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SOLUTIONS OF THE GENERALIZED DIRICHLET PROBLEM FOR THE ITERATED SLICE DIRAC EQUATION

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Abstract. Applying the method of normalized systems of functions we construct solutions of the generalized Dirichlet problem for the iterated slice Dirac operator in Clifford analysis. This problem is a natural generalization of the Dirichlet problem.

Keywords: slice Clifford analysis; slice Dirac equation; Dirichlet problem

MSC 2020: 30G35, 35J40

1. Introduction

The problem of finding a solution of a second-order elliptic equation which is regular in the domain of definition was studied by Dirichlet (see [11]) and is known as the Dirichlet or first boundary value problem. In this paper, we mainly consider a generalized Dirichlet problem of the following form.

Let $\Omega = \{x \in \mathbb{R}^{m+1}\colon\ |x| < 1\}$ denote the unit ball. Find a function u satisfying

$$\begin{cases} D_0^4 u(x) = P(x), & x \in \Omega, \\ u|_{\partial\Omega} = Q(x), & \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = R(x), \end{cases} \tag{1.1}$$

where D_0 is the slice Dirac operator, n is the outward normal to $\partial\Omega$, and $P(x)$, $Q(x)$, $R(x)$ are Clifford-valued polynomials.

The slice Dirac operator is an extension of the well-known Dirac operator. The Dirac operator introduced in the Clifford algebra setting by the work of Romanian mathematicians Moisil and Teodorescu and Swiss mathematician Fueter (see [12])

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was later developed by Delanghe et al., see [2], [9], [10]. The null-solutions of the Dirac operator are called *monogenic functions*. The study of properties of monogenic functions is a very well-established research field, usually called the *Clifford analysis*. A disappointment about monogenic functions is that the identity function or the powers of a variable are not monogenic functions. Gentili et al. in [13] offer a new definition of monogenicity for functions (or slice monogenic functions) based on a generalized Cauchy-Riemann operator (or slice Cauchy-Riemann operator). This new class of monogenic functions contains polynomials (and, more in general, power series) with coefficients in the Clifford algebra Cl_n. However, when taking the step from general Clifford analysis based on the Dirac operator (see e.g. [2], [9], [10], [15]) to slice Clifford analysis (see e.g. [1], [6], [7], [8], [13], [14], [20], [21]), some important properties are lost as well, such as the Fourier transforms based on the differential operator. Cnudde et al. in [5] introduced a particular representation of the slice Dirac operator, which allows to establish the Lie superalgebra structure behind slice Clifford analysis. Furthermore, they studied integral transforms related to the slice Dirac operator, such as the slice Fourier transform and slice Segal-Bargmann transform, see [3], [4]. As far as we know, up till now, Dirichlet type problems for the iterated slice Dirac equations have not been considered. In this paper, we construct solutions of generalized Dirichlet problems for iterated slice Dirac equations by means of the method of normalized systems of functions rather than Poisson formulas.

The method of normalized systems of functions was studied by the second author in [16], [17], [18], [19] to construct and investigate polynomial solutions of initial and boundary value problems for partial differential equations in real analysis, such as Dirichlet problems, Neumann problems, Riqurie problems, etc. However, the study of boundary value problems for partial differential equations in the slice Clifford analysis is different from that in real analysis. Functions in the slice Clifford analysis are not mutually commuting with respect to the pointwise product, see [3], [4], [5]. To overcome the noncommutative properties between functions, we exploit the intertwining relations of differential operators (i.e., differential operators satisfy the defining relations of the Lie superalgebra, see [5]). Applying these relations, we construct the normalized system of functions for the iterated slice Dirac operator. Then we get the Almansi representation for null solutions to the iterated slice Dirac equations by the normalized system. Furthermore, we obtain solutions of inhomogeneous iterated slice Dirac equations. These results help us to investigate generalized Dirichlet problems for the iterated slice Dirac equation. In Section 4 of this paper, we consider the homogeneous boundary value problem for the inhomogeneous iterated slice Dirac equation. Furthermore, we study the inhomogeneous boundary value problem for the inhomogeneous iterated slice Dirac equation. These problems are related to Dirichlet problems.

2. Preliminaries

For the basic facts on Clifford algebras, Clifford analysis and slice Clifford analysis, see, for example, [2], [3], [4], [5], [9], [10], [15]. One of the main aims of the slice Clifford analysis is to construct a first order operator, the so-called *slice Dirac operator*, factorizing the Laplace operator and to study the function-theoretical properties of the null-solutions of this operator.

Let us identify the $(m+1)$-tuple $(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ with the 1-vector $x \in \mathrm{Cl}_{m+1}$. The vector x is defined as $x = x_0e_0 + x_1e_1 + \ldots + x_me_m = x_0e_0 + \underline{x}$, where $e_0, e_1, \ldots, e_m$ are the basis elements of the Clifford algebra Cl_{m+1} satisfying the relations

$$e_ie_j + e_je_i = -2\delta_{i,j}, \quad i, j = 0, 1, \ldots, m.$$

Here $\delta_{i,j}$ denotes the Kronecker symbol. In particular, $e_i^2 = -1$ for $i = 0, \ldots, m$. One thus has $x^2 = \sum\limits_{i=0}^{m} x_i^2 = -|x|^2 = -(x_0^2 + |\underline{x}|^2) = -(x_0^2 + r^2)$.

A first generalization of the classical Cauchy-Riemann operator is given by the Dirac operator defined as

$$D = \sum_{i=0}^{m} e_i\partial_{x_i}.$$

A second generalization of the classical Cauchy-Riemann operator is based on the polar form of the vector. This is given by the slice Dirac operator defined as

$$D_0 = e_0\partial_{x_0} + \underline{\omega}\partial_r = e_0\partial_{x_0} + \frac{\underline{x}}{|\underline{x}|^2}\sum_{i=1}^{m} x_i\partial_{x_i},$$

where $r = \sqrt{x_1^2 + \ldots + x_m^2}$ and $\underline{\omega} = \underline{x}/r$.

The square of the slice Dirac operator is

$$D_0^2 = -(\partial_{x_0}^2 + \partial_r^2).$$

The Euler operator in slice Clifford analysis is given by

$$E = \sum_{i=0}^{m} x_i\partial_{x_i}.$$

The operators x, D_0, E constitute the Lie superalgebra $\mathrm{osp}(1|2)$, see [20]. In particular, the operators x^2, D_0^2 and E satisfy the intertwining relations

$$D_0^2x^2 - x^2D_0^2 = 4E + 4, \tag{2.1}$$

$$Ex^2 - x^2E = 2x^2. \tag{2.2}$$

3. Solutions of iterated slice Dirac equations

In this section, we discuss properties of the Euler operator. Then by these properties and the intertwining relations of differential operators, we construct a normalized system of functions with respect to the operator D_0^2. Applying the system, we get the Almansi expression for null solutions to the iterated slice Dirac equation. Furthermore, the solution of the equations $D_0^2 u(x) = f(x)$ and $D_0^4 u(x) = f(x)$ can be obtained in this context. These results will help us to investigate generalized Dirichlet problems for iterated slice Dirac equations in the next section.

From now on $\Omega = \{x \in \mathbb{R}^{m+1} \colon |x| < 1\}$ denotes the unit ball in $\mathbb{R}^{m+1}$.

Lemma 3.1. *If $f \in C^1(\Omega, \mathrm{Cl}_{m+1})$, then*

$$(E+s)\int_0^1 (1-t)^{s-1} f(tx)\,\mathrm{d}t = (s-1)\int_0^1 (1-t)^{s-2} f(tx)\,\mathrm{d}t, \quad s = 2,3,\dots, \tag{3.1}$$

$$(E+s)\int_0^1 f(tx)\,\mathrm{d}t = f(x), \quad s = 1, \tag{3.2}$$

$$(E+s)\int_0^1 \frac{(1-t)^q}{q!} t^{s-1} f(tx)\,\mathrm{d}t = \int_0^1 \frac{(1-t)^{q-1}}{(q-1)!} t^s f(tx)\,\mathrm{d}t, \quad q, s \in \mathbb{N}. \tag{3.3}$$

P r o o f. By direct calculation, we have

$$\begin{aligned} E\int_0^1 (1-t)^{s-1} f(tx)\,\mathrm{d}t &= \sum_{i=0}^m x_i \partial_{x_i} \int_0^1 (1-t)^{s-1} f(tx)\,\mathrm{d}t = \int_0^1 (1-t)^{s-1} t \frac{\mathrm{d}}{\mathrm{d}t} f(tx) \\ &= -\int_0^1 [(1-t)^{s-1} - (s-1)t(1-t)^{s-2}] f(tx)\,\mathrm{d}t \\ &= (s-1)\int_0^1 (1-t)^{s-2} f(tx)\,\mathrm{d}t - s\int_0^1 (1-t)^{s-1} f(tx)\,\mathrm{d}t, \end{aligned}$$

which proves (3.1). In a similar way, one can obtain the formulas (3.2) and (3.3). □

Lemma 3.2. *Let $f \in C^2(\Omega, \mathrm{Cl}_{m+1})$ be such that $D_0^2 f(x) = 0$. Let $s > 0$ and $F_s \colon \Omega \to \mathrm{Cl}_{m+1}$ be defined as*

$$F_s(x) = \frac{x^{2s}}{4^s s!\,(s-1)!} \int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha, \tag{3.4}$$

and $F_0(x) = f(x)$. Then $D_0^2 F_s(x) = F_{s-1}(x)$.

P r o o f. As the first step, we prove that

$$D_0^2[x^{2s}f(x)] = x^{2s}D_0^2 f(x) + 4sx^{2s-2}(E+s)f(x). \tag{3.5}$$

Using the formulas (2.1) and (2.2), we obtain

$$\begin{aligned} D_0^2[x^{2s}f(x)] &= [x^2D_0^2 + 4E + 4][x^{2s-2}f(x)] \\ &= x^2D_0^2x^{2s-2}f(x) + 4Ex^{2s-2}f(x) + 4x^{2s-2}f(x) \\ &= x^2[x^2D_0^2 + 4E + 4]x^{2s-4}f(x) + 4(x^2E + 2x^2)x^{2s-4}f(x) + 4x^{2s-2}f(x). \end{aligned}$$

By iterating this computation, we get

$$\begin{aligned} D_0^2[x^{2s}f(x)] &= x^{2s}D_0^2f(x) + 4sx^{2s-2}Ef(x) + 4sx^{2s-2}f(x) \\ &\quad + [8(s-1) + \ldots + 8]x^{2s-2}f(x) \\ &= x^{2s}D_0^2f(x) + 4sx^{2s-2}(E+s)f(x). \end{aligned}$$

Let now $f \in C^2(\Omega, \mathrm{Cl}_{m+1})$ be such that $D_0^2 f(x) = 0$. Then we let the operator D_0^2 act on the right side of the equation (3.4): for $s > 1$, it follows by Lemma 3.1 that

$$\begin{aligned} D_0^2[F_s(x)] &= D_0^2\left[\frac{x^{2s}}{4^s s!\,(s-1)!}\int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha\right] \\ &= \frac{x^{2s}}{4^s s!\,(s-1)!} D_0^2 \int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha \\ &\quad + \frac{x^{2(s-1)}(E+s)}{4^{s-1}(s-1)!\,(s-1)!}\int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha \\ &= \frac{x^{2(s-1)}}{4^{s-1}(s-1)!\,(s-1)!} \\ &\quad \times \left[E\int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha + s\int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha\right] \\ &= \frac{x^{2(s-1)}}{4^{s-1}(s-1)!\,(s-1)!} \\ &\quad \times \left[\int_0^1 (1-\alpha)^{s-1}\alpha\,\mathrm{d}f(\alpha x) + s\int_0^1 (1-\alpha)^{s-1} f(\alpha x)\,\mathrm{d}\alpha\right] \\ &= \frac{x^{2(s-1)}}{4^{s-1}(s-1)!\,(s-1)!} \\ &\quad \times \int_0^1 [-(1-\alpha)^{s-1} + (s-1)\alpha(1-\alpha)^{s-2} + s(1-\alpha)^{s-1}] f(\alpha x)\,\mathrm{d}\alpha \\ &= \frac{x^{2(s-1)}}{4^{s-1}(s-1)!\,(s-2)!}\int_0^1 (1-\alpha)^{s-2} f(\alpha x)\,\mathrm{d}\alpha = F_{s-1}f(x). \end{aligned}$$

For $s = 1$, it follows by Lemma 3.1 that

$$D_0^2[F_1(x)] = (E+1)\int_0^1 f(\alpha x)\,\mathrm{d}\alpha = f(x) = F_0(x),$$

which completes the proof. $\square$

Remark 3.3. The sequence of functions $\{F_s, s = 0, 1, \ldots\}$ defined in the previous lemma is the normalized system of functions with respect to the operator D_0^2, that is

$$D_0^2 F_0(x) = 0, \quad D_0^2 F_s(x) = F_{s-1}(x).$$

From Lemma 3.2, it is easy to obtain

$$D_0^{2k}\left[f_0(x) + \sum_{s=1}^{k-1} \frac{x^{2s}}{4^s s!\,(s-1)!}\int_0^1 (1-\alpha)^{s-1} f_s(\alpha x)\,\mathrm{d}\alpha\right] = 0,$$

where $D_0^2 f_s(x) = 0$, $s = 0, 1, \ldots, k-1$.

Theorem 3.4. *Let $G \in C^{2k}(\Omega, \mathrm{Cl}_{m+1})$. If $D_0^{2k}G(x) = 0$, then*

$$G(x) = f_0(x) + \sum_{s=1}^{k-1} \frac{x^{2s}}{4^s s!\,(s-1)!}\int_0^1 (1-\alpha)^{s-1} f_s(\alpha x)\,\mathrm{d}\alpha, \tag{3.6}$$

where $f_s(x)$, $s = 0, 1, 2, \ldots, k-1$, satisfy the equation $D_0^2 f(x) = 0$ and

$$f_s(x) = D_0^{2s}G(x) + \sum_{l=1}^{k-s-1} \frac{(-1)^l x^{2l}}{4^l l!\,(l-1)!}\int_0^1 (1-\beta)^{l-1}\beta^{l-1} D_0^{2(s+l)}G(\beta x)\,\mathrm{d}\beta. \tag{3.7}$$

P r o o f. We first prove that $D_0^2 f_s(x) = 0$. Applying Lemma 3.1, we see that

$$\begin{aligned}
D_0^2 f_s(x) &= D_0^{2(s+1)}G(x) \\
&\quad + \sum_{l=1}^{k-s-1} \frac{(-1)^l}{4^l l!\,(l-1)!} D_0^2\left[x^{2l}\int_0^1 (1-\beta)^{l-1}\beta^{l-1} D_0^{2(s+l)}G(\beta x)\,\mathrm{d}\beta\right] \\
&= D_0^{2(s+1)}G(x) \\
&\quad + \sum_{l=1}^{k-s-2} \frac{(-1)^l}{4^l l!\,(l-1)!}\left[x^{2l}\int_0^1 (1-\beta)^{l-1}\beta^{l+1} D_0^{2(s+l+1)}G(\beta x)\,\mathrm{d}\beta\right] \\
&\quad + \sum_{l=1}^{k-s-1} \frac{(-1)^l x^{2l-2}}{4^{l-1}(l-1)!\,(l-1)!}(E+l)\int_0^1 (1-\beta)^{l-1}\beta^{l-1} D_0^{2(s+l)}G(\beta x)\,\mathrm{d}\beta \\
&= D_0^{2(s+1)}G(x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{k-s-2}\frac{(-1)^l}{4^l l!\,(l-1)!}\left[x^{2l}\int_0^1(1-\beta)^{l-1}\beta^{l+1}D_0^{2(s+l+1)}G(\beta x)\,\mathrm{d}\beta\right]\\
&-(E+1)\int_0^1 D_0^{2(s+1)}G(\beta x)\,\mathrm{d}\beta\\
&+\sum_{l=2}^{k-j-1}\frac{(-1)^l x^{2l-2}}{4^{l-1}(l-1)!\,(l-2)!}\int_0^1(1-\beta)^{l-2}\beta^l D_0^{2(j+l)}G(\beta x)\,\mathrm{d}\beta\\
={}&D_0^{2(s+1)}G(x)\\
&+\sum_{l=1}^{k-s-2}\frac{(-1)^l}{4^l l!\,(l-1)!}\left[x^{2l}\int_0^1(1-\beta)^{l-1}\beta^{l+1}D_0^{2(s+l+1)}G(\beta x)\,\mathrm{d}\beta\right]\\
&-D_0^{2(s+1)}G(x)\\
&+\sum_{l=1}^{k-s-2}\frac{(-1)^{l+1}x^{2l}}{4^l l!\,(l-1)!}\int_0^1(1-\beta)^{l-1}\beta^{l+1}D_0^{2(s+l+1)}G(\beta x)\,\mathrm{d}\beta=0.
\end{aligned}
$$

Let us now prove the formula (3.6), where $f_s(x)$ are given as in (3.7). To do this, we substitute the expression for $f_s(x)$ into (3.6). Then we find

$$
\begin{aligned}
&f_0(x)+\sum_{s=1}^{k-1}\frac{x^{2s}}{4^s s!\,(s-1)!}\int_0^1(1-\alpha)^{s-1}f_s(\alpha x)\,\mathrm{d}\alpha\\
&\quad=G(x)+\sum_{s=1}^{k-1}\frac{(-1)^s x^{2s}}{4^s s!}\int_0^1\frac{(1-\alpha)^{s-1}\alpha^{s-1}}{(s-1)!}D_0^{2s}G(\alpha x)\,\mathrm{d}\alpha\\
&\qquad+\sum_{s=1}^{k-1}\frac{x^{2s}}{4^s s!\,(s-1)!}\int_0^1(1-\alpha)^{s-1}D_0^{2s}G(\alpha x)\,\mathrm{d}\alpha\\
&\qquad+\sum_{s=1}^{k-2}\frac{x^{2s+2l}}{4^s s!\,(s-1)!}\int_0^1\sum_{l=1}^{k-s-1}\frac{(-1)^l(1-\alpha)^{s-1}\alpha^{2l}}{4^l l!}\\
&\qquad\times\int_0^1\frac{(1-\beta)^{l-1}\beta^{l-1}}{(l-1)!}D_0^{2(l+s)}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha.
\end{aligned}
$$

By making the change of variables $\alpha\beta\to t$ and by changing the integration order, the last sum in the above equality becomes

$$
\begin{aligned}
&\sum_{s=1}^{k-2}\sum_{l=1}^{k-s-1}\frac{(-1)^l x^{2s+2l}}{4^{l+s}s!\,l!}\\
&\quad\times\int_0^1\frac{\alpha^2(1-\alpha)^{s-1}}{(s-1)!}\int_0^1\frac{(\alpha-\alpha\beta)^{l-1}(\alpha\beta)^{l-1}}{(l-1)!}D_0^{2(l+s)}G(\alpha\beta x)\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
&=\sum_{s=1}^{k-2}\sum_{l=1}^{k-s-1}\frac{(-1)^l x^{2s+2l}}{4^{l+s}s!\,l!}\int_0^1\frac{\alpha(1-\alpha)^{s-1}}{(s-1)!}\int_0^\alpha\frac{(\alpha-t)^{l-1}t^{l-1}}{(l-1)!}D_0^{2(l+s)}G(tx)\,\mathrm{d}t\,\mathrm{d}\alpha
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{s=1}^{k-2}\sum_{l=1}^{k-s-1} \frac{(-1)^l x^{2s+2l}}{4^{l+s} s!\, l!} \int_0^1 \int_0^\alpha \frac{\alpha(1-\alpha)^{s-1}(\alpha-t)^{l-1}}{(s-1)!\,(l-1)!} t^{l-1} D_0^{2(l+s)} G(tx)\, \mathrm{d}t\, \mathrm{d}\alpha \\
&= \sum_{s=1}^{k-2}\sum_{l=1}^{k-s-1} \frac{(-1)^l x^{2s+2l}}{4^{l+s} s!\, l!} \int_0^1 \int_t^1 \frac{\alpha(1-\alpha)^{s-1}(\alpha-t)^{l-1}}{(s-1)!\,(l-1)!} t^{l-1} D_0^{2(l+s)} G(tx)\, \mathrm{d}\alpha\, \mathrm{d}t \\
&= \sum_{s=1}^{k-2}\sum_{l=1}^{k-s-1} \frac{(-1)^l x^{2s+2l}}{4^{l+s}} \\
&\quad \times \int_0^1 \Big[\frac{(1-t)^{s+l}}{s!\,(l-1)!\,(l+s)!} + \frac{t(1-t)^{l+s-1}}{s!\,l!\,(l+s-1)!}\Big] t^{l-1} D_0^{2(l+s)} G(tx)\, \mathrm{d}t.
\end{aligned}
$$

Let $s+l=i$. Then we can continue as follows

$$
\begin{aligned}
&\sum_{i=2}^{k-1}\sum_{l=1}^{i-1} \frac{(-1)^l x^{2i}}{4^i} \int_0^1 \Big[\frac{t^{l-1}(1-t)^i}{(i-l)!\,(l-1)!\,i!} + \frac{t^l(1-t)^{i-1}}{(i-l)!\,l!\,(i-1)!}\Big] D_0^{2i} G(tx)\, \mathrm{d}t \\
&= \sum_{i=2}^{k-1} \frac{x^{2i}}{4^i} \int_0^1 \frac{(1-t)^{i-1}(-t)^{i-1}}{i!\,(i-1)!} D_0^{2i} G(tx)\, \mathrm{d}t - \sum_{i=2}^{k-1} \frac{x^{2i}}{4^i} \int_0^1 \frac{(1-t)^{i-1}}{i!\,(i-1)!} D_0^{2i} G(tx)\, \mathrm{d}t \\
&= \sum_{i=2}^{k-1} \frac{x^{2i}}{4^i} \int_0^1 \frac{(1-t)^{i-1}(-t)^{i-1}}{i!\,(i-1)!} D_0^{2i} G(tx)\, \mathrm{d}t + \frac{x^2}{4} \int_0^1 D_0^{2i} G(tx)\, \mathrm{d}t \\
&\quad - \sum_{i=2}^{k-1} \frac{x^{2i}}{4^i} \int_0^1 \frac{(1-t)^{i-1}}{i!\,(i-1)!} D_0^{2i} G(tx)\, \mathrm{d}t - \frac{x^2}{4} \int_0^1 D_0^{2i} G(tx)\, \mathrm{d}t \\
&= -\sum_{s=1}^{k-1} \frac{(-1)^s x^{2s}}{4^s} \int_0^1 \frac{(1-t)^{s-1} t^{s-1}}{s!\,(s-1)!} D_0^{2s} G(tx)\, \mathrm{d}t \\
&\quad - \sum_{s=1}^{k-1} \frac{x^{2s}}{4^s} \int_0^1 \frac{(1-t)^{s-1}}{s!\,(s-1)!} D_0^{2s} G(tx)\, \mathrm{d}t
\end{aligned}
$$

and the theorem is proved. □

We now consider the equation

$$
D_0^2 u(x) = f(x), \tag{3.8}
$$

where D_0 is the slice Dirac operator and $f \in C^\infty(\Omega, \mathrm{Cl}_{m+1})$ is a real analytic function. In the sequel, we always assume that infinite series converge absolutely and uniformly in the unit ball Ω.

Theorem 3.5. *The solution of the equation* (3.8) *can be represented as*

$$
u(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!\,s!} \int_0^1 (1-t)^s t^s D_0^{2s} f(tx)\, \mathrm{d}t. \tag{3.9}
$$

P r o o f. We let the operator D_0^2 act on the right side of the equation (3.9). Then from the formula (3.5) it follows that

$$\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}D_0^2\Big[x^{2(s+1)}\int_0^1(1-t)^st^sD_0^{2s}f(tx)\,\mathrm{d}t\Big]$$
$$=\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}\Big[x^{2(s+1)}\int_0^1(1-t)^st^{s+2}D_0^{2(s+1)}f(tx)\,\mathrm{d}t\Big]$$
$$+\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}4(s+1)x^{2s}(E+s+1)\int_0^1(1-t)^st^sD_0^{2s}f(tx)\,\mathrm{d}t$$
$$=\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}\Big[x^{2(s+1)}\int_0^1(1-t)^st^{s+2}D_0^{2(s+1)}f(tx)\,\mathrm{d}t\Big]$$
$$+(E+1)\int_0^1f(tx)\,\mathrm{d}t+\sum_{s=1}^{\infty}\frac{(-1)^s}{4^ss!\,s!}x^{2s}(E+s+1)\int_0^1(1-t)^st^sD_0^{2s}f(tx)\,\mathrm{d}t.$$

By means of Lemma 3.1, we have

$$\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}D_0^2\Big[x^{2(s+1)}\int_0^1(1-t)^st^sD_0^{2s}f(tx)\,\mathrm{d}t\Big]$$
$$=f(x)+\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}\Big[x^{2(s+1)}\int_0^1(1-t)^st^{s+2}D_0^{2(s+1)}f(tx)\,\mathrm{d}t\Big]$$
$$+\sum_{s=1}^{\infty}\frac{(-1)^sx^{2s}}{4^ss!\,(s-1)!}\int_0^1(1-t)^{s-1}t^{s+1}D_0^{2s}f(tx)\,\mathrm{d}t$$
$$=f(x)+\sum_{s=0}^{\infty}\frac{(-1)^s}{4^{s+1}(s+1)!\,s!}\Big[x^{2(s+1)}\int_0^1(1-t)^st^{s+2}D_0^{2(s+1)}f(tx)\,\mathrm{d}t\Big]$$
$$+\sum_{s=0}^{\infty}\frac{(-1)^{s+1}x^{2(s+1)}}{4^{s+1}s!\,(s+1)!}\int_0^1(1-t)^st^{s+2}D_0^{2(s+1)}f(tx)\,\mathrm{d}t=f(x).$$

This implies that the desired result holds. □

Applying Theorem 3.5, we study the iterated slice Dirac equation

$$D_0^4u(x)=f(x),\tag{3.10}$$

where $f\in C^\infty(\Omega,\mathrm{Cl}_{m+1})$ is a real analytic function.

Theorem 3.6. *The solution of the equation* (3.10) *is given by*

$$u(x)=\sum_{s=0}^{\infty}\frac{(-1)^sx^{2(s+2)}}{4^{s+2}(s+2)!\,s!}\int_0^1(1-t)^{s+1}t^sD_0^{2s}f(tx)\,\mathrm{d}t.$$

P r o o f. From Theorem 3.5, we find that

$$g(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!\, s!} \int_0^1 (1-t)^s t^s D_0^{2s} f(tx)\,\mathrm{d}t$$

is the solution of the equation $D_0^2 g(x) = f(x)$.

Similarly, the solution of the equation $D_0^2 u(x) = g(x)$ can be written as

$$u(x) = \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!\, s!} \int_0^1 (1-t)^s t^s D_0^{2s} g(tx)\,\mathrm{d}t.$$

Since $D_0^2 g(x) = f(x)$, we have

$$u(x) = \frac{x^2}{4}\int_0^1 g(tx)\,\mathrm{d}t + \sum_{s=1}^{\infty} \frac{(-1)^s (tx)^{2(s+1)}}{4^{s+1}(s+1)!\, s!} \int_0^1 (1-t)^s t^s D_0^{2(s-1)} f(tx)\,\mathrm{d}t. \tag{3.11}$$

We compute the first term on the right side of (3.11):

$$\begin{aligned}
\frac{x^2}{4}\int_0^1 g(tx)\,\mathrm{d}t &= \frac{x^2}{4}\int_0^1 \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+1)}}{4^{s+1}(s+1)!\, s!} \int_0^1 (1-\beta)^s \beta^s D_0^{2s} f(t\beta x)\,\mathrm{d}\beta\,\mathrm{d}t \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+1)!} \int_0^1\int_0^1 \frac{(1-\beta)^s \beta^s t^{2(s+1)}}{s!} D_0^{2s} f(t\beta x)\,\mathrm{d}\beta\,\mathrm{d}t \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+1)!} \int_0^1\int_0^1 \frac{(t-t\beta)^s (t\beta)^s t^2}{s!} D_0^{2s} f(t\beta x)\,\mathrm{d}\beta\,\mathrm{d}t \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+1)!} \int_0^1\int_0^1 \frac{(t-t\beta)^s (t\beta)^s t}{s!} D_0^{2s} f(t\beta x)\,\mathrm{d}(t\beta)\,\mathrm{d}t.
\end{aligned}$$

By making the change of variables $t\beta \to \widetilde{\beta}$ and by changing the integration order, the first term on the right side of (3.11) becomes

$$\begin{aligned}
\frac{x^2}{4}\int_0^1 g(tx)\,\mathrm{d}t &= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+1)!} \int_0^1\int_0^t \frac{(t-\widetilde{\beta})^s \beta^s t}{s!} D_0^{2s} f(\widetilde{\beta} x)\,\mathrm{d}\beta\,\mathrm{d}t \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+1)!\, s!} \int_0^1 \widetilde{\beta}^s D_0^{2s} f(\widetilde{\beta} x) \int_{\widetilde{\beta}}^1 (t-\widetilde{\beta})^s t\,\mathrm{d}t\,\mathrm{d}\widetilde{\beta} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+1)!\, s!} \int_0^1 \widetilde{\beta}^s \Big[\frac{(1-\widetilde{\beta})^{s+1}}{s+1} - \frac{(1-\widetilde{\beta})^{s+2}}{(s+2)(s+1)}\Big] D_0^{2s} f(\widetilde{\beta} x)\,\mathrm{d}\widetilde{\beta} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+2)!\,(s+1)!} \\
&\quad \times \int_0^1 \widetilde{\beta}^s [(1-\widetilde{\beta})(s+2) - (1-\widetilde{\beta})^{s+2}] D_0^{2s} f(\widetilde{\beta} x)\,\mathrm{d}\widetilde{\beta}.
\end{aligned}$$

We now return to formula (3.11),

$$
\begin{aligned}
u(x) &= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+2)!\,(s+1)!} \int_0^1 \widetilde{\beta}^s [(1-\widetilde{\beta})(s+2) - (1-\widetilde{\beta})^{s+2}] D_0^{2s} f(\widetilde{\beta}x)\,\mathrm{d}\widetilde{\beta} \\
&\quad + \sum_{s=1}^{\infty} \frac{(-1)^s (tx)^{2(s+1)}}{4^{s+1}(s+1)!\,s!} \int_0^1 (1-t)^s t^s D_0^{2(s-1)} f(tx)\,\mathrm{d}t \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+2)!\,(s+1)!} \int_0^1 t^s [(s+2)(1-t)^{s+1} - (1-t)^{s+2}] D_0^{2s} f(tx)\,\mathrm{d}t \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+2)!\,(s+1)!} \int_0^1 [-(1-t)^{s+1} t^{s+1}] D_0^{2s} f(tx)\,\mathrm{d}t \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s x^{2(s+2)}}{4^{s+2}(s+2)!\,s!} \int_0^1 t^s (1-t)^{s+1} D_0^{2s} f(tx)\,\mathrm{d}t,
\end{aligned}
$$

which completes the proof. □

4. Boundary value problems for iterated slice Dirac equations

In [22], the author examined Dirichlet type problems for the iterated slice Dirac equation by integral operators. In this section, we construct generalized Dirichlet problems for inhomogeneous iterated slice Dirac equations by means of a new approach.

4.1. Homogeneous boundary value problems for inhomogeneous iterated slice Dirac equations. In this section, we first consider the following homogeneous boundary value problem for the inhomogeneous iterated slice Dirac equation.

Let f be a Clifford-valued polynomial and let n denote the outward normal to $\partial\Omega$. We look for a function $u \in C^4(\Omega, \mathrm{Cl}_{m+1})$ such that

$$
\begin{cases}
D_0^4 u(x) = f(x), & x \in \Omega, \\
u|_{\partial\Omega} = 0, & \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0.
\end{cases}
\tag{4.1}
$$

In order to obtain the solution of the problem (4.1), we need the following lemma.

Lemma 4.1. *Let P_l be a a Clifford-valued homogeneous polynomial of degree l. Then*

$$
P_l(x) = R_l(x) + x^2 R_{l-2}(x) + \ldots + x^{2j} R_{l-2j}(x),
\tag{4.2}
$$

where $R_{l-2j}(x)$ are homogeneous polynomials of degree $l-2j$ and

$$R_{l-2j}(x)=\frac{(l-2j)!}{4^j j!\,(l-j)!}\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s} D_0^{2(s+j)} P_l(x)}{4^s s!\,(l-2j-s)_s}. \tag{4.3}$$

Here $(a)_s=a(a+1)\dots(a+s-1)$ and $D_0^2 R_{l-2j}(x)=0$.

P r o o f. Using the formulas (3.6) and (3.7), we have

$$P_l(x)=f_0(x)+\sum_{j=1}^{[l/2]}\frac{x^{2j}}{4^j j!\,(j-1)!}\int_0^1(1-\alpha)^{j-1}f_j(\alpha x)\,\mathrm{d}\alpha,$$

$$f_j(x)=D_0^{2j}P_l(x)+\sum_{s=1}^{[l/2-j]}\frac{(-1)^s x^{2s}}{4^s l!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{s-1}D_0^{2(s+j)}P_l(\beta x)\,\mathrm{d}\beta.$$

Then the formula (4.2) can be written as

$$R_{l-2j}(x)=\frac{f_j(x)}{4^j j!\,(j-1)!}\int_0^1(1-\alpha)^{j-1}\alpha^{l-2j}\,\mathrm{d}\alpha=\frac{\mathrm{B}(j,l-2j+1)}{4^j j!\,(j-1)!}f_j(x), \tag{4.4}$$

where $\mathrm{B}(m,n)$ is the Euler beta function. It follows that $f_0(x)=R_l(x)$. Using the formula (3.6), we have

$$\begin{aligned} f_0(x)&=P_l(x)+\sum_{s=1}^{[l/2]}\frac{(-1)^s x^{2s}}{4^s s!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{s-1}D_0^{2s}P_l(\beta x)\,\mathrm{d}\beta\\ &=P_l(x)+\sum_{s=1}^{[l/2]}\frac{(-1)^s x^{2s}D_0^{2s}P_l(x)}{4^s s!\,(s-1)!}\int_0^1(1-\beta)^{s-1}\beta^{l-s-1}\,\mathrm{d}\beta. \end{aligned} \tag{4.5}$$

Applying the relation between the Euler beta function and beta functions, we can continue as follows

$$\begin{aligned} &P_l(x)+\sum_{s=1}^{[l/2]}\frac{(-1)^s x^{2s}D_0^{2s}P_l(x)}{4^s s!\,(s-1)!}\mathrm{B}(s,l-s)\\ &\qquad=P_l(x)+\sum_{s=1}^{[l/2]}\frac{(-1)^s x^{2s}D_0^{2s}P_l(x)}{4^s s!\,(s-1)!}\,\frac{\Gamma(s)\Gamma(l-s)}{\Gamma(l)}\\ &\qquad=P_l(x)+\sum_{s=1}^{[l/2]}\frac{(-1)^s x^{2s}D_0^{2s}P_l(x)}{4^s s!\,(s-1)!}\,\frac{(s-1)!\,(l-s-1)!}{(l-1)!}\\ &\qquad=P_l(x)+\sum_{s=1}^{[l/2]}\frac{(-1)^s x^{2s}D_0^{2s}P_l(x)}{4^s s!\,(l-s)_s}=\sum_{s=0}^{[l/2]}\frac{(-1)^s x^{2s}D_0^{2s}P_l(x)}{4^s s!\,(l-s)_s}. \end{aligned}$$

Therefore,

$$f_j(x) = \sum_{s=0}^{[l/2-j]} \frac{(-1)^s x^{2s} D_0^{2(s+j)} P_l(x)}{4^s s!\,(l-2j-s)_s}.$$

Thus, it follows from (4.4) that

$$R_{l-2j}(x) = \frac{\mathrm{B}(j, l-2j+1)}{4^j j!\,(j-1)!} \sum_{s=0}^{[l/2-j]} \frac{(-1)^s x^{2s} D_0^{2(s+j)} P_l(x)}{4^s s!\,(l-2j-s)_s}.$$

This means that the desired result holds. □

Theorem 4.2. *Let $f(x)$ be an arbitrary Clifford-valued polynomial. Then the solution of the problem* (4.1) *can be written as*

$$u(x) = (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+\alpha x^2)^s (1-\alpha)^{s+1}}{4^{s+2}(s+2)!\,s!} D_0^{2s} f(\alpha x)\,\mathrm{d}\alpha. \tag{4.6}$$

P r o o f. Firstly, we consider the following boundary value problem: For the inhomogeneous iterated slice Dirac equation in the domain Ω, find a function $u(x)$ such that

$$\begin{cases} D_0^4 u(x) = x^{2i} R_{l-2i}(x), \\ u|_{\partial\Omega} = 0, \quad \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \end{cases} \tag{4.7}$$

where the homogenous polynomials $R_{l-2i}(x)$ satisfy the equation $D_0^2 R_{l-2i}(x) = 0$.

Using the formula (3.5), we have

$$\begin{aligned} D_0^4[x^{2(l+2)} P_s(x)] &= D_0^2 D_0^2 [x^{2(l+2)} P_s(x)] \\ &= D_0^2 [x^{2(l+2)} D_0^2 P_s(x) + 4(l+2) x^{2l+2} (E+l+2) P_s(x)] \\ &= 4(l+2)(s+l+2) D_0^2 [x^{2l+2} P_s(x)] \\ &= 4(l+2)(s+l+2)[x^{2l+2} D_0^2 P_s(x) + 4(l+1) x^{2l} (E+l+1) P_s(x)] \\ &= 4(l+2)(s+l+2)[x^{2l+2} D_0^2 P_s(x) + 4(l+1) x^{2l} (s+l+1) P_s(x)] \\ &= 4^2 (l+2)(l+1)(s+l+2)(s+l+1) x^{2l} P_s(x). \end{aligned}$$

This implies that the solution of the equation $D_0^4 u(x) = x^{2l} P_s(x)$ is given by

$$u(x) = \frac{x^{2(l+2)} P_s(x)}{4^2 (l+2)(l+1)(s+l+2)(s+l+1)}$$

with the homogenous polynomial $P_s(x)$ of degree s.

Thus, we can see that

$$g_i(x) = \frac{x^{2i+4} R_{l-2i}(x)}{4^2(i+2)(i+1)(l-i+2)(l-i+1)}$$

is the solution of the equation $D_0^2 u(x) = x^{2i} R_{l-2i}(x)$.

Hence, the solution of the problem (4.7) is

$$u_i(x) = \frac{[x^{2i+4} + (-1)^i(i+1) + (-1)^i(i+2)x^2] R_{l-2i}(x)}{4^2(i+2)(i+1)(l-i+2)(l-i+1)}. \tag{4.8}$$

Then we consider the homogeneous boundary value problem

$$\begin{cases} D_0^4 u(x) = P_l(x), & x \in \Omega, \\ u|_{\partial\Omega} = 0, & \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \end{cases} \tag{4.9}$$

where $P_l(x)$ is a Clifford-valued polynomial of degree l.

Let $u_l(x)$ be the solution of the problem (4.9). Then

$$u_l(x) = \sum_{i=0}^{[l/2]} \frac{[x^{2i+4} + (-1)^i(i+1) + (-1)^i(i+2)x^2] R_{l-2i}(x)}{4^2(i+2)(i+1)(l-i+2)(l-i+1)}.$$

Using the formula (4.3), we have

$$u_l(x) = (x^2+1)^2 \sum_{s=0}^{[l/2]} \frac{D_0^{2s} P_l(x)}{4^{s+2}(s+2)!} \sum_{k=0}^{s} \frac{(s+1)!\, x^{2k}}{k!\,(s-k)!\,(l-2s+k+1)_{s+2}}. \tag{4.10}$$

Since

$$\begin{aligned} \frac{1}{(l-2s+k+1)_{s+2}} &= \frac{1}{(l-2s+k+1)\dots(l-s+k+2)} = \frac{(l-2s+k)!}{(l-s+k+2)!} \\ &= \frac{\mathrm{B}(s+2, l+k-2s+1)}{\Gamma(s+2)} \frac{1}{(s+1)!} \int_0^1 (1-t)^{s+1} t^{l+k-2s}\, \mathrm{d}t, \end{aligned}$$

then

$$\begin{aligned} u_l(x) &= (x^2+1)^2 \sum_{s=0}^{[l/2]} \frac{D_0^{2s} P_l(x)}{4^{s+2}(s+2)!} \int_0^1 (1-t)^{s+1} t^{l+k-2s} x^{2k}\, \mathrm{d}t \\ &= (x^2+1)^2 \sum_{s=0}^{[l/2]} \frac{D_0^{2s} P_l(x)}{4^{s+2}(s+2)!\, s!} \int_0^1 (1-t)^{s+1} t^{l-2s} \sum_{k=0}^{s} \frac{s!}{k!\,(s-k)!} t^k x^{2k}\, \mathrm{d}t \\ &= (x^2+1)^2 \sum_{s=0}^{[l/2]} \frac{1}{4^{s+2}(s+2)!\, s!} \int_0^1 (1-t)^{s+1} (1+tx^2)^s D_0^{2s} P_l(tx)\, \mathrm{d}t. \end{aligned} \tag{4.11}$$

We now turn to the boundary value problem (4.1).

Since $P(x)$ is an arbitrary Clifford-valued polynomial, then

$$P(x) = \sum_l P_l(x),$$

where $P_l(x)$ is a Clifford-valued polynomial of degree l. Let $u(x)$ denote the solution of the problem (4.1). From the formula (4.11) it follows that

$$\begin{aligned} u(x) = \sum_l u_l(x) &= (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+tx^2)^s(1-t)^{s+1}}{4^{s+2}(s+2)!\, s!} D_0^{2s} \sum_l P_l(tx)\, \mathrm{d}t \\ &= (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+tx^2)^s(1-t)^{s+1}}{4^{s+2}(s+2)!\, s!} D_0^{2s} P(tx)\, \mathrm{d}t, \end{aligned}$$

which completes the proof. □

4.2. Inhomogeneous boundary value problems for inhomogeneous iterated slice Dirac equations. In this section, we turn to the problem (1.1).

Theorem 4.3. *The solution of the problem* (1.1) *is given by*

$$\begin{aligned} u(x) = Q(x) &+ \frac{(x^2+1)}{2}[R(x) - EQ(x)] \\ &+ (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+tx^2)^s(1-t)^s}{2\cdot 4^{s+1}(s+1)!\, s!} D_0^{2s}[D_0^2(EQ-R)](tx)\, \mathrm{d}t \\ &- (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+tx^2)^s(1-t)^{s+1}}{4^{s+2}(s+2)!\, s!} D_0^{2s}[P - D_0^2 Q](tx)\, \mathrm{d}t, \end{aligned} \tag{4.12}$$

where E is the Euler operator.

Proof. We first consider the boundary value problem for the iterated slice Dirac equation

$$\begin{cases} D_0^4 u(x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = Q(x), & \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \end{cases} \tag{4.13}$$

with the Clifford-valued polynomial boundary value $Q(x)$.

From Theorem 4.2 it follows that

$$\begin{aligned} u(x) = Q(x) &- \frac{(x^2+1)}{2} EQ(x) \\ &+ (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+tx^2)^s(1-t)^s}{2\cdot 4^{s+1}(s+1)!\, s!} D_0^{2(s+1)}[EQ(tx)]\, \mathrm{d}t \\ &- (x^2+1)^2 \int_0^1 \sum_{s=0}^{\infty} \frac{(1+tx^2)^s(1-t)^{s+1}}{4^{s+2}(s+1)!\, s!} D_0^{2(s+2)} Q(tx)\, \mathrm{d}t \end{aligned} \tag{4.14}$$

is the solution of the problem (4.13). In fact, since $x^2+1=0$, we can prove that the function $u(x)$ in (4.14) satisfies the equation $u|_{\partial\Omega}=Q(x)$. Since $\partial u/\partial n = Eu(x)$, we can prove that the function $u(x)$ in (4.14) satisfies $\partial u/\partial n|_{\partial\Omega}=0$.

Next, we consider the boundary value problem for the iterated slice Dirac equation

$$\begin{cases} D_0^4 u(x)=0, & x\in\Omega,\\ u|_{\partial\Omega}=0, & \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega}=R(x) \end{cases} \tag{4.15}$$

with the Clifford-valued polynomial boundary value $R(x)$. Combining relations (4.6) and (4.16), we have that

$$u(x)=\frac{(x^2+1)}{2}R(x)-(x^2+1)^2\int_0^1\sum_{s=0}^{\infty}\frac{(1+tx^2)^s(1-t)^s}{2\cdot 4^{s+1}(s+1)!\,s!}D_0^{2(s+1)}R(tx)\,\mathrm{d}t \tag{4.16}$$

is the solution of the problem (4.15).

It is easy to see that the solution of the problem (1.1) can be represented as the sum of solutions of the three problems (4.1), (4.13), and (4.15). Combining (4.14) and (4.16), we finally obtain the desired solution (4.12). □

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