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EXPLICIT SOLUTIONS OF INFINITE LINEAR SYSTEMS ASSOCIATED WITH GROUP INVERSE ENDOMORPHISMS

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Abstract. The aim of this note is to offer an algorithm for studying solutions of infinite linear systems associated with group inverse endomorphisms. As particular results, we provide different properties of the group inverse and we characterize EP endomorphisms of arbitrary vector spaces from the coincidence of the group inverse and the Moore-Penrose inverse.

Keywords: infinite linear system; group inverse; Moore-Penrose inverse; EP endomorphism

MSC 2020: 15A06, 15A09, 15A04

1. Introduction

For an arbitrary $(n \times n)$-matrix A with entries in a general field k, the index of A, $i(A) \geqslant 0$, is the smallest integer such that $\operatorname{rk}(A^{i(A)}) = \operatorname{rk}(A^{i(A)+1})$.

If $\operatorname{Mat}_{n\times n}(k)$ is the set of $n \times n$ matrices with entries in k, it is known that, given a matrix $A \in \operatorname{Mat}_{n\times n}(k)$, the system of equations

$$AXA = A, \quad XAX = X, \quad AX = XA$$

has a solution if and only if $i(A) \leqslant 1$ and the solution is unique. This solution is the "group inverse" of A, is denoted by $A^{\#}$ and satisfies several properties, see Subsection 2.2.

This notion can be immediately extended to endomorphisms of finite-dimensional vector spaces over $\mathbb{C}$. Thus, given a finite-dimensional $\mathbb{C}$-vector space E, an endomorphism $f \in \operatorname{End}_{\mathbb{C}} E$ has the index $i(f) \leqslant 1$ ($\operatorname{Im} f^2 = \operatorname{Im} f$) if and only if there

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exists an endomorphism $f^{\#} \in \operatorname{End}_{\mathbb{C}} E$ such that:

$$f \circ f^{\#} \circ f = f, \quad f^{\#} \circ f \circ f^{\#} = f^{\#}, \quad f^{\#} \circ f = f \circ f^{\#}.$$

The endomorphism $f^{\#}$ is the "group inverse" of f. Recently, in [6], the author has extended the notion of group inverse to finite potent endomorphisms of arbitrary vector spaces.

The aim of this note is to offer an algorithm for studying the solutions of infinite linear systems associated with group inverse endomorphisms. The main tools for this algorithm are the explicit expression of the group inverse of a linear map on infinite-dimensional vector spaces offered in Theorem 3.6 and the characterization of the EP endomorphisms obtained from Propositions 4.3 and 4.4.

The paper is organized as follows. For the sake of completeness, in Section 2 we recall the basic definitions and properties of the Drazin inverse, the Moore-Penrose inverse and the group inverse of a square matrix, and of the Moore-Penrose inverse of a linear map on an arbitrary vector space. Section 3 contains the explicit expression of the group inverse of an endomorphism on an infinite-dimensional vector space and various properties of the endomorphisms admissible for the group inverse. Section 4 deals with the determination of the structure of EP endomorphisms from the coincidence of the group inverse and the Moore-Penrose inverse. Finally, Section 5 is devoted to offering the algorithm that allows us to study the solutions of infinite linear systems associated with group inverse endomorphisms.

To finish this introduction, we wish to remark that all the results of this work are valid for infinite matrices associated with group inverse endomorphisms.

2. Preliminaries

2.1. Drazin inverse of $(n \times n)$-matrices. Let A be an $(n \times n)$-matrix with entries in $\mathbb{C}$.

Definition 2.1. The *index of* A, $i(A) \geqslant 0$, is the smallest integer such that $\operatorname{rk}(A^{i(A)}) = \operatorname{rk}(A^{i(A)+1})$.

In 1958, Drazin in [5] showed the existence of a unique $(n \times n)$-matrix A^D satisfying the equations:

- ▷ $A^{k+1}A^D = A^k$ for $k = i(A)$,
- ▷ $A^D A A^D = A^D$,
- ▷ $A^D A = A A^D$.

The Drazin inverse A^D also verifies that

- ▷ $(A^D)^D = A$ if and only if $i(A) \leqslant 1$,
- ▷ if $A^2 = A$, then $A^D = A$.

2.2. Group inverse of a finite matrix. Given a matrix $A \in \mathrm{Mat}_{n\times n}(k)$, the system of equations

$$AXA = A, \quad XAX = X, \quad AX = XA$$

has a solution if and only if $i(A) \leqslant 1$ and the solution is unique. This solution is the "group inverse" of A, is denoted by $A^{\#}$ and coincides with the Drazin inverse A^D.

If $A \in \mathrm{Mat}_{n\times n}(k)$ with $i(A) \leqslant 1$, then the group inverse $A^{\#}$ satisfies the following properties:
▷ If A is nonsingular, then $A^{\#} = A^{-1}$.
▷ $(A^{\#})^{\#} = A$.
▷ $(A^t)^{\#} = (A^{\#})^t$, where A^t is the transpose of A.
▷ If $n \in \mathbb{Z}^+$, then $(A^n)^{\#} = (A^{\#})^n$.
▷ A is EP if and only if $A^{\#} = A^{\dagger}$.

2.3. Moore-Penrose inverse of a linear map over arbitrary vector spaces.

2.3.1. Moore-Penrose inverse of an $(n \times m)$-matrix. Let $\mathbb{C}$ be the field. Given a matrix $A \in \mathrm{Mat}_{n\times m}(\mathbb{C})$, the Moore-Penrose inverse of A is a matrix $A^{\dagger} \in \mathrm{Mat}_{m\times n}(\mathbb{C})$ such that:
▷ $AA^{\dagger}A = A$,
▷ $A^{\dagger}AA^{\dagger} = A^{\dagger}$,
▷ $(AA^{\dagger})^* = AA^{\dagger}$,
▷ $(A^{\dagger}A)^* = A^{\dagger}A$,
and B^* being the conjugate transpose of the matrix B.

The Moore-Penrose inverse of A always exists, it is unique, $[A^{\dagger}]^{\dagger} = A$, and, if $A \in \mathbb{C}^{n\times n}$ is nonsingular, then the Moore-Penrose inverse of A coincides with the inverse matrix A^{-1}.

For details, readers are referred to [3].

2.3.2. Moore-Penrose inverse of a linear map over arbitrary vector spaces. Let (V, g) and $(W, \overline{g})$ be inner product vector spaces over k with $k = \mathbb{C}$ or $k = \mathbb{R}$.

Given a linear map $f\colon V \to W$, the linear map $f^+\colon W \to V$ is a reflexive generalized inverse of f when
▷ $f \circ f^+ \circ f = f$,
▷ $f^+ \circ f \circ f^+ = f^+$.

Definition 2.2. Given a linear map $f\colon V \to W$, we say that f is *admissible for the Moore-Penrose inverse* when $V = \mathrm{Ker}\, f \oplus [\mathrm{Ker}\, f]^{\perp}$ and $W = \mathrm{Im}\, f \oplus [\mathrm{Im}\, f]^{\perp}$.

If (V, g) and $(W, \overline{g})$ are inner product spaces over k and $f\colon V \to W$ is a linear map admissible for the Moore-Penrose inverse; then according to [2], Theorem 3.11, there exists a unique linear map $f^\dagger\colon W \to V$ such that:

(1) $f^\dagger$ is a reflexive generalized inverse of f,

(2) $f^\dagger \circ f$ and $f \circ f^\dagger$ are self-adjoint, that is:

  (a) $g([f^\dagger \circ f](v), v') = g(v, [f^\dagger \circ f](v'))$,

  (b) $\overline{g}([f \circ f^\dagger](w), w') = \overline{g}(w, [f \circ f^\dagger](w'))$

for all $v, v' \in V$ and $w, w' \in W$. The *operator* $f^\dagger$ is named the *Moore-Penrose inverse of* f and is the unique linear map satisfying that

$$f^\dagger(w) = \begin{cases} (f|_{[\operatorname{Ker} f]^\perp})^{-1}(w) & \text{if } w \in \operatorname{Im} f, \\ 0 & \text{if } w \in [\operatorname{Im} f]^\perp. \end{cases} \tag{2.1}$$

The Moore-Penrose inverse $f^\dagger\colon W \to V$ also satisfies the following properties:

▷ $f^\dagger$ is admissible for the Moore-Penrose inverse and $(f^\dagger)^\dagger = f$,

▷ if $f \in \operatorname{End}_k(V)$ and f is an isomorphism, then $f^\dagger = f^{-1}$,

▷ $f^\dagger \circ f = P_{[\operatorname{Ker} f]^\perp}$,

▷ $f \circ f^\dagger = P_{\operatorname{Im} f}$,

where $P_{[\operatorname{Ker} f]^\perp}$ and $P_{\operatorname{Im} f}$ are the projections induced by the decompositions $V = \operatorname{Ker} f \oplus [\operatorname{Ker} f]^\perp$ and $W = \operatorname{Im} f \oplus [\operatorname{Im} f]^\perp$, respectively.

For details on this Moore-Penrose inverse the readers are referred to [2].

3. Group inverse endomorphisms

Let k be a field, let V be an arbitrary vector space (in general, infinite-dimensional) and let $f\colon V \to V$ be an endomorphism.

Definition 3.1. Given an endomorphism $f \in \operatorname{End}_k(V)$, we say that $f^\# \in \operatorname{End}_k(V)$ is a *group inverse of* f when:

▷ $f \circ f^\# \circ f = f$,

▷ $f^\# \circ f \circ f^\# = f^\#$,

▷ $f \circ f^\# = f^\# \circ f$.

Remark 3.2. Note that an endomorphism $f \in \operatorname{End}_k(V)$ satisfies the conditions of Definition 3.1 if and only if

▷ $f^2 \circ f^\# = f$,

▷ $f^\# \circ f \circ f^\# = f^\#$,

▷ $f \circ f^\# = f^\# \circ f$,

which is a generalization of the definition of the Drazin inverse of a square matrix A with $i(A) = 1$, see Subsection 2.1.

The characterization of group inverse endomorphisms on an arbitrary vector space was made in 1968 by Robert. Thus, given an endomorphism $f \in \mathrm{End}_k(V)$, according to [7], Theorem 4, we know that the following conditions are equivalent:

(1) $f^{\#}$ exists,

(2) $V = \operatorname{Ker} f \oplus \operatorname{Im} f$,

(3) $\operatorname{Ker} f = \operatorname{Ker} f^2$ and $\operatorname{Im} f = \operatorname{Im} f^2$.

Moreover, if $f^{\#}$ exists, it follows from [7], Theorem 1 that

▷ $\operatorname{Ker} f = \operatorname{Ker} f^{\#}$ and $\operatorname{Im} f = \operatorname{Im} f^{\#}$,

▷ $(f^{\#})^{\#} = f$,

▷ $f \circ f^{\#}$ is the projection on $\operatorname{Im} f$, along $\operatorname{Ker} f$.

Accordingly, given an endomorphism $f \in \mathrm{End}_k(V)$, we have that f is a group inverse endomorphism if and only if $V = \operatorname{Ker} f \oplus \operatorname{Im} f$.

Remark 3.3. If E is a finite-dimensional k-vector space and $f \in \mathrm{End}_k(E)$, it is known that $E = \operatorname{Ker} f \oplus \operatorname{Im} f$ if and only $\operatorname{Ker} f \cap \operatorname{Im} f = \{0\}$. However, this equivalence is not true for infinite-dimensional vector spaces. Indeed, if we consider the k-vector space $V = \langle v_i \rangle_{i \in \mathbb{N}}$ and the linear map $f\colon V \to V$ defined by:

$$f(v_i) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ v_{i+2} & \text{if } i \text{ is even} \end{cases}$$

for all $i \in \mathbb{N}$, one has that

$$\operatorname{Ker} f = \langle v_{2i-1} \rangle_{i \in \mathbb{N}} \quad \text{and} \quad \operatorname{Im} f = \langle v_{2i+2} \rangle_{i \in \mathbb{N}}.$$

Thus, $\operatorname{Ker} f \cap \operatorname{Im} f = \{0\}$ but $V \neq \operatorname{Ker} f \oplus \operatorname{Im} f$ because $v_2 \notin \operatorname{Ker} f + \operatorname{Im} f$.

Remark 3.4. Similarly to Remark 3.3, if $f \in \mathrm{End}_k(E)$ with E being a finite-dimensional k-vector space, it is clear that $\operatorname{Im} f = \operatorname{Im} f^2$ if and only if $\operatorname{Ker} f = \operatorname{Ker} f^2$, but this property does not hold true for infinite-dimensional vector spaces.

A counter-example is the following: let us consider the vector space $V_{\mathbb{Z}} = \langle v_i \rangle_{i \in \mathbb{Z}}$ over an arbitrary field k and the linear map $f\colon V_{\mathbb{Z}} \to V_{\mathbb{Z}}$ defined by

$$f(v_i) = \begin{cases} v_{i+1} & \text{if } i \leqslant 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Thus, $\operatorname{Im} f = \operatorname{Im} f^2 = \langle v_i \rangle_{i \leqslant 1}$, but $v_0 \in \operatorname{Ker} f^2$ and $v_0 \notin \operatorname{Ker} f$.

The explicit computation of $A^{\#}$ for an $n \times n$-matrix A was given in [7], Section 12. We now provide the explicit expression of $f^{\#}$ for a group inverse endomorphism $f \in \mathrm{End}_k(V)$ on an arbitrary k-vector space V.

Lemma 3.5. *If $f \in \mathrm{End}_k(V)$ is a group inverse endomorphism, then $f_{|\operatorname{Im} f} \in \mathrm{Aut}_k(\operatorname{Im} f)$.*

P r o o f. Consider the linear map $f_{|\operatorname{Im} f}\colon \operatorname{Im} f \to \operatorname{Im} f$. Since $\operatorname{Ker} f \cap \operatorname{Im} f = \{0\}$, then $\operatorname{Ker}(f_{|\operatorname{Im} f}) = \{0\}$ and $f_{|\operatorname{Im} f}$ is injective. Also, if $\overline{v} \in \operatorname{Im} f$ and $\overline{v} = f(v)$, writing

$$v = v_1 + f(v_2) \quad \text{with} \quad v_1 \in \operatorname{Ker} f \quad \text{and} \quad f(v_2) \in \operatorname{Im} f,$$

one has that

$$\overline{v} = f(v) = f(f(v_2))$$

and we conclude that $f_{|\operatorname{Im} f}$ is surjective. Thus, $f_{|\operatorname{Im} f} \in \operatorname{Aut}_k(\operatorname{Im} f)$. □

Theorem 3.6. *If $f \in \operatorname{End}_k(V)$ is a group inverse endomorphism, then $f^{\#}$ is the unique linear map satisfying*

$$f^{\#}(v) = \begin{cases} 0 & \text{if } v \in \operatorname{Ker} f, \\ (f_{|\operatorname{Im} f})^{-1}(v) & \text{if } v \in \operatorname{Im} f. \end{cases} \tag{3.1}$$

P r o o f. Let us assume that $f^{\#}$ exists. Given $\widetilde{v} \in \operatorname{Ker} f$, then

$$f^{\#}(\widetilde{v}) = f^{\#}(f^{\#}(f(\widetilde{v}))) = 0,$$

and if $v \in \operatorname{Im} f$, one has that

$$f(f^{\#}(v)) = v \quad \text{with} \quad f^{\#}(v) \in \operatorname{Im} f \Rightarrow f^{\#}(v) = (f_{|\operatorname{Im} f})^{-1}(v),$$

which makes sense from Lemma 3.5. Hence, $f^{\#}$ is the unique linear map satisfying (3.1).

Accordingly, the statement is proved. □

Given $A \in \operatorname{Mat}_{n\times n}(k)$, note that it follows from Remark 3.3 and Theorem 3.6 that $A^{\#}$ exists if and only if $N(A) \cap R(A) = \{0\}$, which is equivalent to $i(A) \leqslant 1$.

Example 3.7. If $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism with $i(\varphi) \leqslant 1$, it is known that $V = \operatorname{Ker}\varphi \oplus \operatorname{Im}\varphi$, see [6] for details. Thus, every finite potent endomorphism φ with $i(\varphi) \leqslant 1$ is a group inverse endomorphism.

Lemma 3.8. *An endomorphism $f \in \operatorname{End}_k(V)$ is a group inverse endomorphism if and only if $\operatorname{Ker} f = \operatorname{Ker} f^n$ and $\operatorname{Im} f = \operatorname{Im} f^n$ for every $n \in \mathbb{N}$.*

P r o o f. The statement is a direct consequence of [7], Theorem 4 because, given a natural number $n \in \mathbb{N}$, one has that

$$\operatorname{Ker} f = \operatorname{Ker} f^2 \Leftrightarrow \operatorname{Ker} f = \operatorname{Ker} f^n \quad \text{and} \quad \operatorname{Im} f = \operatorname{Im} f^2 \Leftrightarrow \operatorname{Im} f = \operatorname{Im} f^n.$$

□

Corollary 3.9. *If $f \in \mathrm{End}_k(V)$ is a group inverse endomorphism, then f^n is also a group inverse endomorphism for all $n \in \mathbb{N}$.*

P r o o f. The claim is a direct consequence of Lemma 3.8. □

Lemma 3.10. *If $f \in \mathrm{End}_k(V)$ is a group inverse endomorphism, then $(f^n)^{\#} = (f^{\#})^n$ for all $n \in \mathbb{N}$.*

P r o o f. Since $\mathrm{Ker}\, f = \mathrm{Ker}\, f^n$ and $\mathrm{Im}\, f = \mathrm{Im}\, f^n$ for all $n \in \mathbb{N}$ (see Lemma 3.8), one has that $(f^n)^{\#}$ is the unique linear map $(f^n)^{\#} \in \mathrm{End}_k(V)$ satisfying

$$(f^n)^{\#}(v) = \begin{cases} 0 & \text{if } v \in \mathrm{Ker}\, f^n \\ ((f^n)_{|\mathrm{Im}\, f^n})^{-1}(v) & \text{if } v \in \mathrm{Im}\, f^n \end{cases}$$
$$= \begin{cases} 0 & \text{if } v \in \mathrm{Ker}\, f, \\ ((f_{|\mathrm{Im}\, f})^{-1})^n(v) & \text{if } v \in \mathrm{Im}\, f, \end{cases}$$

which coincides with $(f^{\#})^n$. □

Proposition 3.11. *If $f \in \mathrm{End}_k(V)$ is a group inverse endomorphism, and we put $f^{-j} = (f^{\#})^j$ for all $j \in \mathbb{N}$ and $f^0 = f \circ f^{\#}$, one has that*

$$f^h \circ f^s = f^{h+s} \quad \text{for every } h, s \in \mathbb{Z}.$$

Accordingly, the set $\{f^i\}_{i\in\mathbb{Z}}$ is an Abelian group under the composition of linear maps with the identity element f^0.

P r o o f. The statement is a direct consequence of the basic properties of the group inverse of a finite potent endomorphism (see Definition 3.1), the characterization of $f^{\#}$, see Theorem 3.6 and Lemma 3.10. □

Example 3.12. Let $V_{\mathbb{N}}$ be a vector space of countable dimension over an arbitrary field k with the basis $\{v_1, v_2, v_3, \ldots\}$ indexed by natural numbers. If we consider

$$f(v_i) = \begin{cases} 2v_{4j-3} + v_{4j-2} - 4v_{4j} & \text{if } i = 4j-3, \\ 5v_{4j-3} + 3v_{4j-2} - 9v_{4j} & \text{if } i = 4j-2, \\ v_{4j-3} + v_{4j-2} - v_{4j} & \text{if } i = 4j-1, \\ 0 & \text{if } i = 4j \end{cases}$$

for all $j \in \mathbb{N}$, one has that

$$\mathrm{Ker}\, f = \mathrm{Ker}\, f^2 = \langle -2v_{4j-3} + v_{4j-2} - v_{4j-1}, v_{4j} \rangle_{j\in\mathbb{N}}$$

and

$$\mathrm{Im}\, f = \mathrm{Im}\, f^2 = \langle v_{4j-3} - 3v_{4j}, v_{4j-2} + 2v_{4j} \rangle_{j\in\mathbb{N}}.$$

Accordingly, f is a group inverse endomorphism and a nondifficult computation shows that

$$f^{\#}(v_i) = \begin{cases} 3v_{4j-3} - v_{4j-2} - 11v_{4j} & \text{if } i = 4j-3, \\ -5v_{4j-3} + 2v_{4j-2} + 19v_{4j} & \text{if } i = 4j-2, \\ -11v_{4j-3} + 4v_{4j-2} + 41v_{4j} & \text{if } i = 4j-1, \\ 0 & \text{if } i = 4j \end{cases}$$

for all $j \in \mathbb{N}$.

4. Relation of the group inverse with the Moore-Penrose inverse

Henceforth, (V, g) is an inner product vector space over k with $k = \mathbb{C}$ or $k = \mathbb{R}$. This section is devoted to studying the relationship between the group inverse and the Moore-Penrose inverse of an endomorphism $f \in \operatorname{End}_k(V)$.

Recall from Subsection 2.3 that a linear map $f\colon V \to W$ is admissible for the Moore-Penrose inverse when $V = \operatorname{Ker} f \oplus [\operatorname{Ker} f]^{\perp}$ and $W = \operatorname{Im} f \oplus [\operatorname{Im} f]^{\perp}$.

According to [4], Lemma 2.1, the following definition makes sense.

Definition 4.1. We say that an endomorphism $f \in \operatorname{End}_k(V)$ *admissible for the Moore-Penrose inverse is EP* when

$$f \circ f^{\dagger} = f^{\dagger} \circ f.$$

Lemma 4.2. *If $f \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse is EP, then f is a group inverse endomorphism.*

Proof. According to definition of the Moore-Penrose inverse, if f is EP, then

$$f = f^2 \circ f^{\dagger} = f^{\dagger} \circ f^2.$$

Thus, if $v = f(v') \in \operatorname{Im} f$, one has that

$$v = f^2(f^{\dagger}(v')) \in \operatorname{Im} f^2$$

and deduces that $\operatorname{Im} f = \operatorname{Im} f^2$.

Moreover, for every $v \in \operatorname{Ker} f^2$, one has that $f(v) = f^{\dagger}(f^2(v)) = 0$ and, therefore, $\operatorname{Ker} f = \operatorname{Ker} f^2$.

Hence, bearing in mind (see [7], Theorem 4), one concludes that f is a group inverse endomorphism. □

Proposition 4.3. *An endomorphism $f \in \operatorname{End}_k(V)$ admissible for the Moore-Penrose inverse is EP if and only if f is group invertible and $f^{\#} = f^{\dagger}$.*

P r o o f. If $f \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse is EP, since f is admissible for the Moore-Penrose inverse and

- ▷ $f = f \circ f^{\dagger} \circ f$,
- ▷ $f^{\dagger} = f^{\dagger} \circ f \circ f^{\dagger}$,
- ▷ $f^{\dagger} \circ f = f \circ f^{\dagger}$,

it follows from Definition 3.1 that f is group invertible and $f^{\#} = f^{\dagger}$.

Conversely, if f is admissible for the Moore-Penrose inverse, f is group inverse and $f^{\#} = f^{\dagger}$, one has that

$$f \circ f^{\dagger} = f \circ f^{\#} = f^{\#} \circ f = f^{\dagger} \circ f$$

and f is EP. □

Proposition 4.4. *An endomorphism $f \in \mathrm{End}_k(V)$ admissible for the Moore-Penrose inverse is EP if and only if $[\mathrm{Ker}\, f]^{\perp} = \mathrm{Im}\, f$ and $[\mathrm{Im}\, f]^{\perp} = \mathrm{Ker}\, f$.*

P r o o f. If $[\mathrm{Ker}\, f]^{\perp} = \mathrm{Im}\, f$ and $[\mathrm{Im}\, f]^{\perp} = \mathrm{Ker}\, f$, then the explicit expression of the Moore-Penrose inverse (2.1) and the group inverse (3.1) coincide. Hence, $f^{\#} = f^{\dagger}$ and f is EP.

Conversely, if f is EP, it follows from Proposition 4.3 that $f^{\dagger} = f^{\#}$ and, therefore, one has that

$$[\mathrm{Im}\, f]^{\perp} = \mathrm{Ker}\, f^{\dagger} = \mathrm{Ker}\, f^{\#} = \mathrm{Ker}\, f, \quad \mathrm{Im}\, f = \mathrm{Im}\, f^{\#} = \mathrm{Im}\, f^{\dagger} = [\mathrm{Ker}\, f]^{\perp}.$$

Accordingly, the statement is proved. □

5. Infinite linear systems

Finally, we apply the previous results of this work to offer an algorithm that allows us to study the solutions of linear systems associated with group inverse endomorphisms on infinite-dimensional k-vector spaces.

If V is an infinite-dimensional vector space and $f \in \mathrm{End}_k(V)$ is a group inverse endomorphism, a linear system associated with f is an equation

$$f(x) = v, \tag{5.1}$$

where $v \in V$.

The infinite linear system (5.1) is *consistent* when there exists a vector $x_p \in V$ such that $f(x_p) = v$. Indeed, the system (5.1) is consistent if and only if $v \in \mathrm{Im}\, f$. In this case, the vector x_p is named a *particular solution* of this system and the general solution is the set of vectors $\{x_p + \operatorname{Ker} f\} \subset V$.

Since f is a group inverse endomorphism, it is known that:

- ▷ The infinite linear system (5.1) is consistent if and only if $(f \circ f^{\#})(v) = v$.
- ▷ If (5.1) is consistent, then $f^{\#}(v)$ is a particular solution of this system.

Moreover, if (V, g) is an inner product vector space over k, with $k = \mathbb{C}$ or $k = \mathbb{R}$, and f is EP, according to Proposition 4.3 and Proposition 4.4 of [2], one has that $f^{\#}(v)$ is always the minimal least g-norm solution of the system (5.1).

Thus, an algorithm for studying the solutions of the infinite linear system (5.1) is the following:

(1) Compute $\operatorname{Im} f$, $\operatorname{Im} f^2$, $[\operatorname{Im} f]^{\perp}$, $\operatorname{Ker} f$, $\operatorname{Ker} f^2$ and $[\operatorname{Ker} f]^{\perp}$.
(2) Determine whether f is EP from the statement of Proposition 4.4.
(3) Calculate $f^{\#}$ from the explicit expression offered in Theorem 3.6.
(4) Check if $(f \circ f^{\#})(v) = v$ for studying the consistence of the system.
(5) When the system is consistent, compute the general solution

$$x = f^{\#}(v) + \operatorname{Ker} f.$$

(6) If the system is not consistent and f is EP, calculate the minimal least g-norm solution $f^{\#}(v)$.

Example 5.1. Let (V, g) be an inner product vector space of countable dimension over k. Let $\{v_1, v_2, v_3, \dots\}$ be an orthonormal basis of V indexed by the natural numbers. If $(x_i)_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathbb{N}} k$, since $x_i = 0$ for almost all $i \in \mathbb{N}$, we write $x = (x_i)$ to denote the well-defined vector

$$x = \sum_{i\in\mathbb{N}} x_i \cdot v_i \in V.$$

If $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in V$, let us consider the infinite linear system:

$$\begin{aligned} 5x_{6i-5} - 6x_{6i-4} + 22x_{6i-2} + 7x_{6i-1} - 33x_{6i} &= \alpha_{6i-5}, \\ 3x_{6i-5} - 4x_{6i-4} + 16x_{6i-2} + 13x_{6i-1} - 41x_{6i} &= \alpha_{6i-4}, \\ 3x_{6i-3} - 6x_{6i-2} + 8x_{6i-1} - 10x_{6i} &= \alpha_{6i-3}, \\ -4x_{6i-1} + 8x_{6i} &= \alpha_{6i-1}, \\ -4x_{6i-1} + 8x_{6i} &= \alpha_{6i} \end{aligned} \tag{5.2}$$

for all $i \in \mathbb{N}$, which is clearly associated with the group inverse endomorphism

$$f(v_i) = \begin{cases} 5v_i + 3v_{i+1} & \text{if } i = 6j-5, \\ -6v_{i-1} - 4v_i & \text{if } i = 6j-4, \\ 3v_i & \text{if } i = 6j-3, \\ 22v_{i-3} + 16v_{i-2} - 6v_{i-1} & \text{if } i = 6j-2, \\ 7v_{i-4} + 13v_{i-3} + 8v_{i-2} - 4v_i - 4v_{i+1} & \text{if } i = 6j-1, \\ -33v_{i-5} - 41v_{i-4} - 10v_{i-3} + 8v_{i-1} + 8v_i & \text{if } i = 6j, \end{cases}$$

for every $j \in \mathbb{N}$. One can check that

$$\begin{aligned} \operatorname{Ker} f = \operatorname{Ker} f^2 = \langle & 4v_{6i-5} + 7v_{6j-4} + 2v_{6j-3} + v_{6j-2}, \\ & - 7v_{6i-5} - 9v_{6j-4} - 2v_{6j-3} + 2v_{6j-1} + v_{6j} \rangle_{j \in \mathbb{N}} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im} f = \operatorname{Im} f^2 = \langle & 2v_{6j-5} + v_{6j-4}, v_{6j-5} + v_{6j-4}, v_{6j-3}, \\ & - 4v_{6i-5} - 5v_{6j-4} - 2v_{6j-3} + v_{6j-1} + v_{6j} \rangle_{i \in \mathbb{N}}. \end{aligned}$$

Note that $(\operatorname{Ker} f)^{\perp} \neq \operatorname{Im} f$ and $(\operatorname{Im} f)^{\perp} \neq \operatorname{Ker} f$ and, therefore, f is not EP. Hence, from a nondifficult computation, one has that

$$f^{\#}(v_i) = \begin{cases} 2v_i + \frac{3}{2}v_{i+1} & \text{if } i = 6j-5, \\ -3v_{i-1} - \frac{5}{2}v_i & \text{if } i = 6j-4, \\ \frac{1}{3}v_i & \text{if } i = 6j-3, \\ 13v_{i-3} + \frac{23}{2}v_{i-2} - \frac{2}{3}v_{i-1} & \text{if } i = 6j-2, \\ -5v_{i-4} - \frac{17}{4}v_{i-3} + \frac{1}{2}v_{i-2} - \frac{1}{4}v_i - \frac{1}{4}v_{i+1} & \text{if } i = 6j-1, \\ -3v_{i-5} - \frac{7}{2}v_{i-4} - \frac{1}{3}v_{i-3} + \frac{1}{2}v_{i-1} + \frac{1}{2}v_i & \text{if } i = 6j \end{cases}$$

for every $j \in \mathbb{N}$.

Accordingly, one has that

$$(f \circ f^{\#})(v_i) = \begin{cases} v_i & \text{if } i = 6j-5, \\ v_i & \text{if } i = 6j-4, \\ -v_i & \text{if } i = 6j-3, \\ -4v_{i-3} - 7v_{i-2} + 2v_{i-1} & \text{if } i = 6j-2, \\ 7v_{i-4} + 9v_{i-3} + 2v_{i-2} - v_i - v_{i+1} & \text{if } i = 6j-1, \\ -7v_{i-5} - 9v_{i-4} - 2v_{i-3} + 2v_{i-1} + 2v_i & \text{if } i = 6j, \end{cases}$$

for every $j \in \mathbb{N}$ and

$$\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \operatorname{Im} f \Leftrightarrow (f \circ f^{\#})(\alpha) = \alpha \Leftrightarrow \begin{cases} \alpha_{6i-1} = \alpha_{6i}, \\ \alpha_{6i-2} = 0 \end{cases}$$

for all $i \in \mathbb{N}$, and, when $\alpha \in \operatorname{Im} f$, the vector $\beta = (\beta_i)_{i\in\mathbb{N}} \in V$ with

$$\begin{aligned} &\beta_{6i-5} = 2\alpha_{6i-5} - 3\alpha_{6i-4} - 8\alpha_{6i}, && \beta_{6i-4} = \frac{3}{2}\alpha_{6i-5} - \frac{5}{2}\alpha_{6i-4} - \frac{31}{4}\alpha_{6i}, \\ &\beta_{6i-3} = \frac{1}{3}\alpha_{6i-3} + \frac{1}{6}\alpha_{6i}, && \beta_{6i-2} = 0, \\ &\beta_{6i-1} = \frac{1}{4}\alpha_{6i}, && \beta_{6i} = \frac{1}{4}\alpha_{6i} \end{aligned}$$

for all $i \in \mathbb{N}$ is a particular solution of the system (5.2).

Accordingly, when $\alpha \in \operatorname{Im} f$, we have that

$$\begin{aligned} x = \beta + \langle 4v_{6i-5} + 7v_{6j-4} + 2v_{6j-3} + v_{6j-2}, \\ - 7v_{6i-5} - 9v_{6j-4} - 2v_{6j-3} + 2v_{6j-1} + v_{6j}\rangle_{j\in\mathbb{N}} \end{aligned}$$

is the general solution of the infinite system (5.2).

Remark 5.2. We wish to point out that the above method for computing explicit solutions of infinite linear systems is not a particular case of the algorithm offered in [1] because, in general, a group inverse endomorphism is not finite potent.

Example 5.3. With the notation of Example 5.1, if $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in V$, let us now consider the infinite linear system

$$\begin{aligned} &x_{10i} - x_{10i-6} = \alpha_{10i-8}, && x_{10i-2} = \alpha_{10i-6}, && x_{10i-8} = \alpha_{10i-4}, \\ &x_{10i-4} = \alpha_{10i-2}, && x_{10i-6} + x_{10i-2} = \alpha_{10i}, \end{aligned} \tag{5.3}$$

for all $i \in \mathbb{N}$, which is clearly associated with the group inverse endomorphism

$$g(v_i) = \begin{cases} 0 & \text{if } i \text{ is odd}, \\ v_{i+4} & \text{if } i = 10j-8, \\ -v_{i-2} + v_{i+6} & \text{if } i = 10j-6, \\ v_{i+2} & \text{if } i = 10j-4, \\ v_{i-4} + v_{i+2} & \text{if } i = 10j-2, \\ v_{i-8} & \text{if } i = 10j \end{cases}$$

for every $j \in \mathbb{N}$ and where $\operatorname{Ker} g = \langle v_{2i-1}\rangle_{i\in\mathbb{N}}$, $\operatorname{Im} g = \langle v_{2i}\rangle_{i\in\mathbb{N}}$, $(\operatorname{Ker} g)^{\perp} = \operatorname{Im} g$ and $(\operatorname{Im} g)^{\perp} = \operatorname{Ker} g$.

Hence, bearing in mind that the group inverse of g is

$$g^{\#}(v_i)=\begin{cases} 0 & \text{if } i \text{ is odd},\\ v_{i+8} & \text{if } i=10j-8,\\ -v_i+v_{i+4}-v_{i+6} & \text{if } i=10j-6,\\ v_{i-4} & \text{if } i=10j-4,\\ v_{i-2} & \text{if } i=10j-2,\\ v_{i-6}+v_i & \text{if } i=10j \end{cases}$$

for every $j\in\mathbb{N}$, one has that

$$\alpha=(\alpha_i)_{i\in\mathbb{N}}\in\operatorname{Im} g \Leftrightarrow (g\circ g^{\#})(\alpha)=\alpha \Leftrightarrow \alpha_{2i-1}=0 \quad \text{for all } i\in\mathbb{N},$$

and, in this case, the vector $\beta=(\beta_i)_{i\in\mathbb{N}}\in V$ with

$$\begin{aligned} &\beta_{2i-1}=0, &&\beta_{10i-8}=\alpha_{10i-4}, &&\beta_{10i-6}=\alpha_{10i}-\alpha_{10i-6},\\ &\beta_{10i-4}=\alpha_{10i-2}, &&\beta_{10i-2}=\alpha_{10i-6}, &&\beta_{10i}=\alpha_{10i-8}-\alpha_{10i-6}+\alpha_{10i} \end{aligned}$$

for all $i\in\mathbb{N}$ is a particular solution of the system (5.3).

Thus, when $\alpha\in\operatorname{Im} g$, the general solution of the system (5.3) is

$$x=\beta+\langle v_{2i-1}\rangle_{i\in\mathbb{N}}.$$

Also, if $\alpha\notin\operatorname{Im} g$ and this system is not consistent, since g is EP, then β is the minimal least g-norm solution of (5.3).

References

[1] *V. Cabezas Sánchez, F. Pablos Romo*: Explicit solutions of infinite systems of linear equations from reflexive generalized inverses of finite potent endomorphisms. Linear Algebra Appl. *559* (2018), 125–144. zbl MR doi

[2] *V. Cabezas Sánchez, F. Pablos Romo*: Moore-Penrose inverse of some linear maps on infinite-dimensional vector spaces. Electron. J. Linear Algebra *36* (2020), 570–586. zbl MR doi

[3] *S. L. Campbell, C. D. Meyer, Jr.*: Generalized Inverses of Linear Transformations. Classics in Applied Mathematics 56. SIAM, Philadelphia, 2009. zbl MR doi

[4] *S. Cheng, Y. Tian*: Two sets of new characterizations for normal and EP matrices. Linear Algebra Appl. *375* (2003), 181–195. zbl MR doi

[5] *M. P. Drazin*: Pseudo-inverses in associative rings and semigroups. Am. Math. Mon. *65* (1958), 506–514. zbl MR doi

[6] *F. Pablos Romo*: Group inverse of finite potent endomorphisms on arbitrary vector spaces. Oper. Matrices *14* (2020), 1029–1042. zbl MR doi

[7] *P. Robert*: On the group-inverse of a linear transformation. J. Math. Anal. Appl. *22* (1968), 658–669. zbl MR doi

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