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ON BERNSTEIN INEQUALITIES FOR MULTIVARIATE TRIGONOMETRIC POLYNOMIALS IN L_p, $0 \leqslant p \leqslant \infty$

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Abstract. Let $\mathbb{T}_n$ be the space of all trigonometric polynomials of degree not greater than n with complex coefficients. Arestov extended the result of Bernstein and others and proved that $\|(1/n)T_n'\|_p \leqslant \|T_n\|_p$ for $0 \leqslant p \leqslant \infty$ and $T_n \in \mathbb{T}_n$. We derive the multivariate version of the result of Golitschek and Lorentz

$$\Big\|\Big|T_n \cos\alpha + \frac{1}{n}\nabla T_n \sin\alpha\Big|_{l_\infty^{(m)}}\Big\|_p \leqslant \|T_n\|_p, \quad 0 \leqslant p \leqslant \infty$$

for all trigonometric polynomials (with complex coeffcients) in m variables of degree at most n.

Keywords: univariate trigonometric polynomial; multivariate trigonometric polynomial; multivariate algebraic polynomial; Bernstein inequality; L_p-norm

MSC 2020: 41A10, 41A17

1. Introduction

Let $\mathcal{T}_n(\mathbb{R}^m)$ denote the space of all trigonometric polynomials of $(\theta_1, \ldots, \theta_m) \in \mathbb{R}^m$ with degree not greater than n and complex coefficients, and let $\Pi_n(\mathbb{C}^m)$ denote the space of all algebraic polynomials of $(z_1, \ldots, z_m) \in \mathbb{C}^m$ with degree not greater than n and complex coefficients.

The norm in the space $\mathbb{C}^m$ of $z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ is defined by

$$|z|_{l_\infty^{(m)}} = \max_{1\leqslant k\leqslant m} \{|z_k|\}, \tag{1.1}$$

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and the norm (or quasi-norm) in the space $L_p(T^m)$ of a function $f(\theta_1,\theta_2,\ldots,\theta_m)$ is defined by

$$\|f\|_p=\left[\frac{1}{(2\pi)^m}\int_{T^m}|f(\theta_1,\ldots,\theta_m)|^p\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_m\right]^{1/p},\quad 0<p<\infty, \tag{1.2}$$

where $T^m=\overbrace{T\times T\times\ldots\times T}^{m}$, $T=[a,b]$ with $b-a=2\pi$. For the limiting cases, the supremum norm $\|f\|_\infty$ is defined as usual by

$$\|f\|_\infty=\operatorname*{esssup}_{(\theta_1,\ldots,\theta_m)\in\mathbb{R}^{(m)}}|f(\theta_1,\ldots,\theta_m)|,\quad f\in L_\infty(T^m), \tag{1.3}$$

and the quasi-norm $\|f\|_0$ is defined by

$$\|f\|_0=\exp\left[\frac{1}{(2\pi)^m}\int_{T^m}\ln|f(\theta_1,\ldots,\theta_m)|\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_m\right],\quad f\in L_0(T^m). \tag{1.4}$$

In the case $m=1$, Golitschek and Lorentz in [3] established, for each real α and trigonometric polynomial $T_n\in\mathcal{T}(\mathbb{R})$, the inequality

$$\left\|T_n\cos\alpha+\frac{1}{n}T_n'\sin\alpha\right\|_p\leqslant\|T_n\|_p,\quad 0\leqslant p\leqslant\infty. \tag{1.5}$$

For $p=\infty$ and $\alpha=\frac{1}{2}\pi$, the relation (1.5) is called the *Bernstein inequality*; for $1\leqslant p<\infty$ and $\alpha=\frac{1}{2}\pi$, it has been established by Zygmund (see [6]); for $0<p<1$ and $\alpha=\frac{1}{2}\pi$, it has been proved by Arestov (see [1]); and for $p=\infty$, the special case for real polynomials T_n is the inequality of Szegö-van der Corput-Schaake (see [4]),

$$n^2T_n(\theta)^2+T_n'(\theta)^2\leqslant n^2\|T_n\|_\infty^2,\quad \theta\in T. \tag{1.6}$$

In the present paper, using the method of [3], we extend (1.5) to the multivariate $T_n\in\mathcal{T}_n(\mathbb{R}^m)$ and obtain the inequality

$$\left\|\left|T_n\cos\alpha+\frac{1}{n}\nabla T_n\sin\alpha\right|_{l_\infty^{(m)}}\right\|_p\leqslant|C|\|T_n\|_p,\quad 0\leqslant p\leqslant\infty. \tag{1.7}$$

Our proof yields $C=1$, which is the best possible. Here ∇T_n is the gradient of T_n,

$$\nabla T_n=\left(\frac{\partial T_n}{\partial\theta_1},\frac{\partial T_n}{\partial\theta_2},\ldots,\frac{\partial T_n}{\partial\theta_m}\right), \tag{1.8}$$

and

$$\begin{aligned} T_n\cos\alpha+\frac{1}{n}\nabla T_n\sin\alpha &\\ =\Big(T_n\cos\alpha+\frac{\sin\alpha}{n}\frac{\partial T_n}{\partial\theta_1},T_n\cos\alpha&+\frac{\sin\alpha}{n}\frac{\partial T_n}{\partial\theta_2},\ldots,T_n\cos\alpha+\frac{\sin\alpha}{n}\frac{\partial T_n}{\partial\theta_m}\Big).\end{aligned} \tag{1.9}$$

2. Inequalities in L_0

Let A, B be two real numbers with $B \neq 0$. We consider the operator Λ on $\mathcal{T}_n(\mathbb{R}^m)$ defined by

$$\mathcal{S}_n = \Lambda(T_n) = AT_n + \frac{B}{n}\nabla T_n, \quad T_n \in \mathcal{T}_n(\mathbb{R}^m). \tag{2.1}$$

For any trigonometric polynomial $T_n(\theta_1, \theta_2, \dots, \theta_m) \in \mathcal{T}(\mathbb{R}^m)$, we can write

$$T_n(\theta_1, \theta_2, \dots, \theta_m) = \sum_{0 \leqslant |j_1|+|j_2|+\ldots+|j_m| \leqslant n} c_{j_1 j_2 \dots j_m} \mathrm{e}^{\mathrm{i}(j_1\theta_1 + j_2\theta_2 + \ldots + j_m\theta_m)}, \tag{2.2}$$

and with T_n we associate an algebraic polynomial of degree not greater than $2mn$,

$$P_{2mn}(z_1, z_2, \dots, z_m) = \sum_{0 \leqslant |j_1|+|j_2|+\ldots+|j_m| \leqslant n} c_{j_1 j_2 \dots j_m} z_1^{n+j_1} z_2^{n+j_2} \dots z_m^{n+j_m}. \tag{2.3}$$

So

$$P_{2mn}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \dots, \mathrm{e}^{\mathrm{i}\theta_n}) = \mathrm{e}^{\mathrm{i}n(\theta_1+\theta_2+\ldots+\theta_m)} T_n(\theta_1, \theta_2, \dots, \theta_m). \tag{2.4}$$

and

$$\frac{\partial T_n}{\partial \theta_k} = -\mathrm{i}n\mathrm{e}^{-\mathrm{i}n(\theta_1+\theta_2+\ldots+\theta_m)} \Big[P_{2mn}(z) - \frac{z_k}{n}\frac{\partial P_{2mn}}{\partial z_k}\Big], \tag{2.5}$$

$$\frac{\Lambda(T_n)}{T_n} = \Big(A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\frac{z_1}{P_{2mn}}\frac{\partial P_{2mn}}{\partial z_1}, A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\frac{z_2}{P_{2mn}}\frac{\partial P_{2mn}}{\partial z_2}, \dots, A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\frac{z_m}{P_{2mn}}\frac{\partial P_{2mn}}{\partial z_m}\Big) \tag{2.6}$$

where

$$z = (z_1, z_2, \dots, z_m), \quad z_k = \mathrm{e}^{\mathrm{i}\theta_k}, \quad k = 1, 2, \dots, m.$$

For a fixed k, $1 \leqslant k \leqslant m$, we can write

$$P_{2mn}(z) = \sum_{j_k=-n}^{n} c_{j_k}(z'_k) z_k^{j_k+n}, \tag{2.7}$$

where $z'_k = (z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_m) \in \mathbb{C}^{m-1}$, $c_{j_k}(z'_k) \in \Pi_{2mn-n-j_k}(\mathbb{C}^{m-1})$. Then $P_{2mn}(z)$ is an algebraic polynomial of degree not greater than $2n$ in z_k, and let $\alpha_j^{(k)}(z'_k)$, $j = 1, 2, \dots, 2n$ be its zeros. It follows from (2.6) that

$$\frac{\Lambda(T_n)}{T_n} = \Bigg(A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_1}}{\mathrm{e}^{\mathrm{i}\theta_1} - \alpha_j^{(1)}(z'_1)}, \dots, A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_n}}{\mathrm{e}^{\mathrm{i}\theta_n} - \alpha_j^{(n)}(z'_n)}\Bigg), \tag{2.8}$$

where $z'_k = (\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_{k-1}}, \mathrm{e}^{\mathrm{i}\theta_{k+1}}, \dots, \mathrm{e}^{\mathrm{i}\theta_m})$, $k = 1, 2, \dots, m$.

Lemma 2.1. *Let A, B be two real numbers. For a given $\theta_k \in \mathbb{R}$,*

$$\max_{1\leqslant k\leqslant m} \ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}\right|, \tag{2.9}$$

is subharmonic with respect to each z_j in the regions $|z_j| < 1$ and $|z_j| > 1$, $j = \ldots 1, 2, \ldots, m$.

Futhermore,

$$F(z_1, z_2, \ldots, z_m) = \frac{1}{(2\pi)^m}\int_{T^m}\max_{1\leqslant k\leqslant m}\ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}\right|\,\mathrm{d}\theta_1\ldots\,\mathrm{d}\theta_m \tag{2.10}$$

is subharmonic with respect to each z_j in the regions $|z_j| < 1$ and $|z_j| > 1$.

Proof. For a given $\theta_k \in \mathbb{R}$,

$$f_k(z_1, z_2, \ldots, z_m) = A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}$$

is analytic with respect to each z_j in the regions $|z_j| < 1$ and $|z_j| > 1$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, m$. Thus $\ln|f_k(z_1, z_2, \ldots, z_m)|$ is subharmonic with respect to each z_j in the regions, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, m$.

By definition [2] it is easy to see that if $f_k(w)$, $k = 1, 2, \ldots, m$ are subharmonic in some region of the w-plane, then $\max\limits_{1\leqslant k\leqslant m}\{f_k(w)\}$ is subharmonic in the region.

These facts imply that the function in (2.9) and its integrals with respect to parameters (for positive measures) are subharmonic with respect to each z_j in the regions $|z_j| < 1$ and $|z_j| > 1$, $j = 1, 2, \ldots, m$. The proof is complete. □

Lemma 2.2. *Let A, B be two real numbers, and n be any positive integer. Then*

$$\begin{aligned}\frac{1}{(2\pi)^m}\int_{T^m}\ln|\Lambda(T_n)|_{l_\infty^{(m)}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\,\mathrm{d}\theta_m \\ \leqslant m\ln|A - \mathrm{i}B| + \frac{1}{(2\pi)^m}\int_{T^m}\ln|T_n|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\,\mathrm{d}\theta_m\end{aligned} \tag{2.11}$$

for every $T_n \in \mathcal{T}(\mathbb{R}^m)$.

Proof. Since $\ln x$ is an increasing function for $x > 0$,

$$\begin{aligned}\ln\max_{1\leqslant k\leqslant m}\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}\right| &= \max_{1\leqslant k\leqslant m}\ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}\right| \\ &\leqslant \sum_{k=1}^{m}\ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}\right|\end{aligned} \tag{2.12}$$

holds for $|z_j| \neq 1$. By [3], Theorem 2,

$$\frac{1}{2\pi}\int_T \ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - z_j}\right| \mathrm{d}\theta_k \leqslant \ln|A - \mathrm{i}B|, \quad k = 1, 2, \ldots, m \tag{2.13}$$

holds for any $(z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$.

Therefore,

$$I_k(z_k') = \frac{1}{2\pi}\int_T \ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - \alpha_j^{(k)}(z_k')}\right| \mathrm{d}\theta_k \leqslant \ln|A - \mathrm{i}B| \tag{2.14}$$

holds for any $z_k' = (z_1, z_2, \ldots, z_{k-1}, z_{k+1}, \ldots, z_m) \in \mathbb{C}^m$, $k = 1, 2, \ldots, m$.

By (2.8), Lemma 2.1, (2.12) and Fubini's theorem, (2.14) implies

$$\begin{aligned}
&\frac{1}{(2\pi)^m}\int_{T^m} \ln|\Lambda(T_n)|_{l_\infty^{(m)}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_m - \frac{1}{(2\pi)^m}\int_{T^m} \ln|T_n|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_m \\
&\qquad = \frac{1}{(2\pi)^m}\int_{T^m} \ln \max_{1\leqslant k\leqslant m}\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - \alpha_j^{(k)}(z_k')}\right| \mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_m \\
&\qquad \leqslant \sum_{k=1}^{m}\frac{1}{(2\pi)^m}\int_{T^m} \ln\left|A - \mathrm{i}B + \frac{\mathrm{i}B}{n}\sum_{j=1}^{2n}\frac{\mathrm{e}^{\mathrm{i}\theta_k}}{\mathrm{e}^{\mathrm{i}\theta_k} - \alpha_j^{(k)}(z_k')}\right| \mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_m \\
&\qquad = \sum_{k=1}^{m}\frac{1}{(2\pi)^{m-1}}\int_{T^{m-1}} I_k(z_k')\,\mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{k-1}\,\mathrm{d}\theta_{k+1} \ldots \mathrm{d}\theta_m \\
&\qquad \leqslant m \ln|A - \mathrm{i}B|,
\end{aligned}$$

where $z_k' = (\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{k-1}}, \mathrm{e}^{\mathrm{i}\theta_{k+1}}, \ldots, \mathrm{e}^{\mathrm{i}\theta_m})$, $k = 1, 2, \ldots, m$.

The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let A, B and s be real numbers, and n be any positive integer. Then*

$$\begin{aligned}
\frac{1}{(2\pi)^m}\int_{T^m} \ln|\Lambda(T_n^*)|_{l_\infty^{(m)}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_m& \\
\leqslant m \ln|A - \mathrm{i}B| + \frac{1}{(2\pi)^m}&\int_{T^m} \ln|T_n^*|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_m
\end{aligned} \tag{2.15}$$

for every $T_n \in \mathcal{T}(\mathbb{R}^m)$, where

$$T_n^*(\theta_1, \theta_2, \ldots, \theta_m) = T_n(\theta_1, \theta_2, \ldots, \theta_m) + \mathrm{e}^{\mathrm{i}s}\mathrm{e}^{\mathrm{i}n(\theta_1+\theta_2+\ldots+\theta_m)}. \tag{2.16}$$

P r o o f. For any $T_n(\theta_1, \dots, \theta_m) \in \mathcal{T}_n(\mathbb{R}^m)$,

$$T_n(\theta_1, \theta_2, \dots, \theta_m) = \sum_{0 \leqslant |j_1|+|j_2|+\dots+|j_m| \leqslant n} c_{j_1 j_2 \dots j_m} \mathrm{e}^{\mathrm{i}(j_1\theta_1 + j_2\theta_2 + \dots + j_m\theta_m)}.$$

Setting

$$P^*_{2mn}(z_1, z_2, \dots, z_m) = \sum_{0 \leqslant |j_1|+|j_2|+\dots+|j_m| \leqslant n} c_{j_1 j_2 \dots j_m} z_1^{n+j_1} z_2^{n+j_2 \dots z_m^{n+j_m}} + \mathrm{e}^{\mathrm{i}s} z_1^{2n} \dots z_m^{2n},$$

it is not hard to verify that (2.4)–(2.6) and (2.8) hold with T^*_n and P^*_{2mn}.

The same method in the proof of Lemma 2.2 completes the proof of (2.15). The following theorem contains the statement that

$$\Big\| \Big| T_n \cos\alpha + \frac{1}{n} \nabla T_n \sin\alpha \Big|_{l_\infty^{(m)}} \Big\|_0 \leqslant \|T_n\|_0, \tag{2.17}$$

holds for each real number α and each trigonmetric polynomial $T_n \in \mathcal{T}(\mathbb{R}^m)$.

Theorem 2.1. *For all $T_n \in \mathcal{T}_n(\mathbb{R}^m)$, any real s and real A, B satisfying*

$$|A - \mathrm{i}B| = 1, \tag{2.18}$$

$$\frac{1}{(2\pi)^m} \int_{T^m} \ln |\Lambda(T_n)|_{l_\infty^{(m)}} \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2 \dots \mathrm{d}\theta_m \leqslant \frac{1}{(2\pi)^m} \int_{T^m} \ln |T_n| \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2 \dots \mathrm{d}\theta_m, \tag{2.19}$$

and

$$\frac{1}{(2\pi)^m} \int_{T^m} \ln |\Lambda(T^*_n)|_{l_\infty^{(m)}} \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2 \dots \mathrm{d}\theta_m \leqslant \frac{1}{(2\pi)^m} \int_{T^m} \ln |T^*_n| \,\mathrm{d}\theta_1 \,\mathrm{d}\theta_2 \dots \mathrm{d}\theta_m, \tag{2.20}$$

where $T^*_n(\theta_1, \dots, \theta_m)$ is the trigonometric polynomial in (2.16).

P r o o f. By (2.18), (2.19) and (2.20) follow from (2.11) and (2.15), respectively. □

3. Inequalities in L_p

To obtain the inequalities in L_p, $0 < p < \infty$, we first establish the following theorem.

Theorem 3.1. *For all $T_n \in \mathcal{T}_n(\mathbb{R}^m)$, and real A, B satisfying* (2.18)

$$\frac{1}{(2\pi)^m}\int_{T^m} \ln^+ |\Lambda(T_n)(\theta_1,\theta_2,\dots,\theta_m)|_{l_\infty^{(m)}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\dots\mathrm{d}\theta_m \leqslant \frac{1}{(2\pi)^m}\int_{T^m} \ln^+ |T_n(\theta_1,\theta_2\dots,\theta_m)|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\dots\mathrm{d}\theta_m, \tag{3.1}$$

where

$$\ln^+ t = \max\{0,\ln t\},\quad t\in[0,\infty). \tag{3.2}$$

Proof. By (1.7) in [3] for any real α and $\omega\in\mathbb{C}$

$$\ln^+|\omega| = \frac{1}{2\pi}\int_T \ln|\omega+\mathrm{e}^{\mathrm{i}s}|\,\mathrm{d}s = \frac{1}{2\pi}\int_T \ln|\omega+\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}s}|\,\mathrm{d}s. \tag{3.3}$$

We apply (3.2) to $t=|\Lambda(T_n)|_{l_\infty^{(m)}}$, and note that

$$\ln^+|\Lambda(T_n)|_{l_\infty^{(m)}} = \max_{1\leqslant k\leqslant m}\Big\{\ln^+\Big|AT_n+\frac{B}{n}\frac{\partial T_n}{\partial\theta_k}\Big|\Big\}.$$

Thus for any real α,

$$\begin{aligned}\ln^+|\Lambda(T_n)|_{l_\infty^{(m)}} &= \max_{1\leqslant k\leqslant m}\Big\{\frac{1}{2\pi}\int_T \ln\Big|AT_n+\frac{B}{n}\frac{\partial T_n}{\partial\theta_k}+\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{in(\theta_1+\dots+\theta_m)}\mathrm{e}^{\mathrm{i}s}\Big|\,\mathrm{d}s\Big\}\\ &\leqslant \frac{1}{2\pi}\int_T \max_{1\leqslant k\leqslant m}\Big\{\ln\Big|AT_n+\frac{B}{n}\frac{\partial T_n}{\partial\theta_k}+\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{in(\theta_1+\dots+\theta_m)}\mathrm{e}^{\mathrm{i}s}\Big|\Big\}\,\mathrm{d}s,\end{aligned}$$

and Fubini's theorem gives

$$\frac{1}{(2\pi)^m}\int_{T^m}\ln^+|\Lambda(T_n)|_{l_\infty^{(m)}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\dots\mathrm{d}\theta_m \leqslant \frac{1}{2\pi}\int_T\Big[\frac{1}{(2\pi)^m}\int_{T^m}E\,\mathrm{d}\theta_1\dots\mathrm{d}\theta_m\Big]\,\mathrm{d}s, \tag{3.4}$$

where

$$E=\max_{1\leqslant k\leqslant m}\Big\{\ln\Big|AT_n+\frac{B}{n}\frac{\partial T_n}{\partial\theta_k}+\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{in(\theta_1+\dots+\theta_m)}\mathrm{e}^{\mathrm{i}s}\Big|\Big\}.$$

Setting

$$T_n^*(\theta_1,\dots,\theta_n)=T_n(\theta_1,\dots,\theta_m)+\mathrm{e}^{\mathrm{i}s}\mathrm{e}^{in(\theta_1+\dots+\theta_m)}$$

and writing $A + \mathrm{i}B = \mathrm{e}^{\mathrm{i}\alpha}$, (2.20) gives

$$\text{(3.5)}\quad \frac{1}{(2\pi)^m}\int_{T^m}\max_{1\leqslant k\leqslant m}\Big\{\ln\Big|AT_n+\frac{B}{n}\frac{\partial T_n}{\partial\theta_k}+\mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}n(\theta_1+\ldots+\theta_m)}\mathrm{e}^{\mathrm{i}s}\Big|\Big\}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_m \leqslant \frac{1}{(2\pi)^m}\int_T \ln|T_n+\mathrm{e}^{\mathrm{i}s}\mathrm{e}^{\mathrm{i}n(\theta_1+\ldots+\theta_m)}|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_m,$$

where s is a real parameter.

If we combine (3.4) and (3.5), and use Fubini's theorem, we obtain the desired inequality:

$$\begin{aligned}\frac{1}{(2\pi)^m}\int_{T^m}&\ln^+|\Lambda(T_n)|_{l_\infty^{(m)}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_m\\ &\leqslant\frac{1}{2\pi}\int_T\Big[\frac{1}{(2\pi)^m}\int_{T^m}\ln|T_n+\mathrm{e}^{\mathrm{i}s}\mathrm{e}^{\mathrm{i}n(\theta_1+\theta_2+\ldots+\theta_m)}|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_m\Big]\,\mathrm{d}s\\ &=\frac{1}{(2\pi)^m}\int_{T^m}\Big[\frac{1}{2\pi}\int_T\ln|T_n+\mathrm{e}^{\mathrm{i}s}\mathrm{e}^{\mathrm{i}n(\theta_1+\theta_2+\ldots+\theta_m)}|\,\mathrm{d}s\Big]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_m\\ &=\frac{1}{(2\pi)^m}\int_{T^m}\ln^+|T_n|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ldots\mathrm{d}\theta_m.\end{aligned}$$

This completes the proof. □

Let $\Phi(u)$ with $\Phi(0)=0$, and $\Psi(u)=u\Phi'(u)$ be continuous positive increasing functions defined on $[0,\infty)$. By (3.5) in [3], we have

$$\text{(3.6)}\qquad \Phi(u)=\int_0^\infty \ln^+\frac{u}{s}\,\mathrm{d}\Psi(s).$$

Using the method of the proof of Theorem 5 in [3], by Theorem 3.1, we obtain immediately the following theorem:

Theorem 3.2. *Let $\Phi(u)$ with $\Phi(0)=0$ and $\Psi(u)=u\Psi'(u)$ be continuous positive increasing functions defined on $[0,\infty)$. For each $T_n\in\mathcal{T}(\mathbb{R}^m)$, and real A, B satisfying* (2.18),

$$\text{(3.7)}\qquad \int_{T^m}\Phi\Big(\Big|AT_n+\frac{B}{n}\nabla T_n\Big|_{l_\infty^{(m)}}\Big)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_m\leqslant\int_{T^m}\Phi(|T_n|)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_m.$$

Since $\Phi(u)=u^p$, $0<p<\infty$ is a function that satisfies the condition described in Theorem 3.2, by (3.7) we have the following corollary.

Corollary 3.1. *For each real α and each $T_n \in \mathcal{T}_n(\mathbb{R}^m)$, we have*

$$\Big\| \Big| T_n \cos\alpha + \frac{1}{n} \nabla T_n \sin\alpha \Big|_{l_\infty^{(m)}} \Big\|_p \leqslant \|T_n\|_p, \quad 0 < p < \infty. \tag{3.8}$$

In particular, for each $T_n \in \mathcal{T}_n(\mathbb{R}^m)$, we have

$$\| |\nabla T_n|_{l_\infty^{(m)}} \|_p \leqslant n \|T_n\|_p, \quad 0 < p < \infty. \tag{3.9}$$

4. Inequalities in L_∞

For each $T_n \in \mathcal{T}_n(\mathbb{R}^m)$, we can write it in the form

$$T_n(\theta_1, \theta_2, \ldots, \theta_m) = \sum_{0 \leqslant |j_1| + |j_2| + \ldots + |j_m| \leqslant n} c_{j_1 j_2 \ldots j_m} \mathrm{e}^{\mathrm{i}(j_1\theta_1 + j_2\theta_2 + \ldots + j_m\theta_m)}.$$

Fix k, $1 \leqslant k \leqslant n$, and write

$$T_n = \sum_{j_k = -n}^{n} c_{j_k}(z_k') \mathrm{e}^{\mathrm{i} j_k \theta_k}, \tag{4.1}$$

where $z_k' = (\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_{k-1}}, \mathrm{e}^{\mathrm{i}\theta_{k+1}}, \ldots, \mathrm{e}^{\mathrm{i}\theta_m}) \in \mathbb{C}^{m-1}$. Then T_n is a trigonometric polynomial with degree not greater than n in each θ_k, $1 \leqslant k \leqslant n$. The Bernstein-Szegö inequality (see (1.5)) of one variable implies

$$\Big| A T_n + \frac{B}{n} \frac{\partial T_n}{\partial \theta_k} \Big| \leqslant \|T_n\|_\infty, \quad k = 1, 2, \ldots, m, \tag{4.2}$$

where A, B are real with $|A + \mathrm{i}B| = 1$.

From (4.2) we get the following Bernstein-Szegö inequality for several variables.

Theorem 4.1. *For all $T_n \in \mathcal{T}_n(\mathbb{R}^m)$ and real A, B satisfying* (2.18),

$$\Big\| \Big| A T_n + \frac{B}{n} \nabla T_n \Big|_{l_\infty^{(m)}} \Big\|_\infty \leqslant \|T_n\|_\infty. \tag{4.3}$$

Thus far, we establish the inequality

$$\Big\| \Big| A T_n + \frac{B}{n} \nabla T_n \Big|_{l_\infty^{(m)}} \Big\|_p \leqslant \|T_n\|_p, \quad 0 \leqslant p \leqslant \infty, \tag{4.4}$$

for all $T_n \in \mathcal{T}_n(\mathbb{R}^m)$, and real A, B satisfying (2.18).

Remark 4.1. For $z=(z_1,z_2,\ldots,z_m)$, let

$$|z|_{l_r^{(m)}}=\left(\sum_{k=1}^{m}|z_k|^r\right)^{1/r},\quad 1\leqslant r\leqslant\infty. \tag{4.5}$$

It follows from (1.7) that

$$\left\|\left|T_n\cos\alpha+\frac{1}{n}\nabla T_n\sin\alpha\right|_{l_r^{(m)}}\right\|_p\leqslant m^{1/r}\|T_n\|_p,\quad 0\leqslant p\leqslant\infty. \tag{4.6}$$

However, in [5], Tung obtained the following inequality:

$$\left\|\left|T_n\cos\alpha+\frac{1}{n}\nabla T_n\sin\alpha\right|_{l_2^{(m)}}\right\|_p\leqslant \|T_n\|_p, \tag{4.7}$$

where $\alpha=\frac{1}{2}\pi$, $p=\infty$, $T_n(\theta_1,\ldots,\theta_n)=P_n(\mathrm{e}^{\mathrm{i}\theta_1},\ldots,\mathrm{e}^{\mathrm{i}\theta_m})$, and

$$P_n(z_1,z_2,\ldots,z_m)=\sum_{\substack{j_1+\ldots+j_m\leqslant n\\ 0\leqslant j_k,\,k=1,\ldots,m}} c_{j_1j_2\ldots j_m}z_1^{j_1}z_2^{j_2}\ldots z_m^{j_m}.$$

It would be of interest to investigate similar problems for the case of polynomials in several complex variables.

Remark 4.2. Using (1.7) for all trigonometric polynomials that have degree at most n in each variable, i.e., for trigonometric polynomials of the form

$$T_n(\theta_1,\theta_2,\ldots,\theta_m)=\sum_{\substack{|j_m|\leqslant n,\\ k=1,\ldots,m}} c_{j_1j_2\ldots j_m}\mathrm{e}^{\mathrm{i}(j_1\theta_1+j_2\theta_2+\ldots+j_m\theta_m)}$$

for $1\leqslant r\leqslant\infty$, we have

$$\left\|\left|T_n\cos\alpha+\frac{1}{nm}\nabla T_n\sin\alpha\right|_{l_r^{(m)}}\right\|_p\leqslant m^{1/r}\|T_n\|_p,\quad 0\leqslant p\leqslant\infty. \tag{4.8}$$

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