Czechoslovak Mathematical Journal

Kamal Bahmanpour

Local cohomology, cofiniteness and homological functors of modules

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 2, 541–558

Persistent URL: `http://dml.cz/dmlcz/150416`

Terms of use:

© Institute of Mathematics AS CR, 2022

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

Czechoslovak Mathematical Journal, 72 (147) (2022), 541–558

LOCAL COHOMOLOGY, COFINITENESS AND HOMOLOGICAL FUNCTORS OF MODULES

Kamal Bahmanpour, Ardabil

Received February 9, 2021. Published online January 14, 2022.

Abstract. Let I be an ideal of a commutative Noetherian ring R. It is shown that the R-modules $H^j_I(M)$ are I-cofinite for all finitely generated R-modules M and all $j \in \mathbb{N}_0$ if and only if the R-modules $\operatorname{Ext}^i_R(N, H^j_I(M))$ and $\operatorname{Tor}^R_i(N, H^j_I(M))$ are I-cofinite for all finitely generated R-modules M, N and all integers $i, j \in \mathbb{N}_0$.

Keywords: cofinite module; cohomological dimension; ideal transform; local cohomology; Noetherian ring

MSC 2020: 13D45, 14B15, 13E05

1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring and I be an ideal of R. In this paper we denote $\operatorname{Supp} R/I = \{\mathfrak{p} \in \operatorname{Spec} R \colon \mathfrak{p} \supseteq I\}$ by $V(I)$. Also, $\mathbb{N}$ (or $\mathbb{N}_0$) will denote the set of positive (or nonnegative) integers. Furthermore, $\mathbb{Z}$ will denote the set of integers.

The ith local cohomology module of an R-module M with support in $V(I)$ is defined as:

$$H^i_I(M) = \varinjlim_{n \geqslant 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [10] or [18] for more details about local cohomology.

For an R-module M, its cohomological dimension with respect to I, denoted by $\operatorname{cd}(I, M)$, is defined as the supremum of all integers i such that $H^i_I(M) \neq 0$. Also let

$$q(I, M) = \sup\{i \in \mathbb{N}_0 \colon H^i_I(M) \text{ is not Artinian}\}$$

with the usual convention that the supremum of the empty set is interpreted as $-\infty$. Several authors have studied these two notions, see [6], [3], [15], [17], [19], [22].

DOI: 10.21136/CMJ.2022.0050-21

© Institute of Mathematics, Czech Academy of Sciences 2022.

Hartshorne in [20] defined an R-module X to be I-cofinite if the support of X is contained in $V(I)$ and $\operatorname{Ext}^i_R(R/I, X)$ is finitely generated for all $i \in \mathbb{N}_0$ and asked the following question:

Question 1.1. *For which Noetherian rings R and ideals J of R, are the modules $H^i_J(M)$ J-cofinite for all finitely generated R-modules M and all $i \in \mathbb{N}_0$?*

In the sequel, $\mathscr{C}(R, I)_{\rm cof}$ denotes the category of all I-cofinite R-modules, and $\mathscr{C}^1(R, I)_{\rm cof}$ denotes the category of all R-modules $M \in \mathscr{C}(R, I)_{\rm cof}$ such that $\dim M \leqslant 1$. Also, throughout this paper, let $\mathscr{I}(R)$ be the class of all ideals I of R such that $H^i_I(M) \in \mathscr{C}(R, I)_{\rm cof}$ for all finitely generated R-modules M and all $i \in \mathbb{N}_0$.

Concerning Question 1.1, there are several interesting results in the literature containing some sufficient conditions for the ideals of R for being in $\mathscr{I}(R)$, see [3]–[5], [8], [12]–[14], [21], [23], [25], [27], [28], [31].

In [7] the author proved that for each ideal I of a Noetherian ring R, $I \in \mathscr{I}(R)$ if and only if $H^i_I(R) \in \mathscr{C}^1(R, I)_{\rm cof}$ for each integer $i \geqslant 2$. Furthermore, he proved that in the case that R is a local ring, the condition $I \in \mathscr{I}(R)$ is equivalent to the condition that for each minimal prime ideal $\mathfrak{P}$ of $\widehat{R}$, $\dim \widehat{R}/(I\widehat{R} + \mathfrak{P}) \leqslant 1$ or $\operatorname{cd}(I\widehat{R}, \widehat{R}/\mathfrak{P}) \leqslant 1$.

Huneke and Koh proved that for each pair of finitely generated modules C and M over a Noetherian local ring $(R, \mathfrak{m})$, under some special conditions, the R-module $\operatorname{Ext}^1_R(C, H^i_I(M))$ is I-cofinite whenever $\dim R/I \leqslant 1$; see [21], Lemmas 4.3 and 4.7. Subsequently, as a generalization of these results in [1] and [29], it was shown that the R-modules $\operatorname{Ext}^i_R(N, M)$ and $\operatorname{Tor}^R_i(N, M)$ belong to $\mathscr{C}^1(R, I)_{\rm cof}$ for all $i \geqslant 0$, whenever N is a finitely generated R-module and $M \in \mathscr{C}^1(R, I)_{\rm cof}$. Furthermore, by [8], Corollary 2.7 we know that for each finitely generated module M over a Noetherian ring R and each ideal I of R with $\dim R/I \leqslant 1$ the R-module $H^j_I(M)$ belongs to $\mathscr{C}^1(R, I)_{\rm cof}$ for each $j \in \mathbb{N}_0$. Consequently, for each ideal I of a Noetherian ring R with $\dim R/I \leqslant 1$ and each pair of finitely generated R-modules M and N, the R-modules $\operatorname{Ext}^i_R(N, H^j_I(M))$ and $\operatorname{Tor}^R_i(N, H^j_I(M))$ belong to $\mathscr{C}^1(R, I)_{\rm cof}$ for all $i, j \in \mathbb{N}_0$. Also, the author in [6], Corollary 2.14 proved that the R-modules $\operatorname{Ext}^i_R(N, H^j_I(M))$ and $\operatorname{Tor}^R_i(N, H^j_I(M))$ are I-cofinite whenever $q(I, R) \leqslant 1$. But, by [8], Corollary 2.7 and [3], Theorem 4.10 we know that under each of the assumptions $\dim R/I \leqslant 1$ or $q(I, R) \leqslant 1$, the ideal I belongs to $\mathscr{I}(R)$.

Pursuing this point of view further for each Noetherian ring R we define $\mathscr{H}(R)$ as the class of all ideals I of R such that

$$\operatorname{Ext}^i_R(N, H^j_I(M)), \operatorname{Tor}^R_i(N, H^j_I(M)) \in \mathscr{C}(R, I)_{\rm cof}$$

for all finitely generated R-modules M, N and all integers $i, j \in \mathbb{N}_0$. Then we establish the equality $\mathscr{H}(R) = \mathscr{I}(R)$.

In [33] Zöschinger introduced an interesting class of minimax modules, and in [33], [34] he has given many equivalent conditions for a module to be minimax. The R-module N is said to be a *minimax module* if there is a finitely generated submodule L of N such that N/L is Artinian. Hence, the class of minimax modules includes all finitely generated and all Artinian modules. It was shown by Zink (see [32]) and by Enochs (see [16]) that a module over a complete local ring is minimax if and only if it is Matlis reflexive. In [24] the authors proved many interesting results concerning the homological properties of this family of modules. It is well known that in a short exact sequence, the middle module is minimax if and only if the two other ones are.

Recall that the *I-transform functor*, denoted by $D_I(-)$, is defined as:

$$D_I(-) = \varinjlim_{n\geqslant 1} \operatorname{Hom}_R(I^n, -).$$

In this paper we prove the following theorem as well:

Theorem 1.2. *Let I be an ideal of a Noetherian ring R with $I \in \mathscr{I}(R)$. Suppose that*

$$X^\circ\colon \ \ldots \to M_{i+2} \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_i} M_i \to \ldots,$$

is an exact sequence of R-modules and R-homomorphisms such that the R-module M_i is minimax for each $i \in \mathbb{Z}$. Then for each $n \in \mathbb{Z}$ the nth homology module of the complex $D_I(X^\circ)$ belongs to $\mathscr{C}^1(R,I)_{\rm cof}$.

Throughout this paper, for each ideal I of a Noetherian ring R and each R-module M, let $\Gamma_I(M)$ be the submodule $\bigcup\limits_{n=1}^{\infty}(0 :_M I^n)$ of M. Also, for any ideal J of R, the *radical of* J is defined to be the set $\operatorname{Rad}(J) = \{x \in R\colon x^n \in J$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology we refer to [10], [11], [26].

2. Preliminaries

In this section we prove some technical results which will be used later. We start this section with some auxiliary lemmas.

Lemma 2.1. *For each ideal I of a Noetherian ring R the following statements are equivalent:*

(i) $I \in \mathscr{I}(R)$.
(ii) $H^i_I(R) \in \mathscr{C}^1(R,I)_{\rm cof}$ *for all integers* $i \geqslant 2$.
(iii) *For each finitely generated R-module M, $H^i_I(M) \in \mathscr{C}^1(R,I)_{\rm cof}$ for all integers $i \geqslant 2$.*
(iv) *For each finitely generated R-module M, $\bigoplus\limits_{i=2}^{\infty} H^i_I(M) \in \mathscr{C}^1(R,I)_{\rm cof}$.*

Proof. (i) $\Leftrightarrow$ (ii) The assertion holds by [7], Theorem 4.10.

(ii) $\Rightarrow$ (iii) Let M be a finitely generated R-module. Then by using localization and applying Theorem 2.2 of [15] it is straightforward to see that

$$\bigcup_{i\geqslant 2} \operatorname{Supp} H_I^i(M) \subseteq \bigcup_{i\geqslant 2} \operatorname{Supp} H_I^i(R).$$

Since by the hypothesis $\dim H_I^i(R) \leqslant 1$ for each integer $i \geqslant 2$, it can be deduced that $\dim H_I^i(M) \leqslant 1$ for each integer $i \geqslant 2$. Furthermore, from the hypothesis $I \in \mathscr{I}(R)$ we obtain that $H_I^i(M) \in \mathscr{C}(R,I)_{\rm cof}$ for all integers $i \geqslant 2$. Consequently, we have $H_I^i(M) \in \mathscr{C}^1(R,I)_{\rm cof}$ for all integers $i \geqslant 2$.

(iii) $\Rightarrow$ (iv) Let M be a finitely generated R-module. Suppose that I can be generated by t elements. Then by Theorem 3.3.1 of [10] we have $H_I^i(M) = 0$ for all integers $i > t$. Hence,

$$\bigoplus_{i=2}^{\infty} H_I^i(M) \simeq \bigoplus_{i=2}^{t} H_I^i(M),$$

which shows that $\bigoplus\limits_{i=2}^{\infty} H_I^i(M) \in \mathscr{C}^1(R,I)_{\rm cof}$.

(iv) $\Rightarrow$ (ii) The assertion is obvious. □

For each ideal I of a Noetherian ring R with $I \in \mathscr{I}(R)$, it follows from the definition that

$$\Omega_R(I) := \operatorname{Supp} \bigoplus_{i\geqslant 2} H_I^i(R) = \bigcup_{i\geqslant 2} \operatorname{Supp} H_I^i(R)$$

is a closed subset of $\operatorname{Spec} R$ under the Zariski topology. We define $I^* := \bigcap\limits_{\mathfrak{p}\in\Omega_R(I)} \mathfrak{p}$. Note that by Lemma 2.1, always one has either $I^* = R$ or $0 \leqslant \dim R/I^* \leqslant 1$. Also, it is easy to see that $\operatorname{Rad}(I) \subseteq I^*$ and $\Omega_R(I) = V(I^*)$. Furthermore, it is clear that $I^* = R$ if and only if $\operatorname{cd}(I,R) \leqslant 1$. In addition, the reader can see that $\dim R/I^* = 0$ (or $\dim R/I^* = 1$) if and only if $\mathfrak{q}(I,R) \leqslant 1 < \operatorname{cd}(I,R)$ (or $\mathfrak{q}(I,R) > 1$).

For each $I \in \mathscr{I}(R)$, let $\mathscr{C}^*(R,I)_{\rm cof}$ be the category of all I-cofinite modules X such that $\operatorname{Supp} X \subseteq V(I^*)$. It is clear that $\mathscr{C}^*(R,I)_{\rm cof}$ is a subcategory of $\mathscr{C}^1(R,I)_{\rm cof}$. Moreover, if I is an ideal of a Noetherian ring R with $I \in \mathscr{I}(R)$, then it follows from the proof of Lemma 2.1 that $H_I^i(M) \in \mathscr{C}^*(R,I)_{\rm cof}$ for each finitely generated R-module M and each integer $i \geqslant 2$.

In the sequel, for each ideal I of a Noetherian ring R with $I \in \mathscr{I}(R)$ let $\mathscr{B}(R,I)$ (or $\mathscr{B}^*(R,I)$), be the category of all R-modules Y such that $H_I^i(Y) \in \mathscr{C}(R,I)_{\rm cof}$ (or $H_I^i(Y) \in \mathscr{C}^*(R,I)_{\rm cof}$) for each $i \in \mathbb{N}_0$. Obviously, by these definitions always $\mathscr{C}^*(R,I)_{\rm cof}$ is a subcategory of $\mathscr{B}^*(R,I)$.

Lemma 2.2. *Let I be an ideal of a Noetherian ring with $I \in \mathscr{I}(R)$. Then for each minimax R-module M, the R-module $D_I(M)$ belongs to $\mathscr{B}^*(R,I)$.*

P r o o f. By [10], Corollary 2.2.8 (iv), one has

$$H^i_I(D_I(M)) = 0 \quad \text{for } i = 0, 1$$

and by [10], Corollary 2.2.8 (v), Lemma 6.3.1, it can be seen that

$$H^i_I(D_I(M)) \simeq H^i_I(M)$$

for all integers $i \geqslant 2$. Furthermore, according to the definition of minimaxness, the R-module M possesses a finitely generated submodule N such that the R-module M/N is Artinian. So, by Grothendieck's Vanishing Theorem we have $H^i_I(M/N) = 0$ for each $i \in \mathbb{N}$. Therefore, the exact sequence

$$0 \to N \to M \to M/N \to 0$$

yields the isomorphism of R-modules $H^i_I(M) \simeq H^i_I(N)$ for each integer $i \geqslant 2$. Hence, $H^i_I(D_I(M)) \simeq H^i_I(N)$ for each integer $i \geqslant 2$. Since for each integer $i \geqslant 2$ the R-module $H^i_I(N)$ belongs to $\mathscr{C}^*(R,I)_{\rm cof}$, it is concluded that $D_I(M) \in \mathscr{B}^*(R,I)$. □

Let I be an ideal of a Noetherian ring R and let $\mathscr{D}(R,I)$ denote the category of all R-modules M with $\operatorname{Ext}^i_R(R/I,M) = 0$ for all $i \in \mathbb{N}_0$. We recall that in view of [30], Proposition 3.2, $\mathscr{D}(R,I)$ also can be defined as the category of all R-modules M with $\operatorname{Tor}^R_i(R/I,M) = 0$ for all $i \in \mathbb{N}_0$. Moreover, by Theorem 2.9 of [2], it can also be defined as the class of all R-modules M such that $H^i_I(M) = 0$ for all $i \in \mathbb{N}_0$.

Lemma 2.3. *Let I be an ideal of a Noetherian ring R with $\operatorname{cd}(I,R) \leqslant 1$, and M be an R-module. Then the following statements are equivalent:*
(i) $M \in \mathscr{D}(R,I)$.
(ii) $\operatorname{Hom}_R(R/I,M) = 0 = \operatorname{Ext}^1_R(R/I,M)$.

P r o o f. The conclusion (i) $\Rightarrow$ (ii) is obvious.
(ii) $\Rightarrow$ (i) Since $\operatorname{Hom}_R(R/I,M) = 0 = \operatorname{Ext}^1_R(R/I,M)$, by using the relation

$$\operatorname{Supp} R/I^n = \operatorname{Supp} R/I,$$

we obtain that $\operatorname{Hom}_R(R/I^n,M) = 0 = \operatorname{Ext}^1_R(R/I^n,M)$ for all $n \in \mathbb{N}$. Therefore, for $i = 0, 1$,

$$H^i_I(M) = \varinjlim_{n \geqslant 1} \operatorname{Ext}^i_R(R/I^n,M) = 0.$$

Moreover, by using the hypothesis $\operatorname{cd}(I,R) \leqslant 1$ and [10], Lemma 6.3.1, we see that $H^i_I(M) = 0$ for all $i \geqslant 2$. So, $H^i_I(M) = 0$ for all $i \in \mathbb{N}_0$, and hence $M \in \mathscr{D}(R,I)$. □

Lemma 2.4. *Let R be a Noetherian ring and I be an ideal of R with $\operatorname{cd}(I,R) \leqslant 1$. Then $\mathscr{D}(R,I)$ is an Abelian category.*

Proof. Let $M, N \in \mathscr{D}(R,I)$ and $f\colon M \to N$ be an R-homomorphism. Set $K := \ker f$, $B := \operatorname{im} f$ and $C := \operatorname{coker} f$. The exact sequence

$$0 \to B \to N$$

induces the exact sequence

$$0 \to \operatorname{Hom}_R(R/I, B) \to \operatorname{Hom}_R(R/I, N).$$

Since $\operatorname{Hom}_R(R/I, N) = 0$, obviously, $\operatorname{Hom}_R(R/I, B) = 0$. Also, from the exact sequence

$$0 \to K \to M \to B \to 0$$

we obtain the exact sequence

$$0 \to \operatorname{Hom}_R(R/I, K) \to \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, B) \\ \to \operatorname{Ext}^1_R(R/I, K) \to \operatorname{Ext}^1_R(R/I, M).$$

By the assumption,

$$\operatorname{Hom}_R(R/I, M) \simeq \operatorname{Hom}_R(R/I, B) \simeq \operatorname{Ext}^1_R(R/I, M) \simeq 0,$$

and therefore, from this exact sequence we arrive at the following relations:

$$\operatorname{Hom}_R(R/I, K) = 0 = \operatorname{Ext}^1_R(R/I, K).$$

Hence, by Lemma 2.3 we see that $K \in \mathscr{D}(R,I)$. Moreover, from the exact sequence

$$0 \to K \to M \to B \to 0$$

and the assumption that $K, M \in \mathscr{D}(R,I)$, we get $B \in \mathscr{D}(R,I)$. Finally, by using the exact sequence

$$0 \to B \to N \to C \to 0$$

and the fact that $B, N \in \mathscr{D}(R,I)$, we see that $C \in \mathscr{D}(R,I)$. This means that $\mathscr{D}(R,I)$ is an Abelian category, as required. □

Lemma 2.5. *Let R be a Noetherian ring and I be an ideal of R with $\operatorname{cd}(I,R) \leqslant 1$. Let*

$$X^\circ\colon \ldots \to X^i \to X^{i+1} \to X^{i+2} \to \ldots$$

be a complex of R-modules and R-homomorphisms such that for all $i \in \mathbb{Z}$, $X^i \in \mathscr{D}(R,I)$. Then $H^i(X^\circ) \in \mathscr{D}(R,I)$ for all $i \in \mathbb{Z}$.

Proof. The assertion follows from Lemma 2.4. □

Lemma 2.6. *Let I be an ideal of a Noetherian ring R with* $\operatorname{cd}(I,R) \leqslant 1$, *$M \in \mathscr{D}(R,I)$, and N be a finitely generated R-module. Then for each $i \in \mathbb{N}_0$, the R-modules* $\operatorname{Ext}_R^i(N,M)$ *and* $\operatorname{Tor}_i^R(N,M)$ *belong to $\mathscr{D}(R,I)$.*

P r o o f. Let

$$\ldots \to F_2 \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0 \xrightarrow{\pi} N \to 0$$

be a free resolution for N such that for each $i \in \mathbb{N}_0$, the R-module F_i has finite rank. Now by calculating the R-modules $\operatorname{Ext}_R^i(N,M)$ and $\operatorname{Tor}_i^R(N,M)$ with this free resolution, one can obtain the assertion from Lemma 2.5. □

Lemma 2.7. *Let I be an ideal of a Noetherian ring R with* $\operatorname{cd}(I,R) \leqslant 1$, *and N be a finitely generated R-module. Then for each R-module M and each $i \in \mathbb{N}_0$, the R-modules* $\operatorname{Ext}_R^i(N,D_I(M))$ *and* $\operatorname{Tor}_i^R(N,D_I(M))$ *are in $\mathscr{D}(R,I)$.*

P r o o f. For each R-module M, by [10], Corollary 2.2.8 (iv) for $i = 0,1$, one has

$$H_I^i(D_I(M)) = 0,$$

and by [10], Corollary 2.2.8 (v), Lemma 6.3.1 for all integers $i \geqslant 2$, we have

$$H_I^i(D_I(M)) \simeq H_I^i(M) = 0.$$

Hence, $D_I(M) \in \mathscr{D}(R,I)$. So, by Lemma 2.6 the R-modules $\operatorname{Ext}_R^i(N,D_I(M))$ and $\operatorname{Tor}_i^R(N,D_I(M))$ belong to $\mathscr{D}(R,I)$ for each $i \in \mathbb{N}_0$. □

Lemma 2.8. *Let I be an ideal of a Noetherian ring R with $I \in \mathscr{I}(R)$ and* $\Gamma_I(R) = 0$. *Then for each finitely generated R-module M, the R-module $M \otimes_R D_I(R)$ belongs to $\mathscr{B}^*(R,I)$.*

P r o o f. Let M be a finitely generated R-module and set $W := M \otimes_R D_I(R)$. From the assumptions $I \in \mathscr{I}(R)$ and $\Gamma_I(R) = 0$ with the proof of Theorem 4.10 of [7], it follows that the R-module $H_I^i(W)$ is I-cofinite for all $i \in \mathbb{N}_0$.

Now, let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \notin V(I^*)$. Then it is clear that

$$H_{IR_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) \simeq (H_I^i(R))_{\mathfrak{p}} = 0 \quad \text{for all } i \geqslant 2.$$

Thus, by using Lemma 2.7, we get the relations

$$(H_I^i(W))_{\mathfrak{p}} \simeq H_{IR_{\mathfrak{p}}}^i(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} D_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})) = 0 \quad \text{for each } i \in \mathbb{N}_0.$$

So, $\operatorname{Supp} H_I^i(W) \subseteq V(I^*)$ for each $i \in \mathbb{N}_0$. Hence, the R-module $W = M \otimes_R D_I(R)$ belongs to $\mathscr{B}^*(R,I)$. □

Lemma 2.9. *Let R be a commutative ring and $f\colon M \to N$, $g\colon N \to L$ be two R-homomorphisms of R-modules. Then there are two exact sequences of R-modules and R-homomorphisms like*

$$0 \to (\ker g + \operatorname{im} f)/\operatorname{im} f \to \operatorname{coker} f \to \operatorname{coker} g \circ f \to \operatorname{coker} g \to 0$$

and

$$0 \to \ker f \to \ker g \circ f \to \ker g \to (\ker g + \operatorname{im} f)/\operatorname{im} f \to 0.$$

Proof. This is straightforward and left to the reader. □

Lemma 2.10 ([9], Theorem 2.7)**.** *For each ideal I of a Noetherian ring R, $\mathscr{C}^1(R, I)_{\rm cof}$ is an Abelian category.*

Lemma 2.11. *Let I be an ideal of a Noetherian ring $R, M \in \mathscr{C}^1(R, I)_{\rm cof}$ and let N be a finitely generated R-module. Then for each $i \in \mathbb{N}_0$, the R-modules $\operatorname{Ext}^i_R(N, M)$, $\operatorname{Tor}^R_i(N, M)$ belong to $\mathscr{C}^1(R, I)_{\rm cof}$.*

Proof. See [1], Theorem 2.7 and [29], Lemma 3.3. □

Lemma 2.12. *Let I be an ideal of a Noetherian ring. Then for each pair of finitely generated R-modules M and N there is an isomorphism of R-modules*

$$\operatorname{Hom}_R(N, D_I(M)) \simeq D_I(\operatorname{Hom}_R(N, M)).$$

Proof. Since by assumption N is a finitely generated R-module, we see that the functor $\operatorname{Hom}_R(N, -)$ commutes with direct limits. Therefore, we have

$$\begin{aligned}\operatorname{Hom}_R(N, D_I(M)) &= \operatorname{Hom}_R(N, \varinjlim_{n\geqslant 1} \operatorname{Hom}_R(I^n, M)) \simeq \varinjlim_{n\geqslant 1} \operatorname{Hom}_R(N, \operatorname{Hom}_R(I^n, M))\\ &\simeq \varinjlim_{n\geqslant 1} \operatorname{Hom}_R(N \otimes_R I^n, M) \simeq \varinjlim_{n\geqslant 1} \operatorname{Hom}_R(I^n, \operatorname{Hom}_R(N, M))\\ &= D_I(\operatorname{Hom}_R(N, M)),\end{aligned}$$

as required. □

Lemma 2.13. *Let I be an ideal of a Noetherian ring R with $I \in \mathscr{I}(R)$. Let*

$$0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$$

be a short exact sequence of R-modules and R-homomorphisms. If two of the R-modules M, N and L are in $\mathscr{B}^*(R, I)$, then the third R-module is in $\mathscr{B}^*(R, I)$ as well.

P r o o f. Since $\dim R/I^* \leqslant 1$, the assertion follows by applying Lemma 2.10 to the long exact sequence

$$0 \to H^0_I(M) \to H^0_I(N) \to H^0_I(L) \to H^1_I(M) \to H^1_I(N) \to H^1_I(L) \to \dots$$

□

3. Main results

The main purpose of this section is to prove the equality $\mathscr{H}(R) = \mathscr{I}(R)$ for each Noetherian ring R. We will prove this result in Theorem 3.3. But first we need the following two lemmas.

Lemma 3.1. *Let I be an ideal of a Noetherian ring with $I \in \mathscr{I}(R)$. Then for each pair of finitely generated R-modules M and N, the R-modules $\mathrm{Hom}_R(N, D_I(M))$ and $D_I(M) \otimes_R N$ are in $\mathscr{B}^*(R, I)$.*

P r o o f. Assume that M and N are two finitely generated R-modules. Since by Lemma 2.13, $\mathrm{Hom}_R(N, D_I(M)) \simeq D_I(\mathrm{Hom}_R(N, M))$ and by Lemma 2.2 we have

$$D_I(\mathrm{Hom}_R(N, M)) \in \mathscr{B}^*(R, I),$$

it follows that $\mathrm{Hom}_R(N, D_I(M)) \in \mathscr{B}^*(R, I)$.

Now, we prove that $D_I(M) \otimes_R N \in \mathscr{B}^*(R, I)$. Since $\Gamma_I(R)M \subseteq \Gamma_I(M)$, it follows that $\Gamma_I(R) \subseteq \mathrm{Ann}_R M/\Gamma_I(M)$. Furthermore, as $D_I(M) \simeq D_I(M/\Gamma_I(M))$, obviously

$$\Gamma_I(R) \subseteq \mathrm{Ann}_R M/\Gamma_I(M) \subseteq \mathrm{Ann}_R D_I(M/\Gamma_I(M)) = \mathrm{Ann}_R D_I(M).$$

Therefore, $R/\Gamma_I(R) \otimes_R D_I(M) \simeq D_I(M)$. Thus,

$$\begin{aligned} N \otimes_R D_I(M) &\simeq N \otimes_R (R/\Gamma_I(R) \otimes_R D_I(M)) \simeq (N \otimes_R R/\Gamma_I(R)) \otimes_R D_I(M) \\ &\simeq N/\Gamma_I(R)N \otimes_R D_I(M) \simeq N/\Gamma_I(R)N \otimes_{R/\Gamma_I(R)} D_I(M). \end{aligned}$$

Hence, by the Independence Theorem, we have

$$H^i_I(N \otimes_R D_I(M)) \simeq H^i_{(I+\Gamma_I(R))/\Gamma_I(R)}(N/\Gamma_I(R)N \otimes_{R/\Gamma_I(R)} D_I(M)) \quad \text{for each } i \in \mathbb{N}_0.$$

By using [14], Proposition 2, we get that $(I + \Gamma_I(R))/\Gamma_I(R) \in \mathscr{I}(R/\Gamma_I(R))$, and also $N \otimes_R D_I(M) \in \mathscr{B}^*(R, I)$ if and only if

$$N/\Gamma_I(R)N \otimes_{R/\Gamma_I(R)} D_I(M) \in \mathscr{B}^*(R/\Gamma_I(R), (I + \Gamma_I(R))/\Gamma_I(R)).$$

By replacing R with $R/\Gamma_I(R)$, we can make the additional assumption that $\Gamma_I(R) = 0$. The exact sequence

$$0 \to R \to D_I(R) \to H^1_I(R) \to 0$$

induces the exact sequence

$$\operatorname{Tor}_1^R(H_I^1(R), M) \to M \xrightarrow{\alpha} D_I(R) \otimes_R M \to H_I^1(R) \otimes_R M \to 0.$$

Since

$$\operatorname{Supp} \ker \alpha \subseteq \operatorname{Supp} \operatorname{Tor}_1^R(H_I^1(R), M) \subseteq \operatorname{Supp} H_I^1(R) \subseteq V(I)$$

and

$$\operatorname{Supp} \operatorname{coker} \alpha = \operatorname{Supp} H_I^1(R) \otimes_R M \subseteq \operatorname{Supp} H_I^1(R) \subseteq V(I),$$

by [10], Proposition 2.2.11 (i) we can deduce that the map $D_I(\alpha)$ is an isomorphism and hence,

$$D_I(D_I(R) \otimes_R M) \simeq D_I(M).$$

By using this relation, the following exact sequence can be obtained

$$0 \to A \to D_I(R) \otimes_R M \xrightarrow{\beta} D_I(M) \to B \to 0, \tag{3.1}$$

where $A := \Gamma_I(D_I(R) \otimes_R M)$, $B := H_I^1(D_I(R) \otimes_R M)$ and $A, B \in \mathscr{C}^*(R, I)_{\mathrm{cof}}$, by Lemma 2.8. From the exact sequence (3.1) we obtain the exact sequence

$$0 \to A \to D_I(R) \otimes_R M \to C \to 0, \tag{3.2}$$

where $C := \operatorname{im} \beta$. Also, this exact sequence yields the short exact sequence

$$A \otimes_R N \xrightarrow{\gamma} D_I(R) \otimes_R (M \otimes_R N) \to C \otimes_R N \to 0.$$

Lemma 2.11 shows that the R-modules $A \otimes_R N$, $B \otimes_R N$ and $\operatorname{Tor}_1^R(B, N)$ are I-cofinite. On the other hand, the R-modules A and B have supports in $V(I^*)$, which yields that these R-modules have supports in $V(I^*)$ likewise. Therefore, the R-modules $A \otimes_R N$, $B \otimes_R N$ and $\operatorname{Tor}_1^R(B, N)$ are in $\mathscr{C}^*(R, I)_{\mathrm{cof}}$. Moreover, as $\operatorname{Supp} \operatorname{im} \gamma \subseteq \operatorname{Supp} A \otimes_R N \subseteq V(I)$, we see that $\operatorname{im} \gamma \subseteq \Gamma_I(D_I(R) \otimes_R (M \otimes_R N))$.

Let $\overline{\gamma}\colon A \otimes_R N \to \Gamma_I(D_I(R) \otimes_R (M \otimes_R N))$ be the map induced by γ. Then it is clear that $\operatorname{im} \overline{\gamma} = \operatorname{im} \gamma$. By Lemma 2.8 we know that $\Gamma_I(D_I(R) \otimes_R (M \otimes_R N)) \in \mathscr{C}^*(R, I)_{\mathrm{cof}}$. Therefore, applying Lemma 2.10 shows that the R-module $\operatorname{im} \overline{\gamma}$ is in $\mathscr{C}^*(R, I)_{\mathrm{cof}}$.

Since $\operatorname{im} \gamma \in \mathscr{C}^*(R, I)_{\mathrm{cof}}$ and by Lemma 2.8, $D_I(R) \otimes_R (M \otimes_R N) \in \mathscr{B}^*(R, I)$, the short exact sequence

$$0 \to \operatorname{im} \gamma \to D_I(R) \otimes_R (M \otimes_R N) \to C \otimes_R N \to 0$$

together with Lemma 2.13 implies that $C \otimes_R N \in \mathscr{B}^*(R, I)$.

Furthermore, from (3.1) we get the short exact sequence

$$0 \to C \to D_I(M) \to B \to 0 \tag{3.3}$$

which induces the exact sequence

$$\operatorname{Tor}_1^R(B,N) \xrightarrow{\lambda} C \otimes_R N \xrightarrow{\mu} D_I(M) \otimes_R N \to B \otimes_R N \to 0.$$

Since $\operatorname{Supp} \operatorname{im} \lambda \subseteq \operatorname{Supp} \operatorname{Tor}_1^R(B,N) \subseteq V(I)$, clearly $\operatorname{im} \lambda \subseteq \Gamma_I(C \otimes_R N)$.

Let $\bar{\lambda}\colon \operatorname{Tor}_1^R(B,N) \to \Gamma_I(C \otimes_R N)$ be the map induced by λ. Hence, obviously $\operatorname{im} \bar{\lambda} = \operatorname{im} \lambda$. Since $C \otimes_R N \in \mathscr{B}^*(R,I)$, by the definition one has $\Gamma_I(C \otimes_R N) \in \mathscr{C}^*(R,I)_{\mathrm{cof}}$. Therefore, Lemma 2.11 yields that the R-module $\operatorname{im} \bar{\lambda}$ is in $\mathscr{C}^*(R,I)_{\mathrm{cof}}$. Since $\operatorname{im} \lambda \in \mathscr{C}^*(R,I)_{\mathrm{cof}}$ and $C \otimes_R N \in \mathscr{B}^*(R,I)$, the exact sequence

$$0 \to \operatorname{im} \lambda \to C \otimes_R N \to \operatorname{im} \mu \to 0$$

together with Lemma 2.13 implies that $\operatorname{im} \mu \in \mathscr{B}^*(R,I)$.

Finally, with the facts that $\operatorname{im} \mu \in \mathscr{B}^*(R,I)$, $B \otimes_R N \in \mathscr{C}^*(R,I)_{\mathrm{cof}}$, and applying Lemma 2.13 on the exact sequence

$$0 \to \operatorname{im} \mu \to D_I(M) \otimes_R N \to B \otimes_R N \to 0,$$

we obtain that $D_I(M) \otimes_R N \in \mathscr{B}^*(R,I)$, as required. □

Lemma 3.2. *Let I be an ideal of a Noetherian ring with $I \in \mathscr{I}(R)$. Then for each pair of finitely generated R-modules M, N and each $i \in \mathbb{N}_0$, the R-modules $\operatorname{Tor}_i^R(N, D_I(M))$ and $\operatorname{Ext}_R^i(N, D_I(M))$ are in $\mathscr{B}^*(R,I)$.*

Proof. We use induction on i. For $i = 0$ the assertion holds by Lemma 3.1. Now, we prove the assertion for $i = 1$. Select an exact sequence

$$0 \to K \to R^n \to N \to 0 \tag{3.4}$$

with $n \in \mathbb{N}_0$. The exact sequence (3.4) yields the exact sequence

$$0 \to \operatorname{Tor}_1^R(N, D_I(M)) \to K \otimes_R D_I(M) \xrightarrow{\alpha} \bigoplus_{i=1}^{n} D_I(M) \to N \otimes_R D_I(M) \to 0,$$

which gives the two following short exact sequences:

$$0 \to \operatorname{im} \alpha \to \bigoplus_{i=1}^{n} D_I(M) \to N \otimes_R D_I(M) \to 0 \tag{3.5}$$

and

$$0 \to \operatorname{Tor}_1^R(N, D_I(M)) \to K \otimes_R D_I(M) \to \operatorname{im} \alpha \to 0. \tag{3.6}$$

By using Lemmas 2.13 and 3.1, from the short exact sequence (3.5) we get $\operatorname{im}\alpha \in \mathscr{B}^*(R,I)$. Also, by using this result, Lemma 2.13 and Lemma 3.1, from the short exact sequence (3.6) we get $\operatorname{Tor}_1^R(N, D_I(M)) \in \mathscr{B}^*(R,I)$.

In addition, from the exact sequence (3.4) we have the exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}_R(N, D_I(M)) \to \bigoplus_{i=1}^{n} D_I(M) \xrightarrow{\beta} \operatorname{Hom}_R(K, D_I(M)) \\ \to \operatorname{Ext}_R^1(N, D_I(M)) \to 0, \end{aligned}$$

which induces the short exact sequences

$$0 \to \operatorname{Hom}_R(N, D_I(M)) \to \bigoplus_{i=1}^{n} D_I(M) \to \operatorname{im}\beta \to 0 \tag{3.7}$$

and

$$0 \to \operatorname{im}\beta \to \operatorname{Hom}_R(K, D_I(M)) \to \operatorname{Ext}_R^1(N, D_I(M)) \to 0. \tag{3.8}$$

By using Lemmas 2.13, 3.1 and the exact sequence (3.7) we get $\operatorname{im}\beta \in \mathscr{B}^*(R,I)$. Using this result together with Lemmas 2.13, 3.1 and the short exact sequence (3.8) shows that $\operatorname{Ext}_R^1(N, D_I(M)) \in \mathscr{B}^*(R,I)$.

Suppose, inductively, that $i > 1$ and the result has been proved for smaller values of $i-1$. Since $i > 1$, the exact sequence (3.4) yields the isomorphism of R-modules

$$\operatorname{Tor}_i^R(N, D_I(M)) \simeq \operatorname{Tor}_{i-1}^R(K, D_I(M)), \quad \operatorname{Ext}_R^i(N, D_I(M)) \simeq \operatorname{Ext}_R^{i-1}(K, D_I(M)).$$

By the inductive hypothesis, the R-modules

$$\operatorname{Tor}_{i-1}^R(K, D_I(M)), \quad \operatorname{Ext}_R^{i-1}(K, D_I(M))$$

are in $\mathscr{B}^*(R,I)$. Thus, $\operatorname{Tor}_i^R(N, D_I(M)), \operatorname{Ext}_R^i(N, D_I(M)) \in \mathscr{B}^*(R,I)$. □

The following theorem is the main result of this paper.

Theorem 3.3. *Suppose that R is a Noetherian ring. Then $\mathscr{H}(R) = \mathscr{I}(R)$.*

Proof. It is clear that $\mathscr{H}(R) \subseteq \mathscr{I}(R)$. In order to prove $\mathscr{I}(R) \subseteq \mathscr{H}(R)$, let $I \in \mathscr{I}(R)$. We prove that $\operatorname{Ext}_R^i(N, H_I^j(M)), \operatorname{Tor}_i^R(N, H_I^j(M)) \in \mathscr{C}(R,I)_{\mathrm{cof}}$ for all finitely generated R-modules M, N and all integers $i, j \in \mathbb{N}_0$.

Since $H_I^0(M) = \Gamma_I(M)$ is a finitely generated R-module with support in $V(I)$, we see that the assertion holds for $j = 0$. Moreover, using the fact that for each integer $j \geqslant 2$, $H_I^j(M) \in \mathscr{C}^1(R,I)_{\mathrm{cof}}$, and applying Lemma 2.11, the assertion will hold for all integers $j \geqslant 2$. Therefore, we must prove the assertion just for the case $j = 1$.

For each $k \in \mathbb{N}_0$, set

$$A_k := \operatorname{Tor}_k^R(N, M/\Gamma_I(M)), \quad B_k := \operatorname{Tor}_k^R(N, D_I(M)), \quad C_k := \operatorname{Tor}_k^R(N, H_I^1(M)).$$

Now, assume that $i \in \mathbb{N}_0$. Then the exact sequence

$$0 \to M/\Gamma_I(M) \to D_I(M) \to H^1_I(M) \to 0$$

induces the exact sequence

$$A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \xrightarrow{\delta_i} A_{i-1}.$$

Since A_i is a finitely generated R-module, we see that $\operatorname{im}\alpha_i$ is a finitely generated R-module. Therefore, the R-module $\Gamma_I(\operatorname{im}\alpha_i)$ is finitely generated and $H^1_I(\operatorname{im}\alpha_i) \in \mathscr{C}(R, I)_{\text{cof}}$. Since $\Gamma_I(\operatorname{im}\beta_i) = \operatorname{im}\beta_i$, the exact sequence

$$0 \to \operatorname{im}\alpha_i \to B_i \to \operatorname{im}\beta_i \to 0$$

yields the exact sequence

$$0 \to \Gamma_I(\operatorname{im}\alpha_i) \to \Gamma_I(B_i) \xrightarrow{f_i} \operatorname{im}\beta_i \xrightarrow{g_i} H^1_I(\operatorname{im}\alpha_i) \to H^1_I(B_i) \to 0.$$

By Lemma 3.2, the R-module $H^1_I(B_i)$ is I-cofinite and hence the exact sequence

$$0 \to \operatorname{im} g_i \to H^1_I(\operatorname{im}\alpha_i) \to H^1_I(B_i) \to 0$$

shows that $\operatorname{im} g_i$ is I-cofinite. Also, by Lemma 3.2, the R-module $\Gamma_I(B_i)$ is I-cofinite. Therefore, by considering the fact that the R-module $\Gamma_I(\operatorname{im}\alpha_i)$ is finitely generated with the exact sequence

$$0 \to \Gamma_I(\operatorname{im}\alpha_i) \to \Gamma_I(B_i) \to \operatorname{im} f_i \to 0$$

we deduce that the R-module $\operatorname{im} f_i$ is I-cofinite. Now, the exact sequence

$$0 \to \operatorname{im} f_i \to \operatorname{im}\beta_i \to \operatorname{im} g_i \to 0$$

shows that $\operatorname{im}\beta_i$ is I-cofinite. Since the R-module A_{i-1} is finitely generated, it follows that $\operatorname{im}\delta_i$ is a finitely generated I-torsion R-module. Hence, $\operatorname{im}\delta_i$ is an I-cofinite R-module. Finally, the exact sequence

$$0 \to \operatorname{im}\beta_i \to C_i \to \operatorname{im}\delta_i \to 0$$

shows that the R-module $C_i = \operatorname{Tor}^R_i(N, H^1_I(M))$ is I-cofinite.

Now, for each $k \in \mathbb{N}_0$, set

$$A'_k := \operatorname{Ext}^k_R(N, M/\Gamma_I(M)), \quad B'_k := \operatorname{Ext}^k_R(N, D_I(M)), \quad C'_k := \operatorname{Ext}^k_R(N, H^1_I(M)).$$

Assume that $i \in \mathbb{N}_0$. Then the exact sequence

$$0 \to M/\Gamma_I(M) \to D_I(M) \to H^1_I(M) \to 0$$

induces the exact sequence

$$A'_i \xrightarrow{\alpha'_i} B'_i \xrightarrow{\beta'_i} C'_i \xrightarrow{\delta'_i} A'_{i+1}.$$

Since A'_i is a finitely generated R-module, we see that $\operatorname{im}\alpha'_i$ is a finitely generated R-module. Therefore, the R-module $\Gamma_I(\operatorname{im}\alpha'_i)$ is finitely generated and $H^1_I(\operatorname{im}\alpha'_i) \in \mathscr{C}(R,I)_{\mathrm{cof}}$. Because $\Gamma_I(\operatorname{im}\beta'_i) = \operatorname{im}\beta'_i$, the exact sequence

$$0 \to \operatorname{im}\alpha'_i \to B'_i \to \operatorname{im}\beta'_i \to 0$$

yields the exact sequence

$$0 \to \Gamma_I(\operatorname{im}\alpha'_i) \to \Gamma_I(B'_i) \xrightarrow{f'_i} \operatorname{im}\beta'_i \xrightarrow{g'_i} H^1_I(\operatorname{im}\alpha'_i) \to H^1_I(B'_i) \to 0.$$

By Lemma 3.2, the R-module $H^1_I(B'_i)$ is I-cofinite and hence the exact sequence

$$0 \to \operatorname{im} g'_i \to H^1_I(\operatorname{im}\alpha'_i) \to H^1_I(B'_i) \to 0$$

shows that $\operatorname{im} g'_i$ is I-cofinite. Also, by Lemma 3.2, the R-module $\Gamma_I(B'_i)$ is I-cofinite.

Since the R-module $\Gamma_I(\operatorname{im}\alpha'_i)$ is finitely generated, the exact sequence

$$0 \to \Gamma_I(\operatorname{im}\alpha'_i) \to \Gamma_I(B'_i) \to \operatorname{im} f'_i \to 0$$

implies that the R-module $\operatorname{im} f'_i$ is I-cofinite. Now, the exact sequence

$$0 \to \operatorname{im} f'_i \to \operatorname{im}\beta'_i \to \operatorname{im} g'_i \to 0$$

shows that $\operatorname{im}\beta'_i$ is I-cofinite. Since the R-module A'_{i+1} is finitely generated, it follows that $\operatorname{im}\delta'_i$ is a finitely generated I-torsion R-module. Thus, $\operatorname{im}\delta'_i$ is an I-cofinite R-module. At the end, the exact sequence

$$0 \to \operatorname{im}\beta'_i \to C'_i \to \operatorname{im}\delta'_i \to 0$$

shows that the R-module $C'_i = \operatorname{Ext}^i_R(N, H^1_I(M))$ is I-cofinite. Therefore, $I \in \mathscr{H}(R)$. □

The following theorem is the final result of this paper.

Theorem 3.4. *Let I be an ideal of a Noetherian ring R with $I \in \mathscr{I}(R)$. Suppose that*

$$X^\circ\colon \ \ldots \to M_{i+2} \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_i} M_i \to \ldots$$

is an exact sequence of R-modules and R-homomorphisms such that the R-module M_i is minimax for each $i \in \mathbb{Z}$. Then for each $n \in \mathbb{Z}$ the nth homology module of the complex $D_I(X^\circ)$ belongs to $\mathscr{C}^*(R,I)_{\mathrm{cof}}$.

P r o o f. Select an element $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \notin V(I^*)$ and set $S := R \setminus \mathfrak{p}$. Then it is clear that $\operatorname{cd}(IR_{\mathfrak{p}}, R_{\mathfrak{p}}) \leqslant 1$ and hence by [10], Lemma 6.3.1 the functor $D_{IR_{\mathfrak{p}}}(-)$ is exact. Therefore, the complex $S^{-1}D_I(X^{\circ})$ is an exact sequence. This observation shows that all homology modules of the complex $D_I(X^{\circ})$ have supports in $V(I^*)$.

Suppose that $n \in \mathbb{Z}$ and for each $i \in \mathbb{Z}$, set $C_i := \operatorname{coker} f_i$. The exact sequence

$$0 \to C_{n+2} \to M_{n+1} \xrightarrow{\widetilde{f}_n} \operatorname{im} f_n \to 0$$

induces the exact sequence

$$0 \to D_I(C_{n+2}) \to D_I(M_{n+1}) \xrightarrow{D_I(\widetilde{f}_n)} D_I(\operatorname{im} f_n) \to H^2_I(C_{n+2}) \xrightarrow{\alpha_{n+1}} H^2_I(M_{n+1}),$$

which shows that $\operatorname{coker} D_I(\widetilde{f}_n) \simeq \ker \alpha_{n+1}$. Since by the hypothesis that the R-module M_{n+1} is minimax, it follows that the R-module C_{n+2} is minimax too. So, by the proof of Lemma 2.2 we see that both of the R-modules $H^2_I(C_{n+2})$ and $H^2_I(M_{n+1})$ are in $\mathscr{C}^*(R,I)_{\mathrm{cof}}$. Hence, applying Lemma 2.10, we can deduce that the R-module $\ker \alpha_{n+1}$ belongs to $\mathscr{C}^*(R,I)_{\mathrm{cof}}$. Thus, $\operatorname{coker} D_I(\widetilde{f}_n) \in \mathscr{C}^*(R,I)_{\mathrm{cof}}$.

On the other hand, the exact sequence

$$0 \to \operatorname{im} f_n \xrightarrow{\iota_n} M_n \to C_n \to 0$$

induces the following exact sequence

$$0 \to D_I(\operatorname{im} f_n) \xrightarrow{D_I(\iota_n)} D_I(M_n) \to D_I(C_n) \to H^2_I(\operatorname{im} f_n) \xrightarrow{\beta_n} H^2_I(M_n).$$

Since by assumption the R-module M_n is minimax, it follows that the R-modules $\operatorname{im} f_n$ and C_n are minimax as well. So, by the proof of Lemma 2.2 it follows that both of the R-modules $H^2_I(\operatorname{im} f_n)$ and $H^2_I(M_n)$ are in $\mathscr{C}^*(R,I)_{\mathrm{cof}}$. Therefore, by using Lemma 2.10 we can deduce that $\ker \beta_n \in \mathscr{C}^*(R,I)_{\mathrm{cof}}$. Furthermore, by Lemma 2.2 we have $D_I(C_n) \in \mathscr{B}^*(R,I)$. Therefore, the exact sequence

$$0 \to \operatorname{coker} D_I(\iota_n) \to D_I(C_n) \to \ker \beta_n \to 0$$

together with Lemma 2.13 imply that $\operatorname{coker} D_I(\iota_n) \in \mathscr{B}^*(R,I)$.

Since $f_n = \iota_n \circ \widetilde{f}_n$, by Lemma 2.9, we have the exact sequence

$$0 \to U \to \operatorname{coker} D_I(\widetilde{f}_n) \to \operatorname{coker} D_I(f_n) \to \operatorname{coker} D_I(\iota_n) \longrightarrow 0,$$

where $U = (\ker D_I(\iota_n) + \operatorname{im} D_I(\widetilde{f}_n))/\operatorname{im} D_I(\widetilde{f}_n)$.

Since $\ker D_I(\iota_n) = 0$, from the last exact sequence we get the short exact sequence

$$0 \to \operatorname{coker} D_I(\widetilde{f}_n) \to \operatorname{coker} D_I(f_n) \to \operatorname{coker} D_I(\iota_n) \to 0.$$

By Lemma 2.13 the last exact sequence shows that $\operatorname{coker} D_I(f_n) \in \mathscr{B}^*(R,I)$. Also, by Lemma 2.2 we have $D_I(M_n) \in \mathscr{B}^*(R,I)$. So, by using Lemma 2.13 from the exact sequence

$$0 \to \operatorname{im} D_I(f_n) \to D_I(M_n) \to \operatorname{coker} D_I(f_n) \to 0,$$

we deduce that $\operatorname{im} D_I(f_n) \in \mathscr{B}^*(R,I)$.

Furthermore, applying the same method, it is concluded that $\operatorname{im} D_I(f_{n-1}) \in \mathscr{B}^*(R,I)$. Moreover, by Lemma 2.2 we have $D_I(M_{n-1}) \in \mathscr{B}^*(R,I)$. Therefore, the exact sequence

$$0 \to \ker D_I(f_{n-1}) \to D_I(M_{n-1}) \to \operatorname{im} D_I(f_{n-1}) \to 0,$$

together with Lemma 2.13 show that $\ker D_I(f_{n-1}) \in \mathscr{B}^*(R,I)$. So, by applying Lemma 2.13 to the exact sequence

$$0 \to \operatorname{im} D_I(f_n) \to \ker D_I(f_{n-1}) \to H_n(D_I(X^\circ)) \to 0,$$

we obtain that $H_n(D_I(X^\circ)) \in \mathscr{B}^*(R,I)$.

Since the R-module $H_n(D_I(X^\circ))$ has support in $V(I^*)$, obviously,

$$\Gamma_I(H_n(D_I(X^\circ))) = H_n(D_I(X^\circ)),$$

and hence the R-module $H_n(D_I(X^\circ))$ belongs to $\mathscr{C}^*(R,I)_{\rm cof}$. □

Acknowledgments. The author is deeply grateful to the referee for very careful reading of the manuscript and many valuable suggestions and for drawing the author's attention to Lemma 2.12. The author also would like to thank Mohaghegh Ardabili University for its financial support.

References

[1] *R. Abazari, K. Bahmanpour*: Cofiniteness of extension functors of cofinite modules. J. Algebra *330* (2011), 507–516. zbl MR doi
[2] *M. Aghapournahr, L. Melkersson*: Local cohomology and Serre subcategories. J. Algebra *320* (2008), 1275–1287. zbl MR doi
[3] *K. Bahmanpour*: Cohomological dimension, cofiniteness and abelian categories of cofinite modules. J. Algebra *484* (2017), 168–197. zbl MR doi
[4] *K. Bahmanpour*: A study of cofiniteness through minimal associated primes. Commun. Algebra *47* (2019), 1327–1347. zbl MR doi
[5] *K. Bahmanpour*: Cofiniteness over Noetherian complete local rings. Commun. Algebra *47* (2019), 4575–4585. zbl MR doi
[6] *K. Bahmanpour*: A note on abelian categories of cofinite modules. Commun. Algebra *48* (2020), 254–262. zbl MR doi

[7] *K. Bahmanpour*: On a question of Hartshorne. Collect. Math. *72* (2021), 527–568. zbl MR doi
[8] *K. Bahmanpour, R. Naghipour*: Cofiniteness of local cohomology modules for ideals of small dimension. J. Algebra *321* (2009), 1997–2011. zbl MR doi
[9] *K. Bahmanpour, R. Naghipour, M. Sedghi*: On the category of cofinite modules which is Abelian. Proc. Am. Math. Soc. *142* (2014), 1101–1107. zbl MR doi
[10] *M. P. Brodmann, R. Y. Sharp*: Local Cohomology: An Algebraic Introduction with Geometric Applications. Cambridge Studies in Advanced Mathematics 60. Cambridge University Press, Cambridge, 1998. zbl MR doi
[11] *W. Bruns, J. Herzog*: Cohen-Macaulay Rings. Cambridge Studies in Advanced Mathematics 39. Cambridge University Press, Cambridge, 1993. zbl MR doi
[12] *G. Chiriacescu*: Cofiniteness of local cohomology modules over regular local rings. Bull. Lond. Math. Soc. *32* (2000), 1–7. zbl MR doi
[13] *D. Delfino*: On the cofiniteness of local cohomology modules. Math. Proc. Camb. Philos. Soc. *115* (1994), 79–84. zbl MR doi
[14] *D. Delfino, T. Marley*: Cofinite modules and local cohomology. J. Pure Appl. Algebra *121* (1997), 45–52. zbl MR doi
[15] *K. Divaani-Aazar, R. Naghipour, M. Tousi*: Cohomological dimension of certain algebraic varieties. Proc. Am. Math. Soc. *130* (2002), 3537–3544. zbl MR doi
[16] *E. Enochs*: Flat covers and flat cotorsion modules. Proc. Am. Math. Soc. *92* (1984), 179–184. zbl MR doi
[17] *G. Faltings*: Über lokale Kohomologiegruppen hoher Ordnung. J. Reine Angew. Math. *313* (1980), 43–51. (In German.) zbl MR doi
[18] *A. Grothendieck*: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2). Advanced Studies in Pure Mathematics (Amsterdam) 2. North-Holland, Amsterdam, 1968. (In French.) zbl MR
[19] *R. Hartshorne*: Cohomological dimension of algebraic varieties. Ann. Math. (2) *88* (1968), 403–450. zbl MR doi
[20] *R. Hartshorne*: Affine duality and cofiniteness. Invent. Math. *9* (1970), 145–164. zbl MR doi
[21] *C. Huneke, J. Koh*: Cofiniteness and vanishing of local cohomology modules. Math. Proc. Camb. Philos. Soc. *110* (1991), 421–429. zbl MR doi
[22] *C. Huneke, G. Lyubeznik*: On the vanishing of local cohomology modules. Invent. Math. *102* (1990), 73–93. zbl MR doi
[23] *K.-I. Kawasaki*: Cofiniteness of local cohomology modules for principal ideals. Bull. Lond. Math. Soc. *30* (1998), 241–246. zbl MR doi
[24] *B. Kubik, M. J. Leamer, S. Sather-Wagstaff*: Homology of artinian and Matlis reflexive modules I. J. Pure Appl. Algebra *215* (2011), 2486–2503. zbl MR doi
[25] *T. Marley, J. C. Vassilev*: Cofiniteness and associated primes of local cohomology modules. J. Algebra *256* (2002), 180–193. zbl MR doi
[26] *H. Matsumura*: Commutative Ring Theory. Cambridge Studies in Advanced Mathematics 8. Cambridge University Press, Cambridge, 1986. zbl MR doi
[27] *L. Melkersson*: Properties of cofinite modules and applications to local cohomology. Math. Proc. Camb. Philos. Soc. *125* (1999), 417–423. zbl MR doi
[28] *L. Melkersson*: Modules cofinite with respect to an ideal. J. Algebra *285* (2005), 649–668. zbl MR doi
[29] *R. Naghipour, K. Bahmanpour, I. Khalili Gorji*: Cofiniteness of torsion functors of cofinite modules. Colloq. Math. *136* (2014), 221–230. zbl MR doi
[30] *G. Pirmohammadi, K. Ahmadi Amoli, K. Bahmanpour*: Some homological properties of ideals with cohomological dimension one. Colloq. Math. *149* (2017), 225–238. zbl MR doi
[31] *K.-I. Yoshida*: Cofiniteness of local cohomology modules for ideals of dimension one. Nagoya Math. J. *147* (1997), 179–191. zbl MR doi

[32] *T. Zink*: Endlichkeitsbedingungen für Moduln über einem Noetherschen Ring. Math. Nachr. *64* (1974), 239–252. (In German.) zbl MR doi
[33] *H. Zöschinger*: Minimax-moduln. J. Algebra *102* (1986), 1–32. (In German.) zbl MR doi
[34] *H. Zöschinger*: Über die Maximalbedingung für radikalvolle Untermoduln. Hokkaido Math. J. *17* (1988), 101–116. (In German.) zbl MR doi

Author's address: Kamal Bahmanpour, Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Daneshgah Street, Ardabil, 56199-11367, Iran, e-mail: `bahmanpour.k@gmail.com`.