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ON SUMS AND PRODUCTS IN A FIELD

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Abstract. We study sums and products in a field. Let F be a field with $\mathrm{ch}(F) \neq 2$, where $\mathrm{ch}(F)$ is the characteristic of F. For any integer $k \geqslant 4$, we show that any $x \in F$ can be written as $a_1 + \ldots + a_k$ with $a_1, \ldots, a_k \in F$ and $a_1 \ldots a_k = 1$, and that for any $\alpha \in F \setminus \{0\}$ we can write every $x \in F$ as $a_1 \ldots a_k$ with $a_1, \ldots, a_k \in F$ and $a_1 + \ldots + a_k = \alpha$. We also prove that for any $x \in F$ and $k \in \{2, 3, \ldots\}$ there are $a_1, \ldots, a_{2k} \in F$ such that $a_1 + \ldots + a_{2k} = x = a_1 \ldots a_{2k}$.

Keywords: field; rational function; restricted sum; restricted product

MSC 2020: 11D85, 11P99, 11T99

1. Introduction

Let $\mathbb{Q}$ be the field of rational numbers. In 1749 Euler showed that any $q \in \mathbb{Q}$ can be written as $abc(a+b+c)$ with $a, b, c \in \mathbb{Q}$; equivalently, we can always write $x = -q \in \mathbb{Q}$ as $abcd$ with $a, b, c, d \in \mathbb{Q}$ and $a+b+c+d = 0$. Actually, Euler noted that the equation $abc(a+b+c) = q$ has the rational parameter solutions

$$\begin{aligned} a &= \frac{6qst^3(qt^4 - 2s^4)^2}{(4qt^4 + s^4)(2q^2t^8 + 10qs^4t^4 - s^8)}, \\ b &= \frac{3s^5(4qt^4 + s^4)^2}{2t(qt^4 - 2s^4)(2q^2t^8 + 10qs^4t^4 - s^8)}, \\ c &= \frac{2(2q^2t^8 + 10qs^4t^4 - s^8)}{3s^3t(4qt^4 + s^4)}. \end{aligned}$$

The reader may consult Elkies's talk (see [1]) for a nice exposition of this curious discovery of Euler and its connection to modern topics like $K3$ surfaces. Elkies in [1]

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found that $abcd = x$ with $a + b + c + d = 0$, where

$$\begin{aligned} a =& \frac{(s^4+4x)^2}{2s^3(s^4-12x)}, \quad & b =& \frac{2x(3s^4-4x)^2}{s^3(s^4+4x)(s^4-12x)}, \\ c =& \frac{s(s^4-12x)}{2(3s^4-4x)}, \quad & d =& -\frac{2s^5(s^4-12x)}{(s^4+4x)(3s^4-4x)}. \end{aligned}$$

Let F be a field. If $x = a_1 \dots a_k$ with $a_1, \dots, a_k \in F$ and $a_1 + \dots + a_k = 0$, then $a_1 \dots a_k$ is called a *balanced decomposition* of x by Klyachko and Vassilyev, see [3]. Unaware of Euler's above work in 1749, Klyachko and Vassilyev in [3] showed that if $\mathrm{ch}(F)$ (the characteristic of F) is not two then for each $k = 5, 6, \dots$, every $x \in F$ has a balanced decomposition $a_1 \dots a_k$ with $a_1, \dots, a_k \in F$ and $a_1 + \dots + a_k = 0$. When F is a finite field and $k > 1$ is an integer, they are determined completely when any $x \in F$ can be written as $a_1 \dots a_k$ with $a_1, \dots, a_k \in F$ and $a_1 + \dots + a_k = 0$. In 2016, Klyachko, Mazhuga and Ponfilenko in [2] proved that if $\mathrm{ch}(F) \neq 2, 3$ and $|F| \neq 5$ then any $x \in F$ has a balanced decomposition $a_1a_2a_3a_4$ with $a_1, a_2, a_3, a_4 \in F$ and $a_1 + a_2 + a_3 + a_4 = 0$; in fact, for $x \in F \setminus \{\frac{1}{4}, -\frac{1}{8}\}$ they found that $a(x)b(x)c(x)d(x) = x$ and $a(x) + b(x) + c(x) + d(x) = 0$, where

$$\begin{aligned} a(x) =& \frac{2(1-4x)^2}{3(1+8x)}, \quad & b(x) =& -\frac{1+8x}{6}, \\ c(x) =& -\frac{1+8x}{2(1-4x)}, \quad & d(x) =& \frac{18x}{(1-4x)(1+8x)}. \end{aligned}$$

This is much simpler than Euler's and Elkies' rational parameter solutions to the equation $abcd = x$ with the restriction $a + b + c + d = 0$.

2. Main results and proof

Motivated by the above work, we obtain the following new results.

Theorem 2.1. *Let F be any field with $\mathrm{ch}(F) \neq 2$, and let $\alpha \in F \setminus \{0\}$ and $k \in \{4, 5, \dots\}$. Then every $x \in F$ can be written as $a_1 \dots a_k$ with $a_1, \dots, a_k \in F$ and $a_1 + \dots + a_k = \alpha$.*

Remark 2.1. This theorem with $\alpha = 1$ implies that for any $x \in \mathbb{Q}$ and $k \in \{4, 5, \dots\}$ there are $a_1, \dots, a_k \in \mathbb{Q}$ such that $a_1 \dots a_k(a_1 + \dots + a_k) = x$, this extension of Euler's work was asked by van der Zypen, see [4].

P r o o f of Theorem 2.1. We distinguish three cases.

Case 1: $k = 4$. If every $q \in F$ can be written as $abcd$ with $a, b, c, d \in F$ and $a+b+c+d = 1$, then for any $x \in F$ we can write $x/\alpha^4 = abcd$ with $a, b, c, d \in F$ and $a+b+c+d = 1$, and hence $x = (a\alpha)(b\alpha)(c\alpha)(d\alpha)$ with $a\alpha+b\alpha+c\alpha+d\alpha = \alpha$. So, it suffices to work with $\alpha = 1$.

Let $x \in F$ with $x \neq \pm 1$. Put

$$a(x) = -\frac{(1-x)^2}{2(1+x)}, \quad b(x) = \frac{1+x}{2}, \quad c(x) = \frac{1+x}{1-x}, \quad d(x) = \frac{4x}{x^2-1}.$$

It is easy to verify that

$$a(x)b(x)c(x)d(x) = x \quad \text{and} \quad a(x)+b(x)+c(x)+d(x) = 1.$$

For $x = -1$, we note that

$$-1 = \frac{1}{2} \times \frac{1}{2} \times 2 \times (-2) \quad \text{with } \frac{1}{2} + \frac{1}{2} + 2 - 2 = 1.$$

For $x = 1$, if $\mathrm{ch}(F) = 3$ then

$$1 = 1 \times 1 \times 1 \times 1 \quad \text{with } 1+1+1+1 = 1$$

if $\mathrm{ch}(F) \neq 3$ then

$$1 = \frac{3}{2} \times \Big(-\frac{3}{2}\Big) \times \Big(-\frac{1}{3}\Big) \times \frac{4}{3} \quad \text{with } \frac{3}{2} - \frac{3}{2} - \frac{1}{3} + \frac{4}{3} = 1.$$

This proves Theorem 2.1 for $k = 4$.

Case 2: $k = 5$. As $\mathrm{ch}(F) \neq 2$, we have $\alpha - \varepsilon \neq 0$ for some $\varepsilon \in \{\pm 1\}$. Let $x \in F$. By Theorem 2.1 for $k = 4$, we can write εx as $abcd$ with $a, b, c, d \in F$ and $a+b+c+d = \alpha - \varepsilon$. Hence, $x = abcd\varepsilon$ with $a+b+c+d+\varepsilon = \alpha$. So Theorem 2.1 also holds for $k = 5$.

Case 3: $k \geqslant 6$. Let $x \in F$. If k is even, then by Theorem 2.1 for $k = 4$ there are $a, b, c, d \in F$ with $a+b+c+d = \alpha$ such that $abcd = (-1)^{(k-4)/2}x$, hence

$$x = abcd \times 1^{(k-4)/2} \times (-1)^{(k-4)/2} \quad \text{with } a+b+c+d+\frac{k-4}{2}(1-1) = \alpha.$$

When k is odd, by Theorem 2.1 for $k = 5$ there are $a, b, c, d, e \in F$ with $a+b+c+d+e = \alpha$ such that $abcde = (-1)^{(k-5)/2}x$, hence

$$x = abcde \times 1^{(k-5)/2} \times (-1)^{(k-5)/2} \quad \text{with } a+b+c+d+e+\frac{k-5}{2}(1-1) = \alpha.$$

Combining the above, we have completed the proof of Theorem 2.1. $\square$

Theorem 2.2. *Let F be a field with $\mathrm{ch}(F) \neq 2$ and let $k \geqslant 4$ be an integer.*

(i) *If $\mathrm{ch}(F) \neq 3$, then any $x \in F$ can be written as $a_1 + \ldots + a_k$ with $a_1, \ldots, a_k \in F$ and $a_1 \ldots a_k = -1$.*

(ii) *Any $x \in F$ can be written as $a_1 + \ldots + a_k$ with $a_1, \ldots, a_k \in F$ and $a_1 \ldots a_k = 1$.*

Remark 2.2. It seems that there are no $a, b, c \in \mathbb{Q}$ with $a + b + c = 1 = abc$.

Let F be any field and k be a positive integer. Clearly, any $x \in F$ can be written as $a_1 + \ldots + a_{2k+1}$ with $a_1, \ldots, a_{2k+1} \in F$ and $a_1 \ldots a_{2k+1} = (-1)^k x$; in fact, $x + k(1-1) = x$ and $x \times 1^k \times (-1)^k = (-1)^k x$. If $a^2 = -1$ for some $a \in F$, then any $x \in F$ can be written as $a_1 + \ldots + a_{2k+1}$ with $a_1, \ldots, a_{2k+1} \in F$ and $a_1 \ldots a_{2k+1} = (-1)^{k-1} x$; in fact, $x + (a - a) + (k-1)(1-1) = x$ and

$$x \times a \times (-a) \times 1^{k-1} \times (-1)^{k-1} = (-1)^{k-1} x.$$

P r o o f of Theorem 2.2. For any $m \in \mathbb{Z}$, if every $x \in F$ can be written as $a + b + c + d$ with $a, b, c, d \in F$ and $abcd = m$, then for any $x \in F$ and $k \in \{4, 5, \ldots\}$ there are $a_1, a_2, a_3, a_4 \in F$ such that $a_1 + a_2 + a_3 + a_4 = x - (k-4)$ and $a_1 a_2 a_3 a_4 = m$, hence $a_1 + \ldots + a_k = x$ and $a_1 \ldots a_k = m$, where $a_j = 1$ for $4 < j \leqslant k$. Thus, it suffices to show parts (i) and (ii) in the case $k = 4$.

(i) For $x \in F \setminus \{-1, -3\}$, we put

$$a(x) = \frac{(x+1)^2}{2(x+3)}, \quad b(x) = \frac{x+3}{2}, \quad c(x) = -\frac{x+3}{x+1}, \quad d(x) = \frac{4}{(x+1)(x+3)},$$

and it is easy to verify that

$$a(x)b(x)c(x)d(x) = -1 \quad \text{and} \quad a(x) + b(x) + c(x) + d(x) = x.$$

Observe that

$$-1 = 2 - 2 - \frac{1}{2} - \frac{1}{2} \quad \text{with } 2 \times (-2) \times \Bigl(-\frac{1}{2}\Bigr) \times \Bigl(-\frac{1}{2}\Bigr) = -1.$$

If $\mathrm{ch}(F) \neq 3$, then

$$-3 = \frac{2}{3} - \frac{2}{3} - \frac{3}{2} - \frac{3}{2} \quad \text{with } \frac{2}{3} \times \Bigl(-\frac{2}{3}\Bigr) \times \Bigl(-\frac{3}{2}\Bigr) \times \Bigl(-\frac{3}{2}\Bigr) = -1.$$

This concludes the proof of Theorem 2.2 (i).

(ii) If $x \in F \setminus \{0, \pm 1\}$, then it is easy to verify that

$$\frac{1-x^2}{x^2} + \frac{x^2-1}{x^2} + \frac{x}{1-x^2} + \frac{x^3}{x^2-1} = x \quad \text{and} \quad \frac{1-x^2}{x^2} \times \frac{x^2-1}{x^2} \times \frac{x}{1-x^2} \times \frac{x^3}{x^2-1} = 1.$$

Note that $0 = 1 + 1 - 1 - 1$ and $1 \times 1 \times (-1) \times (-1) = 1$. If $\mathrm{ch}(F) = 3$, then $1 + 1 + 1 + 1 = 1$ and $1 \times 1 \times 1 \times 1 = 1$, and also $-1 - 1 - 1 - 1 = -1$ and $(-1)(-1)(-1)(-1) = 1$. If $\mathrm{ch}(F) \neq 3$, then

$$\frac{3}{2} - \frac{3}{2} - \frac{1}{3} + \frac{4}{3} = 1 \quad \text{and} \quad \frac{3}{2} \times \Big(-\frac{3}{2}\Big) \times \Big(-\frac{1}{3}\Big) \times \frac{4}{3} = 1,$$

and also

$$\frac{3}{2} - \frac{3}{2} + \frac{1}{3} - \frac{4}{3} = -1 \quad \text{and} \quad \frac{3}{2} \times \Big(-\frac{3}{2}\Big) \times \frac{1}{3} \times \Big(-\frac{4}{3}\Big) = 1.$$

So Theorem 2.2 (ii) also holds.

In view of the above, the proof of Theorem 2.2 is now complete. $\square$

Theorem 2.3. *Let F be a field with $\mathrm{ch}(F) \neq 2$ and let $k \geqslant 2$ be an integer. Then for any $x \in F$ there are $a_1, \ldots, a_{2k} \in F$ such that $a_1 + \ldots + a_{2k} = x = a_1 \ldots a_{2k}$.*

Remark 2.3. If F is a field with $\mathrm{ch}(F) \neq 2$ and $k \geqslant 2$ is an integer, then for any $x \in F$, by Theorem 2.3 there are $a_1, \ldots, a_{2k} \in F$ with $a_1 + \ldots + a_{2k} = -x = a_1 \ldots a_{2k}$, hence

$$(-a_1) + \ldots + (-a_{2k}) = x \quad \text{and} \quad (-a_1) \ldots (-a_{2k}) = -x.$$

Proof of Theorem 2.3. We first handle the case $k = 2$. For $x \in F \setminus \{\pm 1\}$, we put

$$a(x) = \frac{(x+1)^2}{2(x-1)}, \quad b(x) = \frac{x-1}{2}, \quad c(x) = \frac{1-x}{1+x}, \quad d(x) = \frac{4x}{1-x^2}$$

and it is easy to verify that

$$a(x)b(x)c(x)d(x) = x = a(x) + b(x) + c(x) + d(x).$$

Clearly,

$$-1 = -\frac{1}{2} - \frac{1}{2} + 2 - 2 \quad \text{with} \ -\frac{1}{2} \times \Big(-\frac{1}{2}\Big) \times 2 \times (-2) = -1.$$

If $\mathrm{ch}(F) \neq 3$, then

$$1 = \frac{3}{2} - \frac{3}{2} - \frac{1}{3} + \frac{4}{3} \quad \text{with} \ \frac{3}{2} \times \Big(-\frac{3}{2}\Big) \times \Big(-\frac{1}{3}\Big) \times \frac{4}{3} = 1.$$

When $\mathrm{ch}(F) = 3$, we have

$$1 = 1 + 1 + 1 + 1 \quad \text{with} \ 1 \times 1 \times 1 \times 1 = 1.$$

This proves Theorem 2.3 for $k = 2$.

Now we consider the case $k \geqslant 3$. By Theorem 2.3 for $k = 2$, there are $a, b, c, d \in F$ such that $a + b + c + d = (-1)^k x = abcd$. Thus,

$$(-1)^k a + (-1)^k b + (-1)^k c + (-1)^k d + (k-2)(1-1) = x$$

and

$$(-1)^k a \times (-1)^k b \times (-1)^k c \times (-1)^k d \times 1^{k-2} \times (-1)^{k-2} = abcd(-1)^k = x.$$

This proves Theorem 2.3 for $k \geqslant 3$.

By the above, we have completed the proof of Theorem 2.3. $\square$

Motivated by Theorem 2.3 and Remark 2.3, we propose the following conjecture based on our computation.

Conjecture 2.1. *Let F be any field with $\mathrm{ch}(F) \neq 2, 3$. Then, for any $x \in F$ there are $a, b, c, d \in F$ such that $a + b + c + d - 1 = x = abcd$.*

For example, in any field F with $\mathrm{ch}(F) \neq 2, 3$, we have

$$-2 + \frac{9}{2} - \frac{2}{3} + \frac{1}{6} - 1 = 1 = (-2) \times \frac{9}{2} \times \left(-\frac{2}{3}\right) \times \frac{1}{6}.$$

Motivated by our proof of Theorem 2.2 (ii), we obtain the following result.

Theorem 2.4. *Let F be a field with $\mathrm{ch}(F) \neq 2$ and let m be any nonzero integer. Then any $x \in F \setminus \{0\}$ can be written as $a+b+c+d$ with $a, b, c, d \in F$ and $abcd = x^m$.*

Proof. If m is odd and $x \in F \setminus \{0, 1\}$, then, for $n = \frac{1}{2}(3 - m)$ we have

$$\frac{1-x}{x^n} + \frac{x-1}{x^n} + \frac{x}{1-x} + \frac{x^2}{x-1} = x$$

and

$$\frac{1-x}{x^n} \times \frac{x-1}{x^n} \times \frac{x}{1-x} \times \frac{x^2}{x-1} = x^{3-2n} = x^m.$$

If m is even and $x \in F \setminus \{0, \pm 1\}$, then, for $n = \frac{1}{2}(4 - m)$ we have

$$\frac{1-x^2}{x^n} + \frac{x^2-1}{x^n} + \frac{x}{1-x^2} + \frac{x^3}{x^2-1} = x$$

and

$$\frac{1-x^2}{x^n} \times \frac{x^2-1}{x^n} \times \frac{x}{1-x^2} \times \frac{x^3}{x^2-1} = x^{4-2n} = x^m.$$

As in the proof of Theorem 2.2 (ii), any $x \in \{\pm 1\}$ can be written as $a + b + c + d$ with $a, b, c, d \in F$ and $abcd = 1$. So we have the desired result. $\square$

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