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(GENERALIZED) FILTER PROPERTIES OF THE AMALGAMATED ALGEBRA

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Abstract. Let $R$ and $S$ be commutative rings with unity, $f: R \to S$ a ring homomorphism and $J$ an ideal of $S$. Then the subring $R \bowtie_f J := \{(a, f(a) + j) \mid a \in R \text{ and } j \in J\}$ of $R \times S$ is called the amalgamation of $R$ with $S$ along $J$ with respect to $f$. In this paper, we determine when $R \bowtie_f J$ is a (generalized) filter ring.

1. Introduction

Throughout this paper, let $R$ and $S$ be two commutative rings with identity, $J$ be a non-zero proper ideal of $S$, and $f: R \to S$ be a ring homomorphism.

D’Anna, Finocchiaro, and Fontana in [10] and [11] have introduced the following subring (with standard component-wise operations)

$$R \bowtie_f J := \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\}$$

of $R \times S$, called the amalgamated algebra (or amalgamation) of $R$ with $S$ along $J$ with respect to $f$. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [13]). Moreover, several classical constructions such as Nagata’s idealization (cf. [16, page 2]), the $R + XS[X]$ and the $R + XS[X]$ constructions can be studied as particular cases of this construction (see [10, Example 2.5 and Remark 2.8]). Recently, many properties of amalgamations investigated in several papers (e.g. [3], [4], [6], [20], etc.) and the construction has proved its worth providing numerous (counter)examples in commutative ring theory.

In [9], Cuong et al. introduced the notion of filter regular sequence as an extension of regular sequence, and via this notion, they studied $f$-modules, as an extension of (generalized) Cohen-Macaulay modules. This structure is a well-known structure in commutative algebra and have applications in algebraic geometry. Then, in [17], Nhan extended this notion to generalized regular sequence, which in turn,
leads to the introduction of generalized $f$-modules in [18]. We have the following implications:

Gorenstein ring $\implies$ Cohen-Macaulay ring $\implies$ generalized Cohen-Macaulay ring $\implies$ $f$-ring $\implies$ generalized $f$-ring.

It has already investigated that when $R \bowtie_f J$ is one of the three first in the above list ([2], [4], [5], [6]). In this paper, we investigate when it is one of the two last properties.

The proofs for the two case is almost the same, but for $f$-modules easier. Therefore we deal with case of generalized $f$-modules in details, and the same proof with minor modifications works in the case of $f$-modules. We provide a sketch of proof for this case and leave details for the reader.

2. Results

Let us first fix some notation which we shall use throughout the paper: As mentioned above, $R$ and $S$ are two commutative rings with identity, $J$ is an ideal of the ring $S$, and $f : R \to S$ is a ring homomorphism. In the sequel, we consider contractions and extensions with respect to the natural embedding $\kappa R : R \to R \bowtie_f J$ defined by $\kappa R(x) = (x, f(x))$, for every $x \in R$.

Let $I$ be an ideal of $R$, and $M$ be a finitely generated $R$-module such that $M \neq IM$. We shall refer to the length of a maximal $M$-sequence contained in $I$ as the depth of $M$ in $I$, and we shall denote this by $\text{depth}(I, M)$. It will be convenient to use the $\text{depth}$ to denote $\text{depth}(m, M)$ when $(R, m)$ is a local ring.

(Generalized) $f$-modules are defined in the context of Noetherian local rings for finitely generated modules. Thus we always assume that $(R, m)$ is a Noetherian local ring and $J$ is finitely generated as an $R$-module. We will also assume that $J \subseteq \text{Jac}(S)$. When this is the case, $(R \bowtie_f J, m')$ is also a Noetherian local ring (see [10] Proposition 5.7 and [12] Corollary 2.7).

The notion of $M$-generalized regular sequence of $M$ is defined as a sequence $x_1, \ldots, x_n$ of elements in $m$ such that, for all $i = 1, \ldots, n$, $x_i \notin p$ for all $p \in \text{Ass}(M/(x_1, \ldots, x_{i-1})M)$ satisfying $\text{dim} R/p > 1$. The length of a maximal generalized regular sequence of $M$ in $I$ is called the generalized depth of $M$ in $I$ and denoted by $g\text{-}\text{depth}(I, M)$. In this paper, we use the following characterization for $g\text{-}\text{depth}(I, M)$ by the support of local cohomology module $H^i_I(M)$:

**Lemma 2.1.** Let $I$ be an ideal of $R$, and $M$ be a finitely generated $R$-module. Then the following equality holds.

$$g\text{-}\text{depth}(I, M) = \min \{ r \mid \text{there exists } p \in \text{Supp}_R(H^r_I(M)) \text{ such that } \text{dim} R/p > 1 \}.$$  

**Proof.** If $\text{dim} M/IM > 1$, then the assertion holds by [17] Proposition 4.5]. If $\text{dim} M/IM \leq 1$, then by definition, $g\text{-}\text{depth}(I, M) = \infty$. The other side is also infinite since $\text{Supp}_R(H^r_I(M)) \subseteq \text{Supp}(M) \cap \text{Supp}(R/I) = \text{Supp}(M/IM)$.

The following lemma, which has the key role in the proof of Theorem 2.4, links the $g$-depth of $R \bowtie_f J$ in the extension ideal $a^*$ to the $g$-depth of $R$ and $J$ in the prime ideal $a$.
Lemma 2.2. Let $a \in \text{Spec}(R)$. Then the following holds:

$$\text{g-depth}(a^e, R \langle J \rangle) = \min \{\text{g-depth}(a, R), \text{g-depth}(a, J)\}.$$  

Proof. We first show that the existence of some $P \in \text{Supp}_{R \langle J \rangle} (H_{a^e}^\ast (R \langle J \rangle))$ with the property $\dim R \langle J \rangle / P > 1$ is equivalent to the existence of some $p \in \text{Supp}_R (H_{a^e}^\ast (R \langle J \rangle))$ with the property $\dim R / p > 1$. To achieve this, first we note that, by [11 Lemma 3.6], the extension $\iota: R \to R \langle J \rangle$ is integral since we assume that $J$ is infinitely generated as an $R$-module. Therefore, for any $P \in \text{Spec}(R \langle J \rangle)$, we have $\dim R \langle J \rangle / P > 1$ if and only if $\dim R / P^c > 1$. Next, let $P \in \text{Supp}_{R \langle J \rangle} (H_{a^e}^\ast (R \langle J \rangle))$, then $\alpha/1$ is a non-zero element of $(H_{a^e}^\ast (R \langle J \rangle))_P$. If $r \in R$ such that $r\alpha = 0$, then $f(r) \in P$, i.e. $r \in P^c$. We have thus proved $P^c \in \text{Supp}_R (H_{a^e}^\ast (R \langle J \rangle))$.

Suppose conversely that $p \in \text{Supp}_R (H_{a^e}^\ast (R \langle J \rangle))$. Then, for some ideal $I$ of $R \langle J \rangle$, with the property $R \langle J \rangle / I \subseteq H_{a^e}^\ast (R \langle J \rangle)$, we have $p \in \text{Supp}_R (R \langle J \rangle / I)$. From this we have $I^c \subseteq p$. By lying over property, there exists $P \in \text{Spec}(R \langle J \rangle)$ such that $I \subseteq P$ and $P^c = p$, hence that $P \in \text{Supp}_{R \langle J \rangle} (R \langle J \rangle / I) \subseteq \text{Supp}_{R \langle J \rangle} (H_{a^e}^\ast (R \langle J \rangle))$. This completes the proof of our claim. Now we have:

$$\text{g-depth}(a^e, R \langle J \rangle) = \min \{r \mid \exists P \in \text{Supp}_{R \langle J \rangle} (H_{a^e}^\ast (R \langle J \rangle)) ; \dim R \langle J \rangle / P > 1\}$$

$$= \min \{r \mid \exists p \in \text{Supp}_R (H_{a^e}^\ast (R \langle J \rangle)) ; \dim R / p > 1\}$$

$$= \min \{r \mid \exists p \in \text{Supp}_R (H_{a^e}^\ast (R \langle J \rangle)) ; \dim R / p > 1\}$$

$$= \min \{r \mid \exists p \in \text{Supp}_R (H_{a^e}^\ast (R) \oplus H_{a^e}^\ast (J)) ; \dim R / p > 1\}$$

$$= \min \{\text{g-depth}(a, R), \text{g-depth}(a, J)\}.$$  

The first and last equality hold by Lemma 2.1, while the second one holds by the above observation. The third equality follows by the Independence Theorem of local cohomology [7 Theorem 4.2.1], and the forth equality obtained using the $R$-module isomorphism $R \langle J \rangle \cong R \oplus J$ [10 Lemma 2.3]. \hfill $\square$

Generalized $f$-modules were introduced in [18] as modules for which every system of parameters is a generalized regular sequence. A ring is called a generalized $f$-ring if it is a generalized $f$-module over itself. For more details we refer the reader to [17] and [18]. We define a finitely generated $R$-module $M$ to be maximal generalized $f$-module if $\text{g-depth}(p, M) = \dim(R) - \dim(R / p)$, for any $p \in \text{Supp} M$ satisfying $\dim R / p > 1$. This definition has stem in the following proposition.

Proposition 2.3. Assume that $M$ is a finitely generated $R$-module such that $\dim M > 1$. Then the following statements are equivalent:

1. $M$ is a generalized $f$-module.
2. $\text{g-depth}(p, M) = \dim(M) - \dim(R / p)$ for each $p \in \text{Supp} M$ satisfying $\dim R / p > 1$.
3. $\text{g-depth}(I, M) = \dim(M) - \dim(R / I)$ for any proper ideal $I$ of $R$ satisfying $I \supseteq \text{Ann}(M)$ and $\dim R / I > 1$. 


Proof. (1) ⇒ (2) and (3) ⇒ (1) is by [18, Proposition 2.5]. The proof of (2) ⇒ (3) is similar to the proof of [14, Remark 4.2], using [17, Proposition 4.3 (ii)] and [18, Proposition 2.5]. □

We use the above proposition to investigate when \( R./fJ \) is a generalized \( f \)-ring, which is one of our main results. Recall that a finitely generated module \( M \) over a Noetherian local ring \((R, m)\) is called a \textit{maximal Cohen-Macaulay} \( R \)-module if \( \text{depth} \ M = \dim R \). In the sequel, when we consider \( J \) as a module, we always consider it as an \( R \)-module via the homomorphism \( f : R \to S \).

**Theorem 2.4.** The following statements are equivalent:

1. \( R./fJ \) is a generalized \( f \)-ring.
2. \( R \) is a generalized \( f \)-ring and \( J \) is a maximal generalized \( f \)-module.
3. \( R \) is a generalized \( f \)-ring and \( J_p \) is maximal Cohen-Macaulay for any \( p \in \text{Supp}(J) \) satisfying \( \dim R./p > 1 \).

Proof. We first assume that \( \dim J > 1 \). The process of proof shows that the opposite assumption, \( \dim J \leq 1 \), leads to trivial cases.

(1) ⇒ (2) Assume that \( R \bowtie f J \) is a generalized \( f \)-ring and pick \( p \in \text{Spec}(R) \) satisfying \( \dim R./p > 1 \). By [11, Lemma 3.6], \( \iota_R : R \to R \bowtie f J \) is an integral extension. Hence, by lying over property, \( p = p^e \), hence that \( \dim R \bowtie f J/p^e = \dim R./p > 1 \). Now, by Proposition 2.3 and Lemma 2.2, we have:

\[
\dim R - \dim R./p = \dim R \bowtie f J - \dim R \bowtie f J/p^e \\
= \text{g-depth}(p^e, R \bowtie f J) \\
\leq \text{g-depth}(p, R) \\
\leq \dim R - \dim R./p.
\]

Again we use Proposition 2.3 to see that \( R \bowtie f J \) is a generalized \( f \)-ring, and a similar argument will show that \( J \) is a maximal generalized \( f \)-module.

(2) ⇒ (1) Suppose that \( R \) is a generalized \( f \)-ring and \( J \) is a maximal generalized \( f \)-module. Then, from Lemma 2.2 and Proposition 2.3 we deduce that \( \text{g-depth}(p^e, R \bowtie f J) = \text{g-depth}(p, R) \), for any \( p \in \text{Spec}(R) \). Now, let \( P \in \text{Spec}(R \bowtie f J) \) and \( \dim R \bowtie f J/P > 1 \). Then \( \dim R./P^e > 1 \) and, by Lemma 2.2 and Proposition 2.3 we have:

\[
\dim R \bowtie f J - \dim R \bowtie f J/P = \dim R - \dim R./P^e \\
= \text{g-depth}(P^e, R) \\
= \text{g-depth}(P^e, R \bowtie f J) \\
\leq \text{g-depth}(P, R \bowtie f J) \\
\leq \dim R \bowtie f J - \dim R \bowtie f J/P.
\]

Thus inequalities are equality, and another appeal to Proposition 2.3 gives the desired conclusion.

(2) ⇒ (3) Let \( p \in \text{Supp}(J) \) with the property \( \dim R./p > 1 \). In order to show that \( J_p \) is maximal Cohen-Macaulay, observe that [17, Proposition 4.4] together...
with our assumptions yields the following inequalities:
\[
\text{depth } J_p \geq \text{g-depth}(p, J) = \dim R - \dim R/p \geq \dim R_p \geq \text{depth } J_p.
\]

(3) ⇒ (2) Let \( p \in \text{Supp}(J) \) satisfying \( \dim R/p > 1 \). Then, using [17, Proposition 4.4] and [8, Proposition 1.2.10(a)], we get a prime ideal \( q \) containing \( p \) such that \( q \in \text{Supp}(J) \), \( \dim R/q > 1 \), and \( \text{g-depth}(p, J) = \text{depth } J_q \). The following inequalities complete the proof:
\[
\text{g-depth}(p, J) = \text{depth } J_q = \dim R_q \geq \text{g-depth}(q, R) = \dim R - \dim R/q \geq \dim R - \dim R/p \geq \text{g-depth}(p, J).
\]

□

Recall that if \( f := id_R \) is the identity homomorphism on \( R \), and \( I \) is an ideal of \( R \), then \( R \bowtie I := R \bowtie^{id_R} I \) is called the amalgamated duplication of \( R \) along \( I \). The next corollary deals with this case.

**Corollary 2.5.** \( R \bowtie I \) is a generalized f-ring if and only if \( R \) is a generalized f-ring and \( I \) is maximal generalized f-module if and only if \( R \) is a generalized f-ring and \( I_p \) is maximal Cohen-Macaulay for any \( p \in \text{Supp}(I) \) satisfying \( \dim R/p > 1 \).

Let \( M \) be an \( R \)-module. Nagata (1955) considered a ring extension of \( R \) called the idealization of \( M \) in \( R \), denoted here by \( R \ltimes M \) [16, page 2]. As in [10, Remark 2.8], if \( S := R \ltimes M, J := 0 \ltimes M \), and \( \iota: R \to S \) be the natural embedding, then \( R \bowtie^\iota J \cong R \ltimes M \). It is easy to check that, as \( R \)-modules, \( 0 \ltimes M \cong M \). The following corollary shows when the idealization is generalized f-ring.

**Corollary 2.6.** If \( M \) is a finitely generated \( R \)-module, then \( R \ltimes M \) is a generalized f-ring if and only if \( R \) is a generalized f-ring and \( M \) is a maximal generalized f-module if and only if \( R \) is a generalized f-ring and \( M_p \) is maximal Cohen-Macaulay for any \( p \in \text{Supp}(M) \) satisfying \( \dim R/p > 1 \).

In the remaining part of the paper we investigate when \( R \bowtie^I J \) is an f-ring. The arguments are the same as the ones in the case of generalized f-ring. But, for the readers convenience, we give brief proofs and refer the reader to previous arguments.

The notion of \( M \)-filter regular sequence is defined as a sequence \( x_1, \ldots, x_n \) of elements in \( m \) such that \( x_i \notin p \) for all \( p \in \text{Ass}(M/(x_1, \ldots, x_{i-1})M) \setminus \{m\} \) and for all \( i = 1, \ldots, n \). The filter depth, \( \text{f-depth}(I, M) \), of \( I \) on \( M \) is defined as the length of any maximal \( M \)-filter regular sequence in \( I \). Here, we use the following characterization for \( \text{f-depth}(I, M) \) (see [15, Theorem 3.1] and [14, Theorem 3.10]):
\[
\text{f-depth}(I, M) = \inf \{r | H^r_I(M) \text{ is not an Artinian } R \text{-module}\}.
\]

The following lemma expresses \( \text{f-depth}(p^e, R \bowtie^I J) \), the f-depth of extension of a prime ideal \( p \) of \( R \) in \( R \bowtie^I J \). For the proof, we use the elementary fact that being Artinian as an \( R \bowtie^I J \)-module is the same as being Artinian as an \( R \)-module.

**Lemma 2.7.** Let \( p \in \text{Spec}(R) \). Then the following holds:
\[
\text{f-depth}(p^e, R \bowtie^I J) = \min\{\text{f-depth}(p, R), \text{f-depth}(p, J)\}.
\]
Proof. By [14, Theorem 3.10] (and arguments similar to Lemma 2.2), we have:
\[
f\text{depth}(p^e, R \otimes^f J) = \inf \{ r \mid H^r_{p^e}(R \otimes^f J) \text{ is not Artinian } R \otimes^f J\text{-module} \} \\
= \inf \{ r \mid H^r_{p^e}(R \otimes^f J) \text{ is not Artinian } R\text{-module} \} \\
= \inf \{ r \mid H^r_p(R) \otimes H^r_p(J) \text{ is not Artinian } R\text{-module} \} \\
= \min \{ f\text{-depth}(p, R), f\text{-depth}(p, J) \} .
\]

In [9], the authors introduced \textit{f-modules} as modules for which every system of parameters is a filter regular sequence. The ring \( R \) is called an \textit{f-ring} if it is an \textit{f-module} over itself. This structure is a well-known structure in commutative algebra and have applications in algebraic geometry. For more details we refer the reader to [9], [14], and [21]. We define an \( R \)-module \( M \) to be \textit{maximal f-module} if \( f\text{-depth}(p, M) = \dim(R) - \dim(R/p) \), for any \( p \in \text{Supp}(M \setminus \{ m \}) \). This definition has stem in the following Proposition [14, Theorem 4.1 and Remark 4.2]:

**Proposition 2.8.** For a finitely generated \( R \)-module \( M \), the following statements are equivalent:
1. \( M \) is an \textit{f-module}
2. for any \( p \in \text{Supp}(M \setminus \{ m \}) \), \( f\text{-depth}(p, M) = \dim(M) - \dim(R/p) \)
3. for any proper ideal \( I \) of \( R \) with the property \( I \supseteq \text{Ann}(M) \) and \( \sqrt{I} \neq m \), \( f\text{-depth}(I, M) = \dim(M) - \dim(R/I) \)

We use the above proposition to investigate when \( R \otimes^f J \) is \textit{f-ring}, which is our final result.

**Theorem 2.9.** The following statements are equivalent:
1. \( R \otimes^f J \) is an \textit{f-ring}.
2. \( R \) is an \textit{f-ring} and \( J \) is a maximal \textit{f-module}.
3. \( R \) is an \textit{f-ring} and \( J_p \) is maximal Cohen-Macaulay for any \( p \in \text{Supp}(J \setminus \{ m \}) \).

**Proof.** (1) \( \Rightarrow \) (2) Assume that \( R \otimes^f J \) is an \textit{f-ring} and pick \( p \in \text{Spec}(R \setminus \{ m \}) \). As before, the extension \( \iota_R : R \to R \otimes^f J \) is integral, and so \( p = p^{ec} \). Thus \( \sqrt{p^e} \neq m' \) and \( \dim R \otimes^f J/p^e = \dim R/p \). Then Proposition 2.8 gives the desired conclusion, just as in the proof of Theorem 2.4.

(2) \( \Rightarrow \) (1) Suppose that \( R \) is an \textit{f-ring} and \( J \) is a maximal \textit{f-module}, and let \( P \in \text{Spec}(R \otimes^f J \setminus \{ m' \}) \). Then \( P^c \in \text{Spec}(R \setminus \{ m \}) \) and Proposition 2.8 gives the desired conclusion, as in the case of Theorem 2.4.

(2) \( \Leftrightarrow \) (3) The proof of this part is the same as the proof in Theorem 2.4, using the following equality instead of [17, Proposition 4.4]:
\[
f\text{depth}(p, J) = \min \{ \text{depth}(pR_q, J_q) \mid q \in \text{Supp}(J/pJ \setminus \{ m \}) \} .
\]

For the proof the equality, see the proof of [14, Theorem 3.10].
Corollary 2.10 (cf. [19, Theorem 3.5]). $R \triangleright I$ is an $f$-ring if and only if $R$ is an $f$-ring and $I$ is maximal $f$-module if and only if $R$ is an $f$-ring and $I_p$ is maximal Cohen-Macaulay for any $p \in \text{Supp}(I) \setminus \{m\}$.

Corollary 2.11. If $M$ is a finitely generated $R$-module, then $R \triangleright M$ is an $f$-ring if and only if $R$ is an $f$-ring and $M$ is a maximal $f$-module if and only if $R$ is an $f$-ring and $M_p$ is maximal Cohen-Macaulay for any $p \in \text{Supp}(M) \setminus \{m\}$.

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References


**Department of Mathematics, University of Tabriz, Tabriz, Iran**

AND

**School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran**

*E-mail:* u.azimi@tabrizu.ac.ir