RESIDUATION IN TWIST PRODUCTS AND PSEUDO-KLEENE POSETS

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Abstract. M. Busaniche, R. Cignoli (2014), C. Tsinakis and A. M. Wille (2006) showed that every residuated lattice induces a residuation on its full twist product. For their construction they used also lattice operations. We generalize this problem to left-residuated groupoids which need not be lattice-ordered. Hence, we cannot use the same construction for the full twist product. We present another appropriate construction which, however, does not preserve commutativity and associativity of multiplication. Hence we introduce the so-called operator residuated posets to obtain another construction which preserves the mentioned properties, but the results of operators on the full twist product need not be elements, but may be subsets. We apply this construction also to restricted twist products and present necessary and sufficient conditions under which we obtain a pseudo-Kleene operator residuated poset.

Keywords: left-residuated poset; operator residuated poset; twist product; pseudo-Kleene poset; Kleene poset

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1. Introduction

Busaniche and Cignoli (see [1]) as well as Tsinakis and Wille (see [6]) showed that if $(L, \leqslant, \cdot, \rightarrow, 1)$ is a residuated lattice then $\cdot$ and $\rightarrow$ can be used to define binary operations $\odot$ and $\Rightarrow$ on the full twist product of $(L, \leqslant)$ such that the resulting structure becomes a residuated lattice again. For the construction of such operations $\odot$ and $\Rightarrow$ they used the lattice operations $\vee$ and $\wedge$. When going from lattices

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to residuated posets, the natural question arises whether also in this case the corresponding twist product can be equipped with certain operations $\odot$ and $\Rightarrow$ (without using lattice operations) such that the resulting structure is residuated again. We solve this problem in the positive. We define suitable operations $\odot$ and $\Rightarrow$ on the full twist product such that the arising structure becomes a left-residuated groupoid again. Unfortunately, this construction does preserve neither commutativity nor associativity of the original structure $(Q, \leqslant, \cdot, \rightarrow, 1)$. Hence, we try another approach where instead of operations we use certain operators $\odot$ and $\Rightarrow$ in such a way that the full twist product becomes an operator residuated poset and the commutativity as well as associativity of the original operation $\cdot$ are preserved. As the authors already showed in [3], any poset $\mathbf{Q} = (Q, \leqslant)$ can be embedded into a pseudo-Kleene poset $(P_a(\mathbf{Q}), \leqslant, ')$ where $(P_a(\mathbf{Q}), \leqslant)$ is a certain subposet of the full twist product of $\mathbf{Q}$. This has motivated us to investigate whether our construction of the operators $\odot$ and $\Rightarrow$ can be extended also to this case, i.e. whether we can determine for a bounded commutative residuated monoid $(Q, \leqslant, \cdot, \rightarrow, 0, 1)$ a corresponding pseudo-Kleene poset which is operator residuated and into which $\mathbf{Q}$ can be embedded. We characterize those left-residuated posets for which our construction is possible.

2. Preliminaries

The concept of a Kleene lattice (alias Kleene algebra) was introduced by Kalman (see [5], and also [4]). Recall that a *Kleene lattice* is a distributive lattice $\mathbf{L} = (L, \vee, \wedge, ')$ with an involution $'$ satisfying the so-called *normality condition*, i.e. the inequality

$$x \wedge x' \leqslant y \vee y'$$

which can be rewritten as an identity. This concept was generalized by the first author in [2]: $\mathbf{L}$ is called a *pseudo-Kleene lattice* if it satisfies the above identity, but it need not be distributive.

Let $(P, \leqslant)$ be a poset, $a, b \in P$ and $A, B \subseteq P$. Then the lower cone $L(A)$ of A and the upper cone $U(A)$ of A are introduced as

$$L(A) := \{x \in P\colon\ x \leqslant A\}, \quad U(A) := \{x \in P\colon\ x \geqslant A\}.$$

Here $x \leqslant A$ means $x \leqslant y$ for all $y \in A$ and, similarly, $x \geqslant A$ means $x \geqslant y$ for all $y \in A$. The expression $A \leqslant B$ means $x \leqslant y$ for all $x \in A$ and $y \in B$. Instead of $L(\{a, b\})$ and $L(U(A))$ we simply write $L(a, b)$ and $LU(A)$, respectively. Analogously we proceed in similar cases. Let $\max A$ denote the set of all maximal elements of $(A, \leqslant)$. A unary operation $'$ on P is called *antitone* if $x, y \in P$ and $x \leqslant y$ implies $y' \leqslant x'$, an *involution* if it satisfies the identity $x'' \approx x$.

The concept of a pseudo-Kleene lattice was generalized by the authors in [3] for posets as follows:

A *pseudo-Kleene poset* is a poset $\mathbf{P} = (P, \leqslant, ')$ with an antitone involution satisfying the condition

$$L(x, x') \leqslant U(y, y')$$

for all $x, y \in P$. A *Kleene poset* is a distributive pseudo-Kleene poset. Recall that a *poset* $(P, \leqslant)$ is called *distributive* if it satisfies one of the following equivalent LU-identities:

$$L(U(x, y), z) \approx LU(L(x, z), L(y, z)),$$
$$U(L(x, y), z) \approx UL(U(x, z), U(y, z)).$$

In [3] it was shown that an arbitrary poset can be embedded into a pseudo-Kleene poset by means of the so-called *twist construction*:

The *full twist product* of a poset $\mathbf{Q} = (Q, \leqslant)$ is the poset $(Q^2, \leqslant_t)$ where

$$(x, y) \leqslant_t (z, v) \text{ if and only if } x \leqslant z \text{ and } v \leqslant y$$

for all $(x, y), (z, v) \in Q^2$. We have

$$L((x, y), (z, v)) = L(x, z) \times U(y, v), \quad U((x, y), (z, v)) = U(x, z) \times L(y, v)$$

for all $(x, y), (z, v) \in Q^2$.

3. Left-residuated groupoids

We will investigate when a residuated poset can be transferred to a residuated full twist product. For this purpose we use the twist construction. For residuated lattices such a transfer was already published in [1] by using a construction developed in [6].

From now on, let $(Q, \leqslant, \cdot, \to, 1)$ denote a poset with constant 1 endowed with two binary operations $\cdot$ and $\to$. For our next investigations, consider the following conditions.

(1) $x \leqslant y$ implies $z \cdot x \leqslant z \cdot y$ (right-isotony),
(2) $x \leqslant y$ implies $x \cdot z \leqslant y \cdot z$ (left-isotony),
(3) $x \cdot y \leqslant z$ if and only if $x \leqslant y \to z$ (left-adjointness),
(4) $x \leqslant y$ implies $z \to x \leqslant z \to y$,
(5) $x \leqslant y$ implies $y \to z \leqslant x \to z$,
(6) $x \cdot 1 \approx x$,
(7) $x \cdot y \leqslant x, y$

for all $x, y, z \in Q$.

The above mentioned conditions are related as shown in the following lemmas.

Lemma 3.1. *For* $(Q, \leqslant, \cdot, \rightarrow, 1)$ *the following holds:*

(i) (1) *and* (3) *imply* (5).

(ii) *If* $\cdot$ *is commutative then* (1) *and* (6) *imply* (7).

Proof. Let $a, b, c \in Q$.

(i) If $a \leqslant b$ then every of the following statements implies the next one:

$$\begin{aligned} b \rightarrow c &\leqslant b \rightarrow c, \\ (b \rightarrow c) \cdot b &\leqslant c, \\ (b \rightarrow c) \cdot a &\leqslant c, \\ b \rightarrow c &\leqslant a \rightarrow c. \end{aligned}$$

(ii) We have $a \cdot b \leqslant a \cdot 1 = a$ and $a \cdot b = b \cdot a \leqslant b$. □

Lemma 3.2. *Assume* $(Q, \leqslant, \cdot, \rightarrow, 1)$ *with associative* $\cdot$ *to satisfy* (2) *and* (3). *Then it satisfies*

(8) $$(x \cdot y) \rightarrow z \approx x \rightarrow (y \rightarrow z).$$

Proof. Let $a, b, c \in Q$. Then every of the following statements implies the next one:

$$\begin{aligned} (a \cdot b) \rightarrow c &\leqslant (a \cdot b) \rightarrow c, \\ ((a \cdot b) \rightarrow c) \cdot (a \cdot b) &\leqslant c, \\ (((a \cdot b) \rightarrow c) \cdot a) \cdot b &\leqslant c, \\ ((a \cdot b) \rightarrow c) \cdot a &\leqslant b \rightarrow c, \\ (a \cdot b) \rightarrow c &\leqslant a \rightarrow (b \rightarrow c). \end{aligned}$$

Moreover, every of the following statements implies the next one:

$$\begin{aligned} a \rightarrow (b \rightarrow c) &\leqslant a \rightarrow (b \rightarrow c), \\ (a \rightarrow (b \rightarrow c)) \cdot a &\leqslant b \rightarrow c, \\ ((a \rightarrow (b \rightarrow c)) \cdot a) \cdot b &\leqslant (b \rightarrow c) \cdot b, \\ (a \rightarrow (b \rightarrow c)) \cdot (a \cdot b) &\leqslant (b \rightarrow c) \cdot b. \end{aligned}$$

Together with $(b \rightarrow c) \cdot b \leqslant c$ which follows from $b \rightarrow c \leqslant b \rightarrow c$ we obtain $(a \rightarrow (b \rightarrow c)) \cdot (a \cdot b) \leqslant c$ which implies $a \rightarrow (b \rightarrow c) \leqslant (a \cdot b) \rightarrow c$. □

Now we introduce one of our main concepts.

Definition 3.3. $(Q, \leqslant, \cdot, \rightarrow, 1)$ is called a *left-residuated groupoid* if it satisfies (3) and (6). It is called

bounded if $(Q, \leqslant)$ is bounded (0 is the bottom and 1 the top element),
commutative if $\cdot$ is commutative,
associative if $\cdot$ is associative.

A *commutative residuated monoid* is a commutative and associative left-residuated groupoid.

An example of a bounded residuated monoid which is not a lattice follows.

Example 3.4. The poset visualized in Figure 1

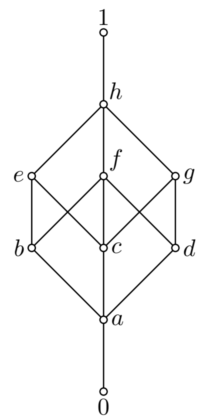

Figure 1.

together with the operations given by

$\cdot$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	a
b	0	0	a	0	0	a	a	0	a	b
c	0	0	0	a	0	a	0	a	a	c
d	0	0	0	0	a	0	a	a	a	d
e	0	0	a	a	0	a	a	a	a	e
f	0	0	a	0	a	a	a	a	a	f
g	0	0	0	a	a	a	a	a	a	g
h	0	0	a	a	a	a	a	a	a	h
1	0	a	b	c	d	e	f	g	h	1

$\rightarrow$	0	a	b	c	d	e	f	g	h	1
0	1	1	1	1	1	1	1	1	1	1
a	h	1	1	1	1	1	1	1	1	1
b	g	h	1	h	h	1	1	h	1	1
c	f	h	h	1	h	1	h	1	1	1
d	e	h	h	h	1	h	1	1	1	1
e	d	h	h	h	h	1	h	h	1	1
f	c	h	h	h	h	h	1	h	1	1
g	b	h	h	h	h	h	h	1	1	1
h	a	h	h	h	h	h	h	h	1	1
1	0	a	b	c	d	e	f	g	h	1

is a bounded commutative residuated monoid which is not a lattice.

The next lemma shows some elementary properties of left-residuated groupoids.

Lemma 3.5. *The following holds:*

(i) *Every left-residuated groupoid* $(Q, \leqslant, \cdot, \to, 1)$ *satisfies* (2), (4) *and*

$$1 \to x \approx x. \tag{9}$$

(ii) *If* $(Q, \leqslant, \cdot, \to, 1)$ *satisfies* (5), (6) *and* (9) *then it satisfies*

$$x \leqslant y \to x. \tag{10}$$

Proof. Let $a, b, c \in Q$.

(i) *Condition* (2): If $a \leqslant b$ then every of the following statements implies the next one:

$$\begin{aligned} b \cdot c &\leqslant b \cdot c, \\ b &\leqslant c \to (b \cdot c), \\ a &\leqslant c \to (b \cdot c), \\ a \cdot c &\leqslant b \cdot c. \end{aligned}$$

Condition (4): If $a \leqslant b$ then every of the following statements implies the next one:

$$\begin{aligned} c \to a &\leqslant c \to a, \\ (c \to a) \cdot c &\leqslant a, \\ (c \to a) \cdot c &\leqslant b, \\ c \to a &\leqslant c \to b. \end{aligned}$$

Condition (9): We have that $1 \to a \leqslant 1 \to a$ implies $1 \to a = (1 \to a) \cdot 1 \leqslant a$ and $a \cdot 1 \leqslant a$ implies $a \leqslant 1 \to a$.

(ii) We have $a = 1 \to a \leqslant b \to a$. □

Now we show that every left-residuated groupoid naturally induces a left-residuated groupoid on its full twist product.

Theorem 3.6. *Let* $(Q, \leqslant, \cdot, \to, 1)$ *be a poset with binary operations* $\cdot$ *and* $\to$ *and a constant* 1, *let* $a, b \in Q$ *and* f, g *be surjective mappings from* Q^2 *to* Q *satisfying* $f(a, b) = g(a, b) = 1$, *and consider the full twist product* $(Q^2, \leqslant_t, \odot, \Rightarrow, (a, b))$ *of* $(Q, \leqslant)$ *with the binary operations* $\odot$ *and* $\Rightarrow$ *defined by*

$$\begin{aligned} (x, y) \odot (z, v) &:= (x \cdot f(z, v), g(z, v) \to y), \\ (x, y) \Rightarrow (z, v) &:= (f(x, y) \to z, v \cdot g(x, y)) \end{aligned}$$

for all $(x, y), (z, v) \in Q^2$ *and the constant* (a, b). *Then* $(Q, \leqslant, \cdot, \to, 1)$ *is a left-residuated groupoid if and only if* $(Q^2, \leqslant_t, \odot, \Rightarrow, (a, b))$ *has this property.*

P r o o f. We investigate when statements $(Q^2, \leqslant_t, \odot, \Rightarrow, (a,b))$ satisfy (3) and (6). The following are equivalent:

$$(Q^2, \leqslant_t, \odot, \Rightarrow, (a,b)) \text{ satisfies (3)},$$
$$(x,y) \odot (z,v) \leqslant_t (t,w) \text{ is equivalent to } (x,y) \leqslant_t (z,v) \Rightarrow (t,w),$$
$$(x \cdot f(z,v), g(z,v) \to y) \leqslant_t (t,w) \text{ is equivalent to } (x,y) \leqslant_t (f(z,v) \to t, w \cdot g(z,v)),$$
$$(x \cdot f(z,v) \leqslant t \text{ and } w \leqslant g(z,v) \to y) \text{ is equivalent to}$$
$$(x \leqslant f(z,v) \to t \text{ and } w \cdot g(z,v) \leqslant y),$$
$$(Q, \leqslant, \cdot, \to, 1) \text{ satisfies (3)}.$$

Moreover, the following statements are equivalent:

$$(Q^2, \leqslant_t, \odot, \Rightarrow, (a,b)) \text{ satisfies (6)},$$
$$(x,y) \odot (a,b) \approx (x,y),$$
$$(x \cdot f(a,b), g(a,b) \to y) \approx (x,y),$$
$$(x \cdot 1, 1 \to y) \approx (x,y),$$
$$x \cdot 1 \approx x \text{ and } 1 \to y \approx y,$$
$$(Q, \leqslant, \cdot, \to, 1) \text{ satisfies (6) and (9)}.$$

Now Lemma 3.5 completes the proof. □

Corollary 3.7. *Let $(Q, \leqslant, \cdot, \to, 1)$ be a poset with binary operations $\cdot$ and $\to$ and a constant 1, and consider the full twist product $(Q^2, \leqslant_t, \odot, \Rightarrow, (1,1))$ of $(Q, \leqslant)$ with the binary operations $\odot$ and $\Rightarrow$ defined by*

$$(x,y) \odot (z,v) := (x \cdot z, v \to y), \quad (x,y) \Rightarrow (z,v) := (x \to z, v \cdot y)$$

for all $(x,y),(z,v) \in Q^2$ and the constant $(1,1)$. Then $(Q, \leqslant, \cdot, \to, 1)$ is a left-residuated groupoid if and only if $(Q^2, \leqslant_t, \odot, \Rightarrow, (1,1))$ has this property.

P r o o f. This is a special case of Theorem 3.6 where $a = b = 1$, f is the first and g the second projection. □

Corollary 3.8. *Let $(Q, \leqslant, \cdot, \to, 1)$ be a poset with binary operations $\cdot$ and $\to$ and a constant 1, and consider the full twist product $(Q^2, \leqslant_t, \odot, \Rightarrow, (1,1))$ of $(Q, \leqslant)$ with the binary operations $\odot$ and $\Rightarrow$ defined by*

$$(x,y) \odot (z,v) := (x \cdot v, z \to y), \quad (x,y) \Rightarrow (z,v) := (y \to z, v \cdot x)$$

for all $(x,y),(z,v) \in Q^2$ and the constant $(1,1)$. Then $(Q, \leqslant, \cdot, \to, 1)$ is a left-residuated groupoid if and only if $(Q^2, \leqslant_t, \odot, \Rightarrow, (1,1))$ has this property.

P r o o f. This is a special case of Theorem 3.6 where $a = b = 1$, f is the second and g the first projection. □

One can easily see that the left-residuated groupoid $(Q^2, \leqslant_t, \odot, \Rightarrow, (a, b))$ from Theorem 3.6 need neither be commutative nor associative even if $(Q, \leqslant, \cdot, \to, 1)$ has this property. Hence, we introduce the next concept.

Definition 4.1. An *operator residuated poset* is an ordered six-tuple $(Q, \leqslant, \odot, \Rightarrow, 0, 1)$ such that

(i) $(Q, \leqslant, 0, 1)$ is a bounded poset,
(ii) $\odot$ and $\Rightarrow$ are mappings from Q^2 to 2^Q (so-called *operators*),
(iii) $x \odot y \approx y \odot x$,
(iv) $\bigcup\limits_{u \in x \odot y} (u \odot z) = \bigcup\limits_{u \in y \odot z} (x \odot u)$ (*operator associativity*),
(v) $x \odot y \leqslant z$ if and only if $x \leqslant y \Rightarrow z$

for all $x, y, z \in Q$.

The following result shows that when using an operator residuated structure on the full twist product, the commutativity and associativity of the original bounded left-residuated groupoid are preserved.

Theorem 4.2. *Let $(Q, \leqslant, \cdot, \to, 0, 1)$ be a bounded commutative residuated monoid and $a_0 \in Q$. Then $(Q^2, \leqslant_t, \odot, \Rightarrow, (0, 1), (1, 0))$, where $(Q^2, \leqslant_t)$ is the full twist product of $(Q, \leqslant)$ and the operators $\odot$ and $\Rightarrow$ on Q^2 are defined by*

$$(x, y) \odot (z, v) := \{(x \cdot z, x \to v), (x \cdot z, z \to y)\},$$
$$(x, y) \Rightarrow (z, v) := \{(x \to z, x \cdot v), (v \to y, x \cdot v)\}$$

for all $(x, y), (z, v) \in Q^2$ is an operator residuated poset and the mapping $x \mapsto (x, a_0)$ is an embedding of $(Q, \leqslant)$ into $(Q^2, \leqslant_t)$.

P r o o f. Let $a, b, c, d, e, f \in Q$. According to Lemmas 3.1, 3.2, and Lemma 3.5, $(Q, \leqslant, \cdot, \to, 0, 1)$ satisfies (1)–(10).

(i) It is evident that $(Q^2, \leqslant_t, (0, 1), (1, 0))$ is a bounded poset.
(ii) $\odot$ and $\Rightarrow$ are mappings from $(Q^2)^2$ to $2^{(Q^2)}$.

We must prove (iii)–(v) of Definition 4.1.

(iii) We have

$$\begin{aligned}(a, b) \odot (c, d) &= \{(a \cdot c, a \to d), (a \cdot c, c \to b)\} \\ &= \{(c \cdot a, c \to b), (c \cdot a, a \to d)\} = (c, d) \odot (a, b).\end{aligned}$$

(iv) We have

$$
\begin{aligned}
\bigcup_{(x,y)\in(a,b)\odot(c,d)} ((x,y)\odot(e,f)) &= \bigcup_{(x,y)\in\{(a\cdot c,a\to d),(a\cdot c,c\to b)\}} ((x,y)\odot(e,f)) \\
&= ((a\cdot c, a\to d)\odot(e,f)) \cup ((a\cdot c, c\to b)\odot(e,f)) \\
&= \{((a\cdot c)\cdot e, (a\cdot c)\to f), ((a\cdot c)\cdot e, e\to(a\to d)), \\
&\qquad ((a\cdot c)\cdot e, (a\cdot c)\to f), ((a\cdot c)\cdot e, e\to(c\to b))\},
\end{aligned}
$$

$$
\begin{aligned}
\bigcup_{(x,y)\in(c,d)\odot(e,f)} ((a,b)\odot(x,y)) &= \bigcup_{(x,y)\in\{(c\cdot e,c\to f),(c\cdot e,e\to d)\}} ((a,b)\odot(x,y)) \\
&= ((a,b)\odot(c\cdot e, c\to f)) \cup ((a,b)\odot(c\cdot e, e\to d)) \\
&= \{(a\cdot(c\cdot e), a\to(c\to f)), (a\cdot(c\cdot e), (c\cdot e)\to b), \\
&\qquad (a\cdot(c\cdot e), a\to(e\to d)), (a\cdot(c\cdot e), (c\cdot e)\to b)\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\{((a\cdot c)\cdot e, (a\cdot c)\to f), ((a\cdot c)\cdot e, e\to(a\to d)), \\
&\qquad\qquad ((a\cdot c)\cdot e, (a\cdot c)\to f), ((a\cdot c)\cdot e, e\to(c\to b))\} \\
&\qquad = \{(a\cdot(c\cdot e), a\to(c\to f)), (a\cdot(c\cdot e), (c\cdot e)\to b), \\
&\qquad\qquad (a\cdot(c\cdot e), a\to(e\to d)), (a\cdot(c\cdot e), (c\cdot e)\to b)\}.
\end{aligned}
$$

(v) The following statements are equivalent:

$$
\begin{gathered}
(a,b)\odot(c,d) \leqslant_t (e,f), \\
\{(a\cdot c, a\to d), (a\cdot c, c\to b)\} \leqslant_t (e,f), \\
a\cdot c\leqslant e, \quad f\leqslant a\to d \quad \text{and} \quad f\leqslant c\to b, \\
a\leqslant c\to e, \quad a\leqslant f\to d \quad \text{and} \quad c\cdot f\leqslant b, \\
(a,b) \leqslant_t \{(c\to e, c\cdot f), (f\to d, c\cdot f)\}, \\
(a,b) \leqslant_t (c,d)\Rightarrow(e,f).
\end{gathered}
$$

Finally, $(a,a_0)\leqslant(b,a_0)$ is equivalent to $a\leqslant b$. □

Example 4.3. If $(Q,\leqslant,\cdot,\to,0,1) := (\{0,1\},\leqslant,\cdot,(x,y)\mapsto 1-x+xy,0,1)$ (where $+$, $-$ and $\cdot$ denote addition, subtraction and multiplication of the reals, respectively) then the tables for $\odot$ and $\Rightarrow$ look as follows:

$\odot$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,0),(0,1)\}$	$\{(0,0),(0,1)\}$
$(0,1)$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$	$\{(0,1)\}$
$(1,0)$	$\{(0,0),(0,1)\}$	$\{(0,1)\}$	$\{(1,0)\}$	$\{(1,0),(1,1)\}$
$(1,1)$	$\{(0,0),(0,1)\}$	$\{(0,1)\}$	$\{(1,0),(1,1)\}$	$\{(1,1)\}$

$\Rightarrow$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$\{(1,0)\}$	$\{(0,0),(1,0)\}$	$\{(1,0)\}$	$\{(0,0),(1,0)\}$
$(0,1)$	$\{(1,0)\}$	$\{(1,0)\}$	$\{(1,0)\}$	$\{(1,0)\}$
$(1,0)$	$\{(0,0),(1,0)\}$	$\{(0,1)\}$	$\{(1,0)\}$	$\{(0,1),(1,1)\}$
$(1,1)$	$\{(0,0),(1,0)\}$	$\{(0,1),(1,1)\}$	$\{(1,0)\}$	$\{(1,1)\}$

5. Pseudo-Kleene posets

It has been shown by the authors in [3] that every poset $\mathbf{Q} = (Q, \leqslant)$ can be embedded into a pseudo-Kleene one. For this we use a certain modification of the full twist product construction as follows.

Let $a \in Q$ and put

$$\begin{aligned} P_a(\mathbf{Q}) &:= \{(x,y) \in Q^2 \colon\ L(x,y) \leqslant a \leqslant U(x,y)\}, \\ (x,y) \leqslant_t (z,v) &:\Leftrightarrow (x \leqslant z \text{ and } v \leqslant y), \\ (x,y)' &:= (y,x) \end{aligned}$$

for all $(x,y),(z,v) \in Q^2$. The following was proved in [3]:

- ▷ $(P_a(\mathbf{Q}), \leqslant_t, ')$ is a pseudo-Kleene poset,
- ▷ the mapping $x \mapsto (x,a)$ is an embedding of $\mathbf{Q}$ into $(P_a(\mathbf{Q}), \leqslant_t)$,
- ▷ $(P_a(\mathbf{Q}), \leqslant_t, ')$ is a Kleene poset if and only if $\mathbf{Q}$ is distributive.

Since $P_a(\mathbf{Q})$ is a subset of the full twist product of $\mathbf{Q}$, it is a question if residuation from $(Q, \leqslant, \cdot, \rightarrow, 1)$ can be transferred to $P_a(\mathbf{Q})$ as shown in Theorem 3.6. Unfortunately, this is not possible in general since $P_a(\mathbf{Q})$ need not be closed under the operators $\odot$ and $\Rightarrow$ defined in Theorem 3.6. However, we can get necessary and sufficient conditions under which $P_a(\mathbf{Q})$ is closed under these operators and hence becomes a pseudo-Kleene operator residuated poset.

If $\mathbf{Q} = (Q, \leqslant)$ is a poset, $a, b \in Q$ and every element of $P_a(\mathbf{Q})$ is comparable with (a,a) then $(a,b) \in P_a(\mathbf{Q})$ and hence $(a,a) \leqslant_t (a,b)$ or $(a,b) \leqslant_t (a,a)$ whence $b \leqslant a$ or $a \leqslant b$, i.e. b is comparable with a.

Theorem 5.1. *Let $(Q, \leqslant, \cdot, \rightarrow, 0, 1)$ be a bounded commutative residuated monoid and $a \in Q$ with $a \cdot a = a$, put $\mathbf{Q} := (Q, \leqslant)$ and assume that all elements of $P_a(\mathbf{Q})$ are comparable with (a, a). Then*

- ▷ $(P_a(\mathbf{Q}), \leqslant_t, \odot, \Rightarrow, (0,1), (1,0))$, *where the operators $\odot$ and $\Rightarrow$ are defined by*

$$(x, y) \odot (z, v) := \{(x \cdot z, x \rightarrow v), (x \cdot z, z \rightarrow y)\},$$
$$(x, y) \Rightarrow (z, v) := \{(x \rightarrow z, x \cdot v), (v \rightarrow y, x \cdot v)\}$$

for all $(x, y), (z, v) \in P_a(\mathbf{Q})$ is an operator residuated poset if and only if the following two conditions hold:

$$a \cdot x < a \Rightarrow a \cdot x = 0, \tag{11}$$
$$a < x \Rightarrow x \rightarrow a = a. \tag{12}$$

- ▷ $(P_a(\mathbf{Q}), \leqslant_t, {}')$ *where $(x, y)' := (y, x)$ for all $(x, y) \in P_a(\mathbf{Q})$ is a pseudo-Kleene poset.*
- ▷ *The mapping $x \mapsto (x, a)$ is an embedding of $\mathbf{Q}$ into $(P_a(\mathbf{Q}), \leqslant_t)$.*
- ▷ $(x, y)' \in (x, y) \Rightarrow (0, 1)$ *for all $(x, y) \in P_a(\mathbf{Q})$.*

Proof. Let $b, c, d, e \in Q$. According to Lemmas 3.1, 3.2 and 3.5, $(Q, \leqslant, \cdot, \rightarrow, 0, 1)$ satisfies (1)–(10). If $a \leqslant x, y$ then $a = a \cdot a \leqslant a \cdot y \leqslant x \cdot y$ according to (1) and (2), i.e. we have

$$a \leqslant x, y \Rightarrow a \leqslant x \cdot y. \tag{13}$$

- ▷ Assume $(b, c), (d, e) \leqslant_t (a, a)$. Then the following holds:
  $(b \cdot d, b \rightarrow e) \leqslant_t (a, a)$ because $b \cdot d \leqslant b \leqslant a$ by (7) and $a \leqslant e \leqslant b \rightarrow e$ by (10).
  $(b \cdot d, d \rightarrow c) \leqslant_t (a, a)$ because $b \cdot d \leqslant b \leqslant a$ by (7) and $a \leqslant c \leqslant d \rightarrow c$ by (10).
  Since $b \leqslant a$ we have $b \cdot e \leqslant a$ according to (7).
  If $b \cdot e = a$ then $(b \rightarrow d, b \cdot e)$ is comparable with (a, a).
  If $b \cdot e < a$ then $(b \rightarrow d, b \cdot e)$ is comparable with (a, a) if and only if $a \leqslant b \rightarrow d$.
  $(e \rightarrow c, b \cdot e) \geqslant_t (a, a)$ because $a \leqslant c \leqslant e \rightarrow c$ by (10) and $b \cdot e \leqslant b \leqslant a$ by (7).
- ▷ Assume $(b, c) \leqslant_t (a, a) \leqslant_t (d, e)$. Then the following holds:
  Since $b \leqslant a$ we have $b \cdot d \leqslant a$ according to (7).
  If $b \cdot d = a$ then $(b \cdot d, b \rightarrow e)$ is comparable with (a, a).
  If $b \cdot d < a$ then $(b \cdot d, b \rightarrow e)$ is comparable with (a, a) if and only if $a \leqslant b \rightarrow e$.
  $(b \cdot d, d \rightarrow c) \leqslant_t (a, a)$ because $b \cdot d \leqslant b \leqslant a$ by (7) and $a \leqslant c \leqslant d \rightarrow c$ by (10).
  $(b \rightarrow d, b \cdot e) \geqslant_t (a, a)$ because $a \leqslant d \leqslant b \rightarrow d$ by (10) and $b \cdot e \leqslant b \leqslant a$ by (7).
  $(e \rightarrow c, b \cdot e) \geqslant_t (a, a)$ because $a \leqslant c \leqslant e \rightarrow c$ by (10) and $b \cdot e \leqslant b \leqslant a$ by (7).

▷ Assume $(d,e) \leqslant_t (a,a) \leqslant_t (b,c)$. Then the following holds:

$(b \cdot d, b \to e) \leqslant_t (a,a)$ because $b \cdot d \leqslant d \leqslant a$ by (7) and $a \leqslant e \leqslant b \to e$ by (10).

Since $d \leqslant a$ we have $b \cdot d \leqslant a$ according to (7).

If $b \cdot d = a$ then $(b \cdot d, d \to c)$ is comparable with (a,a).

If $b \cdot d < a$ then $(b \cdot d, d \to c)$ is comparable with (a,a) if and only if $a \leqslant d \to c$.

Since $a \leqslant b, e$ we have $a \leqslant b \cdot e$ according to (13).

If $b \cdot e = a$ then $(b \to d, b \cdot e)$ is comparable with (a,a).

If $a < b \cdot e$ then $(b \to d, b \cdot e)$ is comparable with (a,a) if and only if $b \to d \leqslant a$.

If $b \cdot e = a$ then $(e \to c, b \cdot e)$ is comparable with (a,a).

If $a < b \cdot e$ then $(e \to c, b \cdot e)$ is comparable with (a,a) if and only if $e \to c \leqslant a$.

▷ Assume $(a,a) \leqslant_t (b,c), (d,e)$. Then the following holds:

Since $a \leqslant b, d$ we have $a \leqslant b \cdot d$ according to (13).

If $b \cdot d = a$ then $(b \cdot d, b \to e)$ is comparable with (a,a).

If $a < b \cdot d$ then $(b \cdot d, b \to e)$ is comparable with (a,a) if and only if $b \to e \leqslant a$.

If $b \cdot d = a$ then $(b \cdot d, d \to c)$ is comparable with (a,a).

If $a < b \cdot d$ then $(b \cdot d, d \to c)$ is comparable with (a,a) if and only if $d \to c \leqslant a$.

$(b \to d, b \cdot e) \geqslant (a,a)$ because $a \leqslant d \leqslant b \to d$ by (10) and $b \cdot e \leqslant e \leqslant a$ by (7).

Since $e \leqslant a$ we have $b \cdot e \leqslant a$ according to (7).

If $b \cdot e = a$ then $(e \to c, b \cdot e)$ is comparable with (a,a).

If $b \cdot e < a$ then $(e \to c, b \cdot e)$ is comparable with (a,a) if and only if $a \leqslant e \to c$.

Hence $(x,y) \odot (z,v) \subseteq P_a(\mathbf{Q})$ and $(x,y) \Rightarrow (z,v) \subseteq P_a(\mathbf{Q})$ for all $(x,y),(z,v) \in P_a(\mathbf{Q})$ if and only if the following statements hold:

(a) $b, d \leqslant a \leqslant c, e$ and $b \cdot e < a$ imply $a \leqslant b \to d$.
(b) $b, e \leqslant a \leqslant c, d$ and $b \cdot d < a$ imply $a \leqslant b \to e$.
(c) $c, d \leqslant a \leqslant b, e$ and $b \cdot d < a$ imply $a \leqslant d \to c$.
(d) $c, d \leqslant a \leqslant b, e$ and $a < b \cdot e$ imply $b \to d \leqslant a$.
(e) $c, d \leqslant a \leqslant b, e$ and $a < b \cdot e$ imply $e \to c \leqslant a$.
(f) $c, e \leqslant a \leqslant b, d$ and $a < b \cdot d$ imply $b \to e \leqslant a$.
(g) $c, e \leqslant a \leqslant b, d$ and $a < b \cdot d$ imply $d \to c \leqslant a$.
(h) $c, e \leqslant a \leqslant b, d$ and $b \cdot e < a$ imply $a \leqslant e \to c$.

Now (a) is equivalent to the statements

$$b \cdot a < a \text{ implies } a \leqslant b \to 0,$$
$$a \cdot b < a \text{ implies } a \cdot b \leqslant 0,$$
$$a \cdot b < a \text{ implies } a \cdot b = 0,$$
$$\text{Condition (11).}$$

In the same way one can see that (b), (c) and (h) are equivalent to (11). Moreover, (d) is equivalent to the statements

$$a \leqslant b, e \text{ and } a < b \cdot e \text{ imply } b \to a \leqslant a,$$
$$a < b \text{ implies } b \to a \leqslant a,$$
$$a < b \text{ implies } b \to a = a,$$
$$\text{Condition (12).}$$

In the same way one can see that (e), (f) and (g) are equivalent to (12). Moreover, we have

$$(b,c)' = (c,b) \in \{(b \to 0, b), (c,b)\} = \{(b \to 0, b \cdot 1), (1 \to c, b \cdot 1)\} = (b,c) \Rightarrow (0,1).$$

The rest follows from Theorem 4.2. □

E x a m p l e 5.2. Consider the bounded commutative residuated semigroup $(Q, \leqslant, \cdot, \to, 0, 1)$ with $Q = \{0, a, 1\}$, $0 < a < 1$, and

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

and put $\mathbf{Q} := (Q, \leqslant)$. It is easy to check that $\mathbf{Q}$ is a distributive lattice and $(Q, \leqslant, \cdot, \to, 0, 1)$ is a bounded commutative residuated monoid satisfying all the assumptions of Theorem 5.1. The poset $(P_a(\mathbf{Q}), \leqslant_t)$ is depicted in Figure 2.

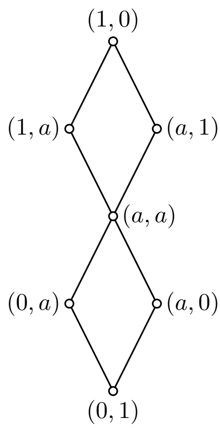

Figure 2.

Then the operators $\odot$ and $\Rightarrow$ have the following tables:

$\odot$	$0a$	01	$a0$	aa	$a1$	10	$1a$
$0a$	01	01	01	01	01	$0a, 01$	$0a, 01$
01	01	01	01	01	01	01	01
$a0$	01	01	$a0$	$a0, a1$	$a0, a1$	$a0$	$a0, a1$
aa	01	01	$a0, a1$	$a1$	$a1$	$a0, aa$	$aa, a1$
$a1$	01	01	$a0, a1$	$a1$	$a1$	$a0, a1$	$a1$
10	$0a, 01$	01	$a0$	$a0, aa$	$a0, a1$	10	$10, 1a$
$1a$	$0a, 01$	01	$a0, a1$	$aa, a1$	$a1$	$10, 1a$	$1a$

$\Rightarrow$	$0a$	01	$a0$	aa	$a1$	10	$1a$
$0a$	10	$a0, 10$	10	10	$a0, 10$	10	10
01	10	10	10	10	10	10	10
$a0$	$0a$	$0a$	10	$0a, 1a$	$0a, 1a$	10	$0a, 1a$
aa	$0a, 1a$	$0a, aa$	10	$1a$	$aa, 1a$	10	$1a$
$a1$	$0a, 1a$	$0a, 1a$	10	$1a$	$1a$	10	$1a$
10	$0a$	01	$a0, 10$	$0a, aa$	$01, a1$	10	$0a, 1a$
$1a$	$0a, 1a$	$01, a1$	$a0, 10$	$aa, 1a$	$a1$	10	$1a$

Hence $(P_a(\mathbf{Q}), \leqslant_t, \odot, \Rightarrow, (0,1), (1,0))$ is an operator residuated poset and $(P_a(\mathbf{Q}), \leqslant_t, ')$ a Kleene lattice.

Example 5.3. On the other hand, the bounded commutative residuated monoid $(Q, \leqslant, \cdot, \to, 1)$ from Example 3.4 has only two idempotents, namely 0 and 1. Every element of $P_0(\mathbf{Q}) = (\{0\} \times Q) \cup (Q \times \{0\})$ is comparable with $(0,0)$. But if $x \neq 0, 1$ then $0 < x$, but $x \to 0 \neq 0$ contradicting (12). Similarly, every element of $P_1(\mathbf{Q}) = (\{1\} \times Q) \cup (Q \times \{1\})$ is comparable with $(1,1)$. But if $x \neq 0, 1$ then $0 < 1 \cdot x < 1$ contradicting (11).

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