I)),\\
&\qquad \operatorname{depth}(R[y_2,\dots,y_n]/((I_1+I_2)y_2,\dots,(I_1+I_n)y_n,I\cap K[y_2,\dots,y_n]))+1\}.
\end{aligned}$$

Set

$$\begin{aligned}
J_1 &= (y_1,\dots,y_n)\bigcap\Bigl(\bigcap_{r=2}^{s}\bigcap_{t=s}^{n}(I_r+I_t,y_1,\dots,y_{r-1},y_{t+1},\dots,y_n,I)\Bigr),\\
J_2 &= (y_1,\dots,y_n)\bigcap\Bigl(\bigcap_{t=s}^{n}(I_1+I_t,y_{t+1},\dots,y_n,I)\Bigr).
\end{aligned}$$

Then by Lemma 2.4, $(I_1y_1,\dots,I_ny_n,I) = J_1\cap J_2$. We see that

$$J_1 = (y_1,I_2y_2,\dots,I_ny_n,I)$$

by putting $I_1 = R$ in Lemma 2.4. Considering the sequence $I_1+I_1 \subseteq \dots \subseteq I_1+I_n$ and applying Lemma 2.4 again, we get that $J_2 = ((I_1+I_1)y_1,\dots,(I_1+I_n)y_n,I)$. Then

$$\begin{aligned}
&\operatorname{depth}(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\
&\quad\geqslant \min\{\operatorname{depth}(R[y_1,\dots,y_n]/J_1),\operatorname{depth}(R[y_1,\dots,y_n]/J_2),\\
&\qquad \operatorname{depth}(R[y_1,\dots,y_n]/(J_1+J_2))+1\}\\
&\quad= \min\{\operatorname{depth}(R[y_1,\dots,y_n]/(y_1,I_2y_2,\dots,I_ny_n,I)),\\
&\qquad \operatorname{depth}(R[y_1,\dots,y_n]/((I_1+I_1)y_1,\dots,(I_1+I_n)y_n,I)),\\
&\qquad \operatorname{depth}(R[y_1,\dots,y_n]/(y_1,(I_1+I_2)y_2,\dots,(I_1+I_n)y_n,I))+1\}\\
&\quad= \min\{\operatorname{depth}(R[y_2,\dots,y_n]/(I_2y_2,\dots,I_ny_n,I\cap K[y_2,\dots,y_n])),\\
&\qquad \operatorname{depth}(R[y_1,\dots,y_n]/((I_1+I_1)y_1,\dots,(I_1+I_n)y_n,I)),\\
&\qquad \operatorname{depth}(R[y_2,\dots,y_n]/((I_1+I_2)y_2,\dots,(I_1+I_n)y_n,I\cap K[y_2,\dots,y_n]))+1\},
\end{aligned}$$

as required. □

Suppose that I is strongly stable. Let us simplify the formulas in Proposition 2.5 and Lemma 2.6.

By Lemma 2.2, we have

$$\begin{aligned}\dim(K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t])) &= t-r+1-(M(I\cap K[y_r,\dots,y_t])-r+1)\\ &= t-M(I\cap K[y_r,\dots,y_t]),\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t])) &= t-r+1-(m(I\cap K[y_r,\dots,y_t])-r+1)\\ &= t-m(I\cap K[y_r,\dots,y_t]).\end{aligned}$$

Corollary 2.8. *Let K be a field, $R=K[x_1,\dots,x_m]$ and $I_1\supseteq\dots\supseteq I_s\subseteq I_{s+1}\subseteq\dots\subseteq I_n$ be ideals of R. Then for any strongly stable monomial ideal I of $K[y_1,\dots,y_n]$,*

$$\begin{aligned}&\dim(R[y_1,\dots,y_n]/(I_1y_1,\dots,I_ny_n,I))\\ &\quad=\max_{s\leqslant t\leqslant n}\{\dim(R),\dim(R/I_t)+t-M(I\cap K[y_s,\dots,y_t])\}.\end{aligned}$$

P r o o f. For a fixed t with $s\leqslant t\leqslant n$, as $I_r+I_t\supseteq I_s+I_t=I_t$ and $M(I\cap K[y_r,\dots,y_t])\geqslant M(I\cap K[y_s,\dots,y_t])$ for all $1\leqslant r\leqslant s$, one has that

$$\dim(R/(I_r+I_t))\leqslant\dim(R/I_t),$$

and

$$\begin{aligned}\dim(K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t])) &= t-M(I\cap K[y_r,\dots,y_t])\\ &\leqslant t-M(I\cap K[y_s,\dots,y_t])\\ &= \dim(K[y_s,\dots,y_t]/(I\cap K[y_s,\dots,y_t])).\end{aligned}$$

Then for all $1\leqslant r\leqslant s$,

$$\begin{aligned}&\dim(R/(I_r+I_t))+\dim(K[y_r,\dots,y_t]/(I\cap K[y_r,\dots,y_t]))\\ &\quad\leqslant\dim(R/I_t)+\dim(K[y_s,\dots,y_t]/(I\cap K[y_s,\dots,y_t]))\\ &\quad=\dim(R/I_t)+t-M(I\cap K[y_s,\dots,y_t]).\end{aligned}$$

Hence, the result follows from Proposition 2.5. □

Corollary 2.9. *Let K be a field, $R = K[x_1, \ldots, x_m]$ and $I_1 \subseteq \ldots \subseteq I_n$ be ideals of R. Then for any strongly stable monomial ideal I of $K[y_1, \ldots, y_n]$,*

$$\begin{aligned}&\operatorname{depth}(R[y_1, \ldots, y_n]/(I_1y_1, \ldots, I_ny_n, I)) \\ &\quad \geqslant \min_{0 \leqslant r \leqslant n} \{\operatorname{depth}(R/I_r) + \operatorname{depth}(K[y_1, \ldots, y_r]/(I \cap K[y_1, \ldots, y_r]))\}.\end{aligned}$$

Proof. By Lemma 2.6, it is enough to show that

$$\begin{aligned}&\operatorname{depth}(K[y_1, \ldots, y_r]/I \cap K[y_1, \ldots, y_r]) \\ &\quad \leqslant \operatorname{depth}(K[y_1, \ldots, y_{r-1}]/(I \cap K[y_1, \ldots, y_{r-1}])) + 1.\end{aligned}$$

It is true because

$$\operatorname{depth}(K[y_1, \ldots, y_r]/I \cap K[y_1, \ldots, y_r]) = r - m(I \cap K[y_1, \ldots, y_r]),$$
$$\operatorname{depth}(K[y_1, \ldots, y_{r-1}]/I \cap K[y_1, \ldots, y_{r-1}]) = r - 1 - m(I \cap K[y_1, \ldots, y_{r-1}]),$$

and

$$m(I \cap K[y_1, \ldots, y_r]) \geqslant m(I \cap K[y_1, \ldots, y_{r-1}]).$$

□

3. Gröbner Basis

Let K be a field, $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $\mathfrak{M} = (x_1, \ldots, x_n)$ be the graded maximal ideal of S. Let

$$S^m \to S^n \to \mathfrak{M} \to 0$$

be a presentation of $\mathfrak{M}$ as an S-module and $e_1, \ldots, e_n$ be the canonical basis of S^n. Then $\operatorname{Syz}_1(\mathfrak{M})$ is generated by the $\binom{n}{2}$ syzygies $\{x_ie_j - x_je_i\colon 1 \leqslant i < j \leqslant n\}$. Now, consider the presentation of $\operatorname{Syz}_1(\mathfrak{M})$

$$S^a \to S^{\binom{n}{2}} \to \operatorname{Syz}_1(\mathfrak{M}) \to 0.$$

Let $\sigma_{ij} \mapsto x_ie_j - x_je_i$, $1 \leqslant i < j \leqslant n$, be the canonical basis of $S^{\binom{n}{2}}$. It is known (cf. [1]) that $\operatorname{Syz}_2(\mathfrak{M})$ is generated by the set of cyclic syzygies:

$$\{x_i\sigma_{jk} - x_j\sigma_{ik} + x_k\sigma_{ij}\colon 1 \leqslant i < j < k \leqslant n\}$$

and they are $\binom{n}{3}$. The symmetric algebra of $\operatorname{Syz}_1(\mathfrak{M})$ has the presentation:

$$\operatorname{Sym}_S(\operatorname{Syz}_1(\mathfrak{M})) = S[y_{ij}\colon 1 \leqslant i < j \leqslant n]/T,$$

where $y_{ij} \mapsto \sigma_{ij}$ and T is the relation ideal generated by the set

$$\{x_i y_{jk} - x_j y_{ik} + x_k y_{ij} \colon 1 \leqslant i < j < k \leqslant n\}.$$

Proposition 3.1. *One Gröbner basis of T with respect to a term order $<$ on $S[y_{ij} \colon 1 \leqslant i < j \leqslant n]$ induced by $x_n > x_{n-1} > \ldots > x_1 > y_{1n} > y_{1,n-1} > \ldots > y_{12} > y_{2n} > \ldots > y_{n-1,n}$ is the following:*

$$\begin{aligned}&\{x_i y_{jk} - x_j y_{ik} + x_k y_{ij} \colon 1 \leqslant i < j < k \leqslant n\}\\ &\qquad \cup \{x_r(y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk}) \colon 1 \leqslant i < j < k < l \leqslant n,\ 1 \leqslant r \leqslant n\}.\end{aligned}$$

Proof. See the proof of Lemma 3.1 of [6]. □

Now assume that $R = S/I$, where $I \subseteq \mathfrak{M}^2$ is a monomial ideal of S with $G(I) = \{u_1, \ldots, u_t\}$, i.e., R is a standard K-algebra with monomial relations. Set $m_i = \max(u_i)$ and $u_i' = u_i/x_{m_i}$, $i = 1, \ldots, t$. Let $\mathfrak{m}$ be the graded maximal ideal of R.

Notice that for any R-module N,

$$\mathrm{Sym}_R(N) = R \otimes_S \mathrm{Sym}_S(N) = \mathrm{Sym}_S(N)/I\mathrm{Sym}_S(N).$$

Lemma 3.2. *Suppose that I is strongly stable. Then*

$$\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m})) \cong S[y_{ij} \colon 1 \leqslant i < j \leqslant n]/J,$$

where

$$J = (u_1, \ldots, u_t; x_i y_{jk} - x_j y_{ik} + x_k y_{ij},\ i < j < k; u_i' y_{j,m_i},\ j < m_i, 1 \leqslant i \leqslant t).$$

Proof. Set $I^{\oplus n} = \oplus_{i=1}^n I$. From

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n} & \to & I^{\oplus n} & \to & I & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Syz}_1(\mathfrak{M}) & \to & S^n & \to & \mathfrak{M} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Syz}_1(\mathfrak{m}) & \to & R^n & \to & \mathfrak{m} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we have an exact sequence

$$0 \to \mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n} \to \mathrm{Syz}_1(\mathfrak{M}) \to \mathrm{Syz}_1(\mathfrak{m}) \to 0.$$

Then $\mathrm{Sym}_S(\mathrm{Syz}_1(\mathfrak{m})) \cong \mathrm{Sym}_S(\mathrm{Syz}_1(\mathfrak{M}))/(\widetilde{\mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n}})$. Hence,

$$\begin{aligned}\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m})) &= R \otimes_S \mathrm{Sym}_S(\mathrm{Syz}_1(\mathfrak{m})) = \mathrm{Sym}_S(\mathrm{Syz}_1(\mathfrak{m}))/I\mathrm{Sym}_S(\mathrm{Syz}_1(\mathfrak{m}))\\ &\cong \mathrm{Sym}_S(\mathrm{Syz}_1(\mathfrak{M}))/(I, \widetilde{\mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n}})\\ &= S[y_{ij}\colon\ 1 \leqslant i < j \leqslant n]\\ &\quad /(u_1, \dots, u_t; x_i y_{jk} - x_j y_{ik} + x_k y_{ij}, i < j < k; \widetilde{\mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n}}).\end{aligned}$$

Note that

$$\mathrm{Syz}_1(\mathfrak{M}) = (x_i e_j - x_j e_i\colon\ 1 \leqslant i < j \leqslant n)$$

and

$$a(x_i e_j - x_j e_i) \in I^{\oplus n} \Leftrightarrow a \in I\colon\ (x_i, x_j).$$

It follows that $(\widetilde{\mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n}})) = (((u_1, \dots, u_t) : (x_i, x_j))y_{ij}\colon\ i < j)$. Then $u_i' y_{j,m_i}$ belongs to this set for any $j < m_i$ and $1 \leqslant i \leqslant t$. Set

$$J = (u_1, \dots, u_t; x_i y_{jk} - x_j y_{ik} + x_k y_{ij},\ i < j < k; u_i' y_{j,m_i},\ j < m_i,\ 1 \leqslant i \leqslant t).$$

Let us show that $(\widetilde{\mathrm{Syz}_1(\mathfrak{M}) \cap I^{\oplus n}})) \subseteq J$. Then the lemma follows.

It is clear that

$$(u_1, \dots, u_t) : x_i = \Big(\frac{u_1}{[u_1, x_i]}, \dots, \frac{u_t}{[u_t, x_i]}\Big)$$

and

$$(u_1, \dots, u_t) : (x_i, x_j) = \Big(\frac{u_s u_k}{[u_s[u_k, x_j], u_k[u_s, x_i]]}\colon\ s, k = 1, \dots, t;\ 1 \leqslant i < j \leqslant n\Big).$$

Then it is enough to show that $(u_s u_k/[u_s[u_k, x_j], u_k[u_s, x_i]])y_{ij} \in J$. Notice that if $x_i \nmid u_s$ or $x_j \nmid u_k$, then $u_s u_k/[u_s[u_k, x_j], u_k[u_s, x_i]]$ is divided by u_s or u_k, which implies that $(u_s u_k/[u_s[u_k, x_j], u_k[u_s, x_i]])y_{ij} \in (u_1, \dots, u_t)$. Hence, we may assume that $x_i \mid u_s$ and $x_j \mid u_k$.

Since $(u_s u_k/[u_s x_j, u_k x_i])y_{ij}$ is divided by $(u_k/x_j)y_{ij}$, it is enough to show that $(u_k/x_j)y_{ij} \in J$ for any $i < j$. If $j = m_k$, the result is clear. Now assume that $j < m_k$. Then one has

$$\frac{u_k}{x_j}y_{ij} = \frac{u_k}{x_{m_k}x_j}(x_{m_k}y_{ij} - x_j y_{i,m_k} + x_i y_{j,m_k}) + \frac{u_k}{x_{m_k}}y_{i,m_k} - \frac{u_k x_i}{x_j x_{m_k}}y_{j,m_k}.$$

By the strong stability of I, we have that $u_k x_i/x_j \in I$. But $\max(u_k x_i/x_j) = m_k$, which implies that $(u_k x_i/x_j x_{m_k})y_{j,m_k} \in J$. The result follows. □

Remark 3.3. Notice that from the above proof, $(u/x_j)y_{ij} \in J$ for any $u \in G(I)$ with $x_j \mid u$. Furthermore, if $x_{j_0} \mid u$ and $i < j \leqslant j_0$, then $(u/x_{j_0})y_{ij} \in J$ also holds because

$$\frac{u}{x_{j_0}}y_{ij} = \frac{ux_j/x_{j_0}}{x_j}y_{ij}$$

with $ux_j/x_{j_0} \in I$.

From now on, we will fix a term order $<$ on $S[y_{ij}\colon\ 1 \leqslant i < j \leqslant n]$ induced by

$$x_n > x_{n-1} > \ldots > x_1 > y_{1n} > y_{1,n-1} > \ldots > y_{12} > y_{2n} > \ldots > y_{n-1,n}.$$

The main result of this section is the following theorem.

Theorem 3.4. *Suppose that I is strongly stable. Then*

$$\Big\{G(I); \frac{u}{x_i}y_{jk}, u \in G(I), x_i \mid u, j < k \leqslant i; x_iy_{jk} - x_jy_{ik} + x_ky_{ij}, i < j < k;$$
$$x_s(y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk}), i < j < k < l, 1 \leqslant s \leqslant n\Big\}$$

is a Gröbner basis of J with respect to the above term order.

Proof. Firstly, notice that to show that one set is a Gröbner basis, it is sufficient to prove that for any two elements α and β of this set, the S-pair

$$S(\alpha, \beta) := \frac{\text{in}(\alpha)}{[\text{in}(\alpha), \text{in}(\beta)]}\beta - \frac{\text{in}(\beta)}{[\text{in}(\alpha), \text{in}(\beta)]}\alpha$$

has a standard expression with zero remainder with respect to the above term order. We may assume that $[\text{in}(\alpha), \text{in}(\beta)] \neq 1$. We will use the following property: If $u, v \in S$ are monomials and $f, g \in K[y_{ij}\colon\ 1 \leqslant i < j \leqslant n]$, then $S(uf, vg) = (uv/[u, v])S(f, g)$.

Denote the above four groups in the set of the theorem by (I)–(IV), respectively.

Since (I) and (II) are monomials and (III)$\cup$(IV) is a Gröbner basis by Proposition 3.1, it is enough to consider the following cases:

(a) $\alpha \in$ (I) and $\beta \in$ (III). Let $u \in G(I)$ with $x_k \mid u$. Then $(u/x_k)x_i, (u/x_k)x_j \in I$ for $i < j < k$ by the strong stability of I. Hence,

$$S(u, x_iy_{jk} - x_jy_{ik} + x_ky_{ij}) = \frac{u}{x_k}x_iy_{jk} - \frac{u}{x_k}x_jy_{ik} \in (G(I)).$$

(b) $\alpha \in$ (I) and $\beta \in$ (IV). Let $u \in G(I)$ with $x_s \mid u$. Then

$$S(u, x_s(y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk})) = u(y_{ij}y_{kl} - y_{ik}y_{jl}) \in (G(I)).$$

(c) $\alpha \in$ (II) and $\beta \in$ (III). For the S-pair $S((u^*/x_{i'})y_{j'k'}, x_iy_{jk} - x_jy_{ik} + x_ky_{ij})$, where $j' < k' \leqslant i'$ and $i < j < k$, there are three possibilities:

(c_1) $(j', k') \neq (i, j)$ and $x_k \mid u^*/x_{i'}$;

(c_2) $(j', k') = (i, j)$ and $x_k \mid u^*/x_{i'}$;

(c_3) $(j', k') = (i, j)$ and $x_k \nmid u^*/x_{i'}$.

In (c_1), the S-pair is $(u_1^*/x_{i'})y_{j'k'}y_{jk} - (u_2^*/x_{i'})y_{j'k'}y_{ik}$, where $u_1^* = (u^*/x_k)x_i$ and $u_2^* = (u^*/x_k)x_j$ are all in I, hence, $(u_1^*/x_{i'})y_{j'k'}$ and $(u_2^*/x_{i'})y_{j'k'}$ belong to (II). Similarly in (c_2), the S-pair is $(u_1^*/x_k)y_{jk} - (u_2^*/x_k)y_{ik}$, where $u_1^* = (u^*/x_{i'})x_{j'}$ and $u_2^* = (u^*/x_{i'})x_{k'}$ are all in I. In (c_3), the S-pair becomes $u_1^*y_{jk} - u_2^*y_{ik}$, where u_1^* and u_2^* are as in (c_2). Then the S-pair belongs to $(G(I))$. Therefore, the S-pair has a standard expression with zero remainder in any possibilities.

(d) $\alpha \in$ (II) and $\beta \in$ (IV). We note that

$$S\Big(\frac{u}{x_{i'}}y_{j'k'}, x_s(y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk})\Big) = \frac{x_s}{[u/x_{i'}, x_s]}\frac{u}{x_{i'}}S(y_{j'k'}, y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk}),$$

which is divided by $(u/x_{i'})y_{j'k'}$ if $y_{j'k'}$ is coprime with $y_{il}y_{jk}$, and divided by $(u/x_{i'})y_{kk'}y_{j'j} - (u/x_{i'})y_{jk'}y_{j'k}$ or $(u/x_{i'})y_{ij'}y_{k'l} - (u/x_{i'})y_{ik'}y_{j'l}$ if $(j', k') = (i, l)$ or (j, k). Since $(u/x_{i'})y_{kk'}, (u/x_{i'})y_{jk'}, (u/x_{i'})y_{ij'}$ and $(u/x_{i'})y_{ik'}$ are all in (II), the S-pair has a standard expression with zero remainder in any cases.

Then the result follows. $\square$

Using this Gröbner basis, we get immediately the following corollary.

Corollary 3.5. *Suppose that I is strongly stable. Then*

$$\mathrm{in}(J) = \Big(G(I), \Big\{\frac{u}{x_i}y_{jk} \colon u \in G(I),\ x_i \mid u,\ j < k \leqslant i\Big\},$$
$$\{x_ky_{ij} \colon i < j < k\}, \{x_sy_{il}y_{jk} \colon i < j < k < l,\ 1 \leqslant s \leqslant n\}\Big).$$

4. Dimension and depth

Suppose that I is strongly stable and its minimal generators $u_1, \ldots, u_t$ are all of degree 2. Then by Corollary 3.5, we have

$$\mathrm{in}(J) = (u_1, \ldots, u_t, I_1x_1, \ldots, I_nx_n),$$

where I_r, $r = 1, \ldots, n$, are ideals of $Q := K[y_{ij} \colon 1 \leqslant i < j \leqslant n]$. Let us identify these ideals I_r and then calculate the dimension and depth of the symmetric algebra $\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))$.

Set $I_{\geqslant r} = I \cap K[x_r, \ldots, x_n]$. Put $m(0) = M(0) = 0$. From Corollary 3.5, we see that the generating set of I_r consists of three parts A, B and C given by

$$\begin{aligned} Ax_r &= Qx_r \cap \{x_k y_{ij}\colon\ i < j < k\}, \\ Bx_r &= Qx_r \cap \{x_s y_{il} y_{jk}\colon\ i < j < k < l,\ 1 \leqslant s \leqslant n\}, \\ Cx_r &= Qx_r \cap \Big\{\frac{u}{x_i} y_{jk}\colon\ u \in G(I),\ x_i \mid u,\ j < k \leqslant i\Big\}. \end{aligned}$$

It is clear that

$$\begin{aligned} A &= \{y_{ij}\colon\ i < j < r\}, \\ B &= \{y_{il} y_{jk}\colon\ i < j < k < l\}, \\ C &= \{y_{jk}\colon\ x_i x_r \in G(I),\ j < k \leqslant i\}. \end{aligned}$$

Since $y_{ij} \in A$ for $i < j < r$, we may assume that $i \geqslant r$ in C. Furthermore, notice that by the strong stability of I, if $x_i x_l \in I$ with $l > r$, then $x_i x_r, x_r x_l \in I$. It follows that the maximal i in C is just $M(I_{\geqslant r})$. Hence, $C = \{y_{ij}\colon\ i < j \leqslant M(I_{\geqslant r})\}$. Then

$$I_r = (y_{ij}\colon\ i < j < \max\{r, M(I_{\geqslant r}) + 1\};\ y_{il} y_{jk}\colon\ i < j < k < l).$$

Notice that $I = I_{\geqslant 1} \supseteq I_{\geqslant 2} \supseteq \ldots \supseteq I_{\geqslant n}$, which implies that $M(I) = M(I_{\geqslant 1}) \geqslant M(I_{\geqslant 2}) \geqslant \ldots \geqslant M(I_{\geqslant n})$ and if $I_{\geqslant r} \neq 0$, then $M(I_{\geqslant r}) \geqslant r$, so $\max\{r, M(I_{\geqslant r}) + 1\} = M(I_{\geqslant r}) + 1$. On the other hand, it is easy to see that $\max\{r\colon\ I_{\geqslant r} \neq 0\} = m(I)$. Then we have the following conclusions:

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_{m(I)} \subseteq I_{m(I)+1} \subseteq \ldots \subseteq I_n.$$

Lemma 4.1. *Q/I_r is Cohen-Macaulay with*

$$\dim(Q/I_r) = \begin{cases} 2n - 2 - M(I_{\geqslant r}), & r = 1, \ldots, m(I), \\ 2n - 1 - r, & r = m(I) + 1, \ldots, n. \end{cases}$$

Proof. Set $r^* = \max\{r, M(I_{\geqslant r}) + 1\}$ and $Q_r = K[y_{ij}\colon\ 1 \leqslant i < j \leqslant n, j \geqslant r^*]$. Then $Q/I_r = Q_r/I'_r$, where $I'_r = (y_{il} y_{jk}\colon\ i < j < k < l, j \geqslant r^*)$. Denote

$$Y_{r^*} = \begin{pmatrix} y_{1r^*} & y_{1,r^*+1} & \cdots & y_{1n} \\ & \cdots & \cdots & \\ y_{r^*-1,r^*} & y_{r^*-1,r^*+1} & \cdots & y_{r^*-1,n} \\ & y_{r^*,r^*+1} & \cdots & y_{r^*,n} \\ & & \cdots & \\ & & & y_{n-1,n} \end{pmatrix}.$$

Then $I'_r = (\mathrm{in}(m)\colon\ m$ is a 2-minor of $Y_{r^*})$.

As shown in the proof of Proposition 3.4 of [7], Q_r/I'_r is Cohen-Macaulay of dimension $2n - 1 - r^*$. Furthermore, if $I_{\geqslant r} \neq 0$, then $r^* = M(I_{\geqslant r}) + 1$ and if $I_{\geqslant r} = 0$, then $r^* = r$. Then the lemma follows. □

Now we can prove the main theorem.

Theorem 4.2. *Let $R = K[x_1,\ldots,x_n]/I$, $n \geqslant 3$, be a standard K-algebra with a strongly stable monomial relation ideal $I \subseteq (x_1,\ldots,x_n)^2$ whose generators are all of degree two, and $\mathfrak{m}$ be the graded maximal ideal of R. Then*

$$\dim(\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))) = \max\bigl\{\tfrac{1}{2}n(n-1), 2n-1-M(I\cap K[x_{m(I)}, x_{m(I)+1}])\bigr\}$$

and

$$\operatorname{depth}(\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))) \geqslant 2n-1-M(I)-m(I).$$

P r o o f. We keep the notations as before. Then

$$\begin{aligned}\dim(\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))) &= \dim(S[y_{ij}\colon\ 1\leqslant i<j\leqslant n]/J)\\ &= \dim(S[y_{ij}\colon\ 1\leqslant i<j\leqslant n]/\mathrm{in}(J))\\ &= \dim(Q[x_1,\ldots,x_n]/(I_1x_1.\ldots,I_nx_n,I)).\end{aligned}$$

It follows from Corollary 2.8 that

$$\begin{aligned}&\dim(\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m})))\\ &\quad= \max_{m(I)\leqslant t\leqslant n}\{\dim(Q), \dim(Q/I_t)+t-M(I\cap K[x_{m(I)},\ldots,x_t])\}\\ &\quad= \max_{m(I)\leqslant t\leqslant n}\bigl\{\tfrac{1}{2}n(n-1)\dim(Q/I_t)+t-M(I\cap K[x_{m(I)},\ldots,x_t])\bigr\}.\end{aligned}$$

By Lemma 4.1, $\dim(Q/I_{m(I)}) = 2n-2-M(I_{\geqslant m(I)})$ and $\dim(Q/I_t) = 2n-1-t$ for $t > m(I)$. Notice that $M(I\cap K[x_{m(I)}]) = m(I)$ and $M(I\cap K[x_{m(I)},\ldots,x_t]) \geqslant M(I\cap K[x_{m(I)}, x_{m(I)+1}])$ for all $m(I) < t \leqslant n$. Then

$$\begin{aligned}\dim(\mathrm{Sym}_R(\mathrm{Syz}_1(\mathfrak{m}))) = \max\bigl\{&\tfrac{1}{2}n(n-1), 2n-2-M(I_{\geqslant m(I)}),\\ &2n-1-M(I\cap K[x_{m(I)}, x_{m(I)+1}])\bigr\}.\end{aligned}$$

It is easy to see that if $M(I_{\geqslant m(I)}) = m(I)$, then $M(I\cap K[x_{m(I)}, x_{m(I)+1}]) = m(I)$, and if $M(I_{\geqslant m(I)}) > m(I)$, then $M(I\cap K[x_{m(I)}, x_{m(I)+1}]) = m(I)+1$. Thus, in any case,

$$\begin{aligned}\max\{2n-2-M(I_{\geqslant m(I)}), 2n-1-M(I\cap K[x_{m(I)}, x_{m(I)+1}])\}&\\ = 2n-1-M(I\cap K[x_{m(I)}, x_{m(I)+1}]).&\end{aligned}$$

Then the equality for the dimension follows.

For the depth, by Proposition 2.7, we have

$$\begin{aligned}
&\operatorname{depth}(\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m})))\\
&\quad = \operatorname{depth}(S[y_{ij}\colon 1 \leqslant i < j \leqslant n]/J) \geqslant \operatorname{depth}(S[y_{ij}\colon 1 \leqslant i < j \leqslant n]/\operatorname{in}(J))\\
&\quad = \operatorname{depth}(Q[x_1, \dots, x_n]/(I_1x_1, \dots, I_nx_n, I))\\
&\quad \geqslant \min_{1\leqslant r\leqslant m(I)-1}\{\operatorname{depth}(Q[x_{m(I)}, \dots, x_n]\\
&\qquad\qquad /(I_{m(I)}x_{m(I)}, \dots, I_nx_n, I \cap K[x_{m(I)}, \dots, x_n])),\\
&\qquad \operatorname{depth}(Q[x_r, \dots, x_n]/((I_r + I_r)x_r, \dots, (I_r + I_n)x_n, I \cap K[x_r, \dots, x_n])),\\
&\qquad \operatorname{depth}(Q[x_{r+1}, \dots, x_n]\\
&\qquad\qquad /((I_r + I_{r+1})x_{r+1}, \dots, (I_r + I_n)x_n, I \cap K[x_{r+1}, \dots, x_n])) + 1\}.
\end{aligned}$$

By Corollary 2.9 and Lemma 4.1, one has

$$\begin{aligned}
&\operatorname{depth}(Q[x_{m(I)}, \dots, x_n]/(I_{m(I)}x_{m(I)}, \dots, I_nx_n, I \cap K[x_{m(I)}, \dots, x_n]))\\
&\quad \geqslant \min_{m(I)\leqslant t\leqslant n}\{\operatorname{depth}(Q),\\
&\qquad \operatorname{depth}(Q/I_t) + \operatorname{depth}(K[x_{m(I)}, \dots, x_t]/(I \cap K[x_{m(I)}, \dots, x_t]))\}\\
&\quad = \min_{m(I)\leqslant t\leqslant n}\{\operatorname{depth}(Q), \operatorname{depth}(Q/I_t) + t - m(I \cap K[x_{m(I)}, \dots, x_t])\}\\
&\quad = \min_{m(I)+1\leqslant t\leqslant n}\bigl\{\tfrac{1}{2}n(n-1), 2n-2-M(I_{\geqslant m(I)}), 2n-1-m(I \cap K[x_{m(I)}, \dots, x_t])\bigr\}\\
&\quad = \min\bigl\{\tfrac{1}{2}n(n-1), 2n-2-M(I_{\geqslant m(I)}), 2n-1-m(I \cap K[x_{m(I)}, \dots, x_n])\bigr\}\\
&\quad = \min\bigl\{\tfrac{1}{2}n(n-1), 2n-2-M(I_{\geqslant m(I)}), 2n-1-m(I_{\geqslant m(I)})\bigr\}\\
&\quad = \min\bigl\{\tfrac{1}{2}n(n-1), 2n-2-M(I_{\geqslant m(I)})\bigr\}\\
&\quad \geqslant \min\bigl\{\tfrac{1}{2}n(n-1), 2n-2-M(I)\bigr\},
\end{aligned}$$

$$\begin{aligned}
&\operatorname{depth}(Q[x_r, \dots, x_n]/((I_r + I_r)x_r, \dots, (I_r + I_n)x_n, I \cap K[x_r, \dots, x_n]))\\
&\quad \geqslant \min_{r\leqslant t\leqslant n}\{\operatorname{depth}(Q),\\
&\qquad \operatorname{depth}(Q/(I_r + I_t)) + \operatorname{depth}(K[x_r, \dots, x_t]/(I \cap K[x_r, \dots, x_t]))\}\\
&\quad = \min_{r\leqslant t\leqslant n}\{\operatorname{depth}(Q), \operatorname{depth}(Q/(I_r + I_t)) + t - m(I \cap K[x_r, \dots, x_t])\}\\
&\quad = \min\Bigl\{\operatorname{depth}(Q), \min_{r\leqslant t\leqslant m(I)}\{\operatorname{depth}(Q/I_r) + t - m(I \cap K[x_r, \dots, x_t])\},\\
&\qquad \min_{m(I)+1\leqslant t\leqslant n}\{\operatorname{depth}(Q/I_{\max\{M(I_{\geqslant r})+1,t\}}) + t - m(I \cap K[x_r, \dots, x_t])\}\Bigr\}\\
&\quad = \min\Bigl\{\operatorname{depth}(Q), \min_{r\leqslant t\leqslant m(I)}\{2n-2-M(I_{\geqslant r}) + t - m(I \cap K[x_r, \dots, x_t])\},\\
&\qquad \min_{m(I)+1\leqslant t\leqslant n}\{2n-1-\max\{M(I_{\geqslant r})+1, t\} + t - m(I \cap K[x_r, \dots, x_t])\}\Bigr\}\\
&\quad \geqslant \min\bigl\{\tfrac{1}{2}n(n-1), 2n-2-M(I_{\geqslant r}) + r - m(I_{\geqslant r}), 2n-1-M(I_{\geqslant r})\bigr\},
\end{aligned}$$

where $t-\max\{M(I_{\geqslant r})+1,t\}\geqslant m(I)+1-(M(I_{\geqslant r})+1)$ for $t=m(I)+1,\dots,n$, is used, and similarly,

$$\begin{aligned}
&\operatorname{depth}(Q[x_{r+1},\dots,x_n]/((I_r+I_{r+1})x_{r+1},\dots,(I_r+I_n)x_n,I\cap K[x_{r+1},\dots,x_n]))\\
&\quad\geqslant \min_{r+1\leqslant t\leqslant n}\{\operatorname{depth}(Q),\operatorname{depth}(Q/(I_r+I_t))+t-m(I\cap K[x_{r+1},\dots,x_t]))\}\\
&\quad= \min\Bigl\{\operatorname{depth}(Q),\min_{r+1\leqslant t\leqslant m(I)}\{2n-2-M(I_{\geqslant r})+t-m(I\cap K[x_{r+1},\dots,x_t])\},\\
&\qquad\min_{m(I)+1\leqslant t\leqslant n}\{2n-1-\max\{M(I_{\geqslant r})+1,t\}+t-m(I\cap K[x_{r+1},\dots,x_t])\}\Bigr\}\\
&\quad\geqslant \min\bigl\{\tfrac12 n(n-1),2n-2-M(I_{\geqslant r})+r+1-m(I_{\geqslant r+1}),2n-1-M(I_{\geqslant r})\bigr\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\operatorname{depth}(\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m})))\\
&\quad\geqslant \min_{1\leqslant r\leqslant m(I)-1}\bigl\{\tfrac12 n(n-1),2n-2-M(I),2n-2-M(I_{\geqslant r})+r-m(I_{\geqslant r}),\\
&\qquad 2n-1-M(I_{\geqslant r}),2n-M(I_{\geqslant r})+r-m(I_{\geqslant r+1}),2n-M(I_{\geqslant r})\bigr\}\\
&\quad= \min\bigl\{\tfrac12 n(n-1),2n-1-M(I)-m(I),2n-2-M(I)\bigr\}\\
&\quad= 2n-1-M(I)-m(I).
\end{aligned}$$

$\square$

Remark 4.3. As $\frac12 n(n-1)\geqslant 2n-2$ for $n\geqslant 4$, it follows that

$$\dim(\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m})))=\tfrac12 n(n-1)$$

for $n\geqslant 4$. Suppose that $M(I)=1$, i.e. $I=(x_1^2)$. Then

$$\operatorname{depth}(\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m})))\geqslant 2n-3.$$

When $n=3$, by Lemma 3.2,

$$\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m}))=K[x_1,x_2,x_3,y_{12},y_{13},y_{23}]/(x_1^2,x_1y_{23}-x_2y_{13}+x_3y_{12}).$$

It is easy to see that $x_1^2, x_1y_{23}-x_2y_{13}+x_3y_{12}$ is a regular sequence. Then $\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m}))$ is Cohen-Macaulay of dimension 4.

Assume that $n\geqslant 4$. Since $\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m}))=\operatorname{Sym}_S(\operatorname{Syz}_1(\mathfrak{M}))/(x_1^2)$, x_1^2 is a regular element in $\operatorname{Sym}_S(\operatorname{Syz}_1(\mathfrak{M}))$, and $\operatorname{Sym}_S(\operatorname{Syz}_1(\mathfrak{M}))$ has depth $2n-2$ by [7], Theorem 4.1, it follows that $\operatorname{depth}(\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m})))=2n-3$. Hence, the lower bound for depth in Theorem 4.2 is reached. Notice that the dimension and depth are different in this case, hence, $\operatorname{Sym}_R(\operatorname{Syz}_1(\mathfrak{m}))$ is not Cohen-Macaulay.

Acknowledgments. All authors would also like to express their sincere thanks to the referees for their valuable comments.

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