Applications of Mathematics

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Applications of Mathematics, Vol. 67 (2022), No. 5, 593–613

Persistent URL: `http://dml.cz/dmlcz/151027`

67 (2022) APPLICATIONS OF MATHEMATICS No. 5, 593–613

STEADY BOUSSINESQ SYSTEM WITH MIXED BOUNDARY CONDITIONS INCLUDING FRICTION CONDITIONS

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Received February 11, 2021. Published online January 26, 2022.

Abstract. In this paper we are concerned with the steady Boussinesq system with mixed boundary conditions. The boundary conditions for fluid may include Tresca slip, leak, one-sided leak, velocity, vorticity, pressure and stress conditions together and the conditions for temperature may include Dirichlet, Neumann and Robin conditions together. For the problem involving the static pressure and stress boundary conditions, it is proved that if the data of the problem are small enough, then there exists a solution and the solution with small norm is unique. For the problem involving the total pressure and total stress boundary conditions, the existence of a solution is proved without smallness of the data.

Keywords: heat-convection; variational inequality; mixed boundary conditions; Tresca slip; leak boundary conditions; one-sided leak; pressure boundary condition; existence and uniqueness

MSC 2020: 35Q35, 35J87, 76D03, 76D05, 49J40

1. Introduction

In this paper we are concerned with the steady Boussinesq system

$$\begin{cases} -2\nabla\cdot(\mu(\theta)\mathcal{E}(v))+(v\cdot\nabla)v+\nabla p=(1-\alpha_0\theta)f, \\ \nabla\cdot v=0, \\ -\nabla\cdot(\kappa(\theta)\nabla\theta)+v\cdot\nabla(\gamma(\theta)\theta)=g \end{cases} \tag{1.1}$$

under mixed boundary conditions. Here v, p and θ are, respectively, velocity, pressure and temperature, and α_0 is a parameter for buoyancy effect, f is a body force, g is a heat source. The strain tensor $\mathcal{E}(v)$ is the one with the components $\varepsilon_{ij}(v)=\frac{1}{2}(\partial_{x_i}v_j+\partial_{x_j}v_i)$. Viscosity $\mu(\theta)$, thermal conductivity $\kappa(\theta)$ and specific heat $\gamma(\theta)$ of fluid depend on the temperature. The boundary conditions for fluid may include

DOI: 10.21136/AM.2022.0031-21

Tresca slip, leak, one-sided leaks, velocity, vorticity, pressure and stress conditions together and the conditions for temperature may include Dirichlet, Neumann and Robin conditions together.

Several papers are concerned with (1.1). In [11], [12] under homogeneous Dirichlet boundary condition for velocity and mixture of nonhomogeneous Dirichlet and Neumann conditions for temperature, the existence of a solution to (1.1) was studied. In [2] under mixture of nonhomogeneous Dirichlet, total pressure and vorticity boundary conditions for velocity and mixture of nonhomogeneous Dirichlet, Neumann and Robin boundary conditions for temperature, the existence of a solution was studied. In [4] variational inequalities for Navier-Stokes type operators were studied, which can describe (1.1) with one-sided flow boundary conditions for fluid and heat on a portion of boundary. In [15] under nonhomogeneous Dirichlet boundary condition for velocity and mixture of nonhomogeneous Dirichlet and Neumann conditions for temperature, where smoothness of boundary data is weaker than [11] and [12], the existence of a solution to (1.1) was studied. In [10] under homogeneous Dirichlet boundary condition for velocity and mixture of nonhomogeneous Dirichlet and homogeneous Neumann boundary conditions for temperature, the existence, uniqueness and smoothness of a weak solution were studied. In [13] when the boundary consists of several connected components, boundary value problem of (1.1) with nonhomogeneous Dirichlet boundary condition was studied. In [6] Dirichlet problem of (1.1) for arbitrarily large and very weak boundary data was studied. In [1] when the boundary consists of several connected components, Dirichlet problem of (1.1) under a weaker condition than [13] was studied.

In [14] a more general equation for heat conducting fluid with dissipative heating was studied under homogeneous Dirichlet condition for velocity and mixture of nonhomogeneous Dirichlet and homogeneous Neumann boundary conditions for temperature. In [9] the equation as in [14] was studied under more complicated mixed boundary conditions including friction conditions. From the result of [9] one can get the existence of solutions to (1.1) with the boundary conditions as in [9]. However, the result for the case of boundary conditions including the total pressure demands that the parameter for buoyancy effect α_0 is small enough in accordance with the data of the problem as in [14] (see (5.13) of [9]), and the result for the case of boundary conditions including the static pressure demands that the data of the problem satisfy two smallness conditions together (see (4.37) and (4.84) of [9]).

In this paper we get the existence of a solution to (1.1) under one smallness condition on the data of the problem for the case of static pressure, and without restriction on α_0 for the case of total pressure.

This paper consists of 5 sections. In Section 2, the problems and assumptions are stated. According to the boundary conditions for the fluid, Problem 1 and

Problem 2 are distinguished. Problem 1 includes the static pressure (correspondingly, the stress) in the boundary conditions, whereas Problem 2 includes the total pressure (correspondingly, the total stress).

In Section 3, we get the variational formulations, which consist of one variational inequality for velocity and a variational equation for temperature (Problems 1–VI and 2–VI). In the end of Section 3, the main results of this paper are stated (Theorems 3.1, 3.2). Theorem 3.1 for Problem 1 involving the static pressure and stress boundary conditions asserts that if the data of the problem are small enough, then there exists a solution and the solution with small norm is unique. However, Theorem 3.2 for Problem 2 involving the total pressure and total stress boundary conditions asserts the existence of a solution without smallness of the data.

Section 4 is devoted to the proof of Theorem 3.1. First in Subsection 4.1 we consider an auxiliary problem involving two parameters δ, ζ concerned with the norm of velocity (which is useful when there is fluid flux across a portion of boundary), one parameter λ concerned with the norm of temperature (which is useful to deal with buoyancy effect) and a parameter ε for approximation. We prove the existence of a solution to the auxiliary problem with parameters δ, ζ, λ, ε (Theorem 4.2). In Subsection 4.2 when the data of the problem are small enough, we determine the parameters δ, ζ, λ, and we get estimates independent of ε of solutions to an approximate problem. In Subsection 4.3 passing to the limits as ε goes to zero, we get the existence, uniqueness and estimates of a solution to the problem. Section 5 is devoted to the proof of Theorem 3.2 for Problem 2. To this end, we consider another auxiliary problem involving parameters ζ, λ, ε. Then, without smallness of the data, we determine parameters ζ, λ, and we get estimates independent of ε of solutions to an auxiliary problem. In this way we get the existence of a solution to the problem.

We will use the following notation. Let Ω be a connected bounded open subset of $\mathbb{R}^l$, $l = 2, 3$, $\partial\Omega \in C^{0,1}$,

$$\partial\Omega = \bigcup_{i=1}^{11} \overline{\Gamma}_i = \overline{\Gamma}_D \cup \overline{\Gamma}_R,$$

$\Gamma_D \cap \Gamma_R = \emptyset$, $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, $\Gamma_i = \bigcup_j \Gamma_{ij}$, where Γ_{ij} are connected open subsets of $\partial\Omega$ and $\Gamma_{ij} \in C^2$ for $i = 2, 3, 7$ and $\Gamma_{ij} \in C^1$ for others. When X is a Banach space, $\mathbf{X} = X^l$. Let $W^{k,p}(\Omega)$ be Sobolev spaces, $H^1(\Omega) = W^{1,2}(\Omega)$, and so $\mathbf{H}^1(\Omega) = \{H^1(\Omega)\}^l$. An inner product in the space $\mathbf{L}^2(\Omega)$ or $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$, and $(\cdot, \cdot)_{\Gamma_i}$ is an inner product in $\mathbf{L}^2(\Gamma_i)$ or $L^2(\Gamma_i)$. The duality pairing between a Sobolev space X and its dual one is denoted by $\langle \cdot, \cdot \rangle$, and $\langle \cdot, \cdot \rangle_{\Gamma_i}$ means the duality pairing between $\mathbf{H}^{1/2}(\Gamma_i)$ and $\mathbf{H}^{-1/2}(\Gamma_i)$ or between $H^{1/2}(\Gamma_i)$ and $H^{-1/2}(\Gamma_i)$. The inner product and norms in $\mathbb{R}^l$, respectively, are denoted by $(\cdot, \cdot)_{\mathbb{R}^l}$ and $|\cdot|$.

Let $n(x)$ and $\tau(x)$ be, respectively, outward normal and tangent unit vectors at x in $\partial\Omega$. When $f \in H^{-1/2}(\Gamma_i)$, if $\langle f, w\rangle_{\Gamma_i} \geqslant 0$ ($\leqslant 0$) for all $w \in C_0^\infty(\Gamma_i)$ with $w \geqslant 0$, then we denote it by $f \geqslant 0$ ($\leqslant 0$) on Γ_i. For convergence in spaces, $\to$ and $\rightharpoonup$ mean strong and weak convergence, respectively.

2. Problems and assumptions

For temperature we consider the boundary conditions

(bc$_1$) $\theta|_{\Gamma_D} = 0$,

(bc$_2$) $(\kappa(\theta)\frac{\partial\theta}{\partial n} + \beta(x)\theta)|_{\Gamma_R} = g_R(x)$, $\beta(x)$, $g_R(x)$-given functions on Γ_R.

Stress tensor $S(\theta, v, p)$ and total stress tensor $S^t(\theta, v, p)$ are, respectively, the ones with components $s_{ij} = -p\delta_{ij} + 2\mu(\theta)\varepsilon_{ij}(v)$ and $s^t_{ij} = -(p + \frac{1}{2}|v|^2)\delta_{ij} + 2\mu(\theta)\varepsilon_{ij}(v)$. Stress vector and total stress vector on the boundary surface, respectively, are $\sigma(\theta, v, p) = S \cdot n$ and $\sigma^t(\theta, v, p) = S^t \cdot n$. The values of normal stress vector and total normal stress vector on the boundary surface are respectively $\sigma_n(\theta, v, p) = \sigma \cdot n$ and $\sigma^t_n(\theta, v, p) = \sigma^t \cdot n$. And $\sigma_\tau(\theta, v, p) = \sigma(\theta, v, p) - \sigma_n(\theta, v, p)n$, $\sigma^t_\tau(\theta, v, p) = \sigma^t(\theta, v, p) - \sigma^t_n(\theta, v, p)n$.

Problem 1 is the one with the boundary conditions including the static pressure and stress

(bcs$_1$) $v|_{\Gamma_1} = 0$,

(bcs$_2$) $v_\tau|_{\Gamma_2} = 0$, $-p|_{\Gamma_2} = \phi_2$,

(bcs$_3$) $v_n|_{\Gamma_3} = 0$, $\operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\mu(\theta)$,

(bcs$_4$) $v_\tau|_{\Gamma_4} = 0$, $(-p + 2\mu(\theta)\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4$,

(bcs$_5$) $v_n|_{\Gamma_5} = 0$, $2(\mu(\theta)\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5$, α: a matrix,

(bcs$_6$) $(-pn + 2\mu(\theta)\varepsilon_n(v))|_{\Gamma_6} = \phi_6$,

(bcs$_7$) $v_\tau|_{\Gamma_7} = 0$, $(-p + \mu(\theta)\frac{\partial v}{\partial n} \cdot n)|_{\Gamma_7} = \phi_7$,

(bcs$_8$) $v_n = 0$, $|\sigma_\tau(\theta, v)| \leqslant g_\tau$, $\sigma_\tau(\theta, v) \cdot v_\tau + g_\tau|v_\tau| = 0$ on Γ_8,

(bcs$_9$) $v_\tau = 0$, $|\sigma_n(\theta, v, p)| \leqslant g_n$, $\sigma_n(\theta, v, p)v_n + g_n|v_n| = 0$ onΓ_9,

(bcs$_{10}$) $v_\tau = 0$, $v_n \geqslant 0$, $\sigma_n(\theta, v, p) + g_{+n} \geqslant 0$, $(\sigma_n(\theta, v, p) + g_{+n})v_n = 0$ on Γ_{10},

(bcs$_{11}$) $v_\tau = 0$, $v_n \leqslant 0$, $\sigma_n(\theta, v, p) - g_{-n} \leqslant 0$, $(\sigma_n(\theta, v, p) - g_{-n})v_n = 0$ on Γ_{11}.

Problem 2 is the one with the conditions including the total pressure and total stress

(bct$_1$) $v|_{\Gamma_1} = 0$,

(bct$_2$) $v_\tau|_{\Gamma_2} = 0$, $-(p + \frac{1}{2}|v|^2)|_{\Gamma_2} = \phi_2$,

(bct$_3$) $v_n|_{\Gamma_3} = 0$, $\operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\mu(\theta)$,

(bct$_4$) $v_\tau|_{\Gamma_4} = 0$, $(-p - \frac{1}{2}|v|^2 + 2\mu(\theta)\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4$,

(bct$_5$) $v_n|_{\Gamma_5} = 0$, $2(\mu(\theta)\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5$, α: a matrix,

(bct$_6$) $(-pn - \frac{1}{2}|v|^2 n + 2\mu(\theta)\varepsilon_n(v))|_{\Gamma_6} = \phi_6$,

(bct$_7$) $v_\tau|_{\Gamma_7} = 0$, $(-p - \frac{1}{2}|v|^2 + \mu(\theta)\frac{\partial v}{\partial n} \cdot n)|_{\Gamma_7} = \phi_7$,
(bct$_8$) $v_n = 0$, $|\sigma^t_\tau(\theta, v)| \leqslant g_\tau$, $\sigma^t_\tau(\theta, v) \cdot v_\tau + g_\tau|v_\tau| = 0$ on Γ_8,
(bct$_9$) $v_\tau = 0$, $|\sigma^t_n(\theta, v, p)| \leqslant g_n$, $\sigma^t_n(\theta, v, p)v_n + g_n|v_n| = 0$ on Γ_9,
(bct$_{10}$) $v_\tau = 0$, $v_n \geqslant 0$, $\sigma^t_n(\theta, v, p) + g_{+n} \geqslant 0$, $(\sigma^t_n(\theta, v, p) + g_{+n})v_n = 0$ onΓ_{10},
(bct$_{11}$) $v_\tau = 0$, $v_n \leqslant 0$, $\sigma^t_n(\theta, v, p) - g_{-n} \leqslant 0$, $(\sigma^t_n(\theta, v, p) - g_{-n})v_n = 0$ on Γ_{11},
where $\varepsilon_n(v) = \mathcal{E}(v)n$, $\varepsilon_{nn}(v) = (\mathcal{E}(v)n, n)_{\mathbb{R}^l}$, $\varepsilon_{n\tau}(v) = \mathcal{E}(v)n - \varepsilon_{nn}(v)n$, $v_n = v \cdot n$, $v_\tau = v-(v\cdot n)n$ and h_i, ϕ_i, α_{ij} (components of matrix α) are given functions or vectors of functions. Finally, $g_\tau \in L^2(\Gamma_8)$, $g_n \in L^2(\Gamma_9)$, $g_{+n} \in L^2(\Gamma_{10})$, $g_{-n} \in L^2(\Gamma_{11})$, $g_\tau > 0$, $g_n > 0$, $g_{+n} > 0$, $g_{-n} > 0$, at a.e. x of the portions of the boundary.

Remark 2.1. On the wall of a domain of fluid the stick boundary condition $v = 0$ is common, but on the inlets and outlets the pressure boundary condition is more common, because it is difficult to know the velocity profile except special cases and it is natural to prescribe the value of the pressure. According to measurement instruments, we can know the static pressure p or total pressure $\frac{1}{2}|v|^2+p$ (Bernoulli's pressure). Boundary conditions (bct$_1$)–(bct$_{11}$) are obtained from (bcs$_1$)–(bcs$_{11}$) by replacing p with $\frac{1}{2}|v|^2 + p$. For the meanings and physical background of the boundary conditions of friction types (bcs$_8$)–(bcs$_{11}$) and (bct$_8$)–(bct$_{11}$), we refer to Introduction of [8] and the references therein. For the other boundary conditions we refer the reader to Fig. 1 and the explanation on page 92 of [9].

For convenience in what follows, the problems with boundary conditions (bcs$_1$)–(bcs$_{11}$) and (bct$_1$)–(bct$_{11}$) are called, respectively, the case of static pressure and the case of total pressure.

We use the following assumption.

Assumption 2.1. We assume the following:

(1) $\Gamma_1 \neq \emptyset$ and $\Gamma_D \neq \emptyset$.

(2) If Γ_i, where i is 10 or 11, is nonempty, then at least one of $\{\Gamma_j\colon j \in \{\{2, 4, 7, 9, 10, 11\} \setminus \{i\}\}$ is nonempty and there exists a diffeomorphism in C^1 between Γ_i and Γ_j.

Also, Γ_{2j}, Γ_{3j}, Γ_{7j} are convex and

$$\Gamma_R \subset \Big(\bigcup_{i=1,3,5,8} \Gamma_i\Big). \tag{2.1}$$

(3) For the functions of (1.1) $f \in \mathbf{L}^3(\Omega)$, $g \in L^{6/5}(\Omega)$ and

$$\begin{aligned}
&\mu \in C(\mathbb{R}),\ 0 < \mu_0 \leqslant \mu(\xi) \leqslant \mu_1 < \infty \quad \forall\, \xi \in \mathbb{R},\\
&\kappa \in C(\mathbb{R}),\ 0 < \kappa_0 \leqslant \kappa(\xi) \leqslant \kappa_1 < \infty \quad \forall\, \xi \in \mathbb{R},\\
&\gamma \in C(\mathbb{R}),\ |\gamma(\xi)| \leqslant \gamma_0 \quad \forall\, \xi \in \mathbb{R}.
\end{aligned}$$

(4) For the functions of (bc_1)–(bc_2), (bcs_1)–(bcs_{11}), (bct_1)–(bct_{11}), $g_R \in L^{4/3}(\Gamma_R)$, $\beta_0 \geqslant \beta(x) \geqslant 0$, β_0-a constant, $\beta(x)$-measurable, $\phi_i \in H^{-1/2}(\Gamma_i)$, $i = 2,4,7$, and $\phi_i \in \mathbf{H}^{-1/2}(\Gamma_i)$, $i = 3,5,6$, the matrix α is positive, $\alpha_{ij} \in L^\infty(\Gamma_5)$.

Remark 2.2. For the necessity of (2) in Assumption 2.1 we refer the reader to Remark 2.1 of [9]. Applying Theorems 2.1, 2.2 of [7] on $\Gamma_{ij}, i = 2,3,7$, we will embed the boundary conditions for pressure, vorticity and an artificial condition into the variational formulations of the problem, and for the theorems the condition $\Gamma_{ij} \in C^2$, $i = 2,3,7$, is necessary. The condition for $\gamma(\xi)$ of (3) in Assumption 2.1 admits negative specific heat of the fluid.

3. Variational formulations and main results

Let

$$\begin{aligned}\mathbf{V} = \{u \in \mathbf{H}^1(\Omega)\colon\ \operatorname{div} u = 0,\ u|_{\Gamma_1} = 0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_7\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11})} = 0,\\ u_n|_{(\Gamma_3\cup\Gamma_5\cup\Gamma_8)} = 0\},\end{aligned}$$

$$K(\Omega) = \{u \in \mathbf{V}\colon\ u_n|_{\Gamma_{10}} \geqslant 0,\ u_n|_{\Gamma_{11}} \leqslant 0\}, \quad W^{1,2}_{\Gamma_D}(\Omega) = \{y \in W^{1,2}(\Omega)\colon\ y|_{\Gamma_D} = 0\}.$$

By (2.1), $v_n = 0$ on Γ_R, and so for $v \in \mathbf{V}$, $\theta \in W^{1,2}(\Omega)$ and $\varphi \in W^{1,2}_{\Gamma_D}(\Omega)$ we have

$$\langle v\cdot\nabla(\gamma(\theta)\theta), \varphi\rangle = (v_n\gamma(\theta)\theta, \varphi)_{\Gamma_R} - (\gamma(\theta)\theta v, \nabla\varphi) = -(\gamma(\theta)\theta v, \nabla\varphi). \tag{3.1}$$

3.1. Variational formulations: the case of static pressure. Applying Theorems 2.1, 2.2 of [7] and (3.1), we can see that smooth solutions (v,p,θ) of problem (1.1), (bc_1)–(bc_2), (bcs_1)–(bcs_{11}) satisfy the following (see (3.1)–(3.5) of [9]).

$$\begin{cases} 2(\mu(\theta)\mathcal{E}(v), \mathcal{E}(u)) + \langle (v\cdot\nabla)v, u\rangle + 2(\mu(\theta)k(x)v, u)_{\Gamma_2} \\ \qquad +2(\mu(\theta)\widetilde{S}\tilde v, \tilde u)_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} + (\mu(\theta)k(x)v, u)_{\Gamma_7} \\ \qquad -2(\mu(\theta)\varepsilon_{n\tau}(v), u)_{\Gamma_8} + (p - 2\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} \\ \qquad = \langle (1-\alpha_0\theta)f, u\rangle + \sum\limits_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum\limits_{i=3,5,6}\langle\phi_i, u\rangle_{\Gamma_i} \quad \forall\, u \in \mathbf{V}, \\ (\kappa(\theta)\nabla\theta, \nabla\varphi) - (\gamma(\theta)\theta v, \nabla\varphi) + (\beta\theta, \varphi)_{\Gamma_R} \\ \qquad = \langle g_R, \varphi\rangle_{\Gamma_R} + \langle g, \varphi\rangle\ \forall\,\varphi \in W^{1,2}_{\Gamma_D}(\Omega), \\ |\sigma_\tau(\theta, v)| \leqslant g_\tau,\ \sigma_\tau(\theta,v)\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\ |\sigma_n(\theta,v,p)| \leqslant g_n,\ \sigma_n(\theta,v,p)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\ \sigma_n(\theta,v,p) + g_{+n} \geqslant 0,\ (\sigma_n(\theta,v,p) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10}, \\ \sigma_n(\theta,v,p) - g_{-n} \leqslant 0,\ (\sigma_n(\theta,v,p) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}, \end{cases} \tag{3.2}$$

where $\widetilde{S}$ is the shape operator of boundary surface, $\tilde{v}$ and $\tilde{u}$ are expressions of the vectors v and u in a local orthogonal curvilinear coordinates on Γ_3 and $k(x) = \operatorname{div} n(x)$ (see Theorems 2.1 and 2.2 of [7]).

Define $a_0(\theta;\cdot,\cdot), a_1(\cdot,\cdot,\cdot)$, $f_1 \in \mathbf{V}^*$, $b_0(\theta;\cdot,\cdot)$, and $g_1 \in (W^{1,2}_{\Gamma_D}(\Omega))^*$ by

$$
\begin{aligned}
(3.3)\quad a_0(\theta; w, u) &= 2(\mu(\theta)\mathcal{E}(w), \mathcal{E}(u)) + 2(\mu(\theta)k(x)w, u)_{\Gamma_2} + 2(\mu(\theta)\widetilde{S}\widetilde{w}, \tilde{u})_{\Gamma_3} \\
&\quad + 2(\alpha(x)w, u)_{\Gamma_5} + (\mu(\theta)k(x)w, u)_{\Gamma_7} \quad \forall\, w, u \in \mathbf{V},\ \theta \in W^{1,2}(\Omega), \\
a_1(v, w, u) &= \langle (v\cdot\nabla)w, u\rangle \quad \forall\, v, w, u \in \mathbf{V}, \\
\langle f_1, u\rangle &= \sum_{i=2,4,7} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall\, u \in \mathbf{V}, \\
b_0(\theta; \tilde{\theta}, \varphi) &= (\kappa(\theta)\nabla\tilde{\theta}, \nabla\varphi) + (\beta(x)\tilde{\theta}, \varphi)_{\Gamma_R} \quad \forall\, \theta, \tilde{\theta} \in W^{1,2}(\Omega), \varphi \in W^{1,2}_{\Gamma_D}(\Omega), \\
\langle g_1, \varphi\rangle &= \langle g_R, \varphi\rangle_{\Gamma_R} + \langle g, \varphi\rangle \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega).
\end{aligned}
$$

Then, taking into account (3.2) and

$$
\sigma_\tau(\theta, v) = 2\mu(\theta)\varepsilon_{n\tau}(v), \quad \sigma_n(\theta, v, p) = -p + 2\mu(\theta)\varepsilon_{nn}(v),
$$

we introduce the following variational formulation for problem (1.1), (bc_1)–(bc_2), (bcs_1)–(bcs_{11}).

Problem 1–VE. Find $(v, \theta, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}) \in K(\Omega) \times W^{1,2}_{\Gamma_D}(\Omega) \times \mathbf{L}^2_\tau(\Gamma_8) \times L^2(\Gamma_9) \times H^{-1/2}(\Gamma_{10}) \times H^{-1/2}(\Gamma_{11})$ such that

$$
\begin{cases}
a_0(\theta; v, u) + a_1(v, v, u) - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9} - \langle \sigma_{+n}, u_n\rangle_{\Gamma_{10}} \\
\qquad\qquad - \langle \sigma_{-n}, u_n\rangle_{\Gamma_{11}} - \langle f - \alpha_0\theta f, u\rangle = \langle f_1, u\rangle \quad \forall\, u \in \mathbf{V}, \\
b_0(\theta; \theta, \varphi) - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle = \langle g_1, \varphi\rangle \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega), \\
|\sigma_\tau| \leqslant g_\tau,\ \sigma_\tau\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\
|\sigma_n| \leqslant g_n,\ \sigma_n v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\
\sigma_{+n} + g_{+n} \geqslant 0,\ \langle \sigma_{+n} + g_{+n}, v_n\rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10}, \\
\sigma_{-n} - g_{-n} \leqslant 0,\ \langle \sigma_{-n} - g_{-n}, v_n\rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11},
\end{cases}
$$

where $\mathbf{L}^2_\tau(\Gamma_8)$ is the subspace of $\mathbf{L}^2(\Gamma_8)$ consisting of functions so that $(u, n)_{\mathbf{L}^2(\Gamma_8)} = 0$.

We will find another variational formulation consisting of a variational inequality and a variational equation, which is equivalent to Problem 1–VE.

Define the functionals $\phi_\tau, \phi_n, \phi_+, \phi_-$, respectively, by

$$\begin{aligned}
\phi_\tau(\eta) &= \int_{\Gamma_8} g_\tau|\eta| \,\mathrm{d}x \quad \forall\, \eta \in \mathbf{L}^2_\tau(\Gamma_8),\\
\phi_n(\eta) &= \int_{\Gamma_9} g_n|\eta| \,\mathrm{d}x \quad \forall\, \eta \in L^2(\Gamma_9),\\
\phi_+(\eta) &= \int_{\Gamma_{10}} g_{+n}\eta \,\mathrm{d}x \quad \forall\, \eta \in L^2(\Gamma_{10}),\\
\phi_-(\eta) &= -\int_{\Gamma_{11}} g_{-n}\eta \,\mathrm{d}x \quad \forall\, \eta \in L^2(\Gamma_{11}).
\end{aligned}$$

Since if $u \in K(\Omega)$, then $u|_{\Gamma_8} \in \mathbf{L}^2_\tau(\Gamma_8)$, $u_n|_{\Gamma_9} \in L^2(\Gamma_9)$, $u_n|_{\Gamma_{10}} \in L^2(\Gamma_{10})$, $u_n|_{\Gamma_{11}} \in L^2(\Gamma_{11})$, in what follows we use the notation

$$\phi_\tau(u) = \phi_\tau(u|_{\Gamma_8}),\ \phi_n(u) = \phi_n(u_n|_{\Gamma_9}),\ \phi_+(u) = \phi_+(u_n|_{\Gamma_{10}}),\ \phi_-(u) = \phi_-(u_n|_{\Gamma_{11}})$$

for $u \in K(\Omega)$.

Define a functional $\Phi\colon\ \mathbf{V} \to \overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\infty\}$ by

$$\Phi(u) = \begin{cases} \phi_\tau(u) + \phi_n(u) + \phi_+(u) + \phi_-(u) & \forall\, u \in K(\Omega),\\ \infty & \forall\, u \notin K(\Omega). \end{cases}$$

Then Φ is proper, convex lower weak semi-continuous. Note $\Phi \geqslant 0$, since $u_n|_{\Gamma_{10}} \geqslant 0$, $u_n|_{\Gamma_{11}} \leqslant 0$ for all $u \in K(\Omega)$.

In the same way as for Problem I–VI of [8], we get the following variational formulation for Problem 1 (the case of static pressure) equivalent to Problem 1–VE which consists of a variational inequality and a variational equation (cf. Problem I–VI of [9]).

Problem 1–VI. Find $(v, \theta) \in \mathbf{V} \times W^{1,2}_{\Gamma_D}(\Omega)$ such that

$$\text{(3.4)} \qquad \begin{cases} a_0(\theta; v, u - v) + a_1(v, v, u - v) + \Phi(u) - \Phi(v) - \langle f - \alpha_0\theta f, u - v\rangle \\ \qquad\qquad\qquad\qquad \geqslant \langle f_1, u - v\rangle \quad \forall\, u \in \mathbf{V},\\ b_0(\theta; \theta, \varphi) - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle = \langle g_1, \varphi\rangle \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega). \end{cases}$$

Remark 3.1. When $(v, \theta, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ is a smooth solution to Problem 1–VE, for the existence of p such that (v, θ, p) satisfies (1.1) and boundary condition (bcs_1)–(bcs_{11}), we refer the reader to Theorem 3.1 of [8] (cf. Theorem 3.1 of [9]).

When (v, θ) is a solution to Problem 1–VI, for the existence of $\sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}$ such that $(v, \theta, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ satisfies Problem 1–VE, we refer the reader to Theorem 3.3 of [8].

3.2. Variational formulations: the case of total pressure. Taking $(v\cdot\nabla)v = \operatorname{rot} v\times v+\frac{1}{2}\operatorname{grad}|v|^2$ into account, we can see that smooth solutions (v,p,θ) of problem (1.1), (bc_1)–(bc_2), (bct_1)–(bct_{11}) satisfy the following:

$$
(3.5)\quad
\begin{cases}
2(\mu(\theta)\mathcal{E}(v),\mathcal{E}(u)) + \langle \operatorname{rot} v\times v, u\rangle + 2(\mu(\theta)k(x)v,u)_{\Gamma_2} \\
\qquad +2(\mu(\theta)\widetilde{S}\tilde{v},\tilde{u})_{\Gamma_3} + 2(\alpha(x)v,u)_{\Gamma_5} + (\mu(\theta)k(x)v,u)_{\Gamma_7} \\
\qquad -2(\mu(\theta)\varepsilon_{n\tau}(v),u)_{\Gamma_8} + \Big(p+\dfrac{1}{2}|v|^2 - 2\mu(\theta)\varepsilon_{nn}(v),u_n\Big)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} \\
\qquad = \langle (1-\alpha_0\theta)f,u\rangle + \displaystyle\sum_{i=2,4,7}\langle \phi_i,u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle\phi_i,u\rangle_{\Gamma_i} \quad \forall\, u\in\mathbf{V}, \\
(\kappa(\theta)\nabla\theta,\nabla\varphi) - (\gamma(\theta)\theta v,\nabla\varphi) + (\beta\theta,\varphi)_{\Gamma_R} \\
\qquad = \langle g_R,\varphi\rangle_{\Gamma_R} + \langle g,\varphi\rangle \ \forall\,\varphi\in W^{1,2}_{\Gamma_D}(\Omega), \\
|\sigma^t_\tau(\theta,v)|\leqslant g_\tau,\ \sigma^t_\tau(\theta,v)\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\
|\sigma^t_n(\theta,v,p)|\leqslant g_n,\ \sigma^t_n(\theta,v,p)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\
\sigma^t_n(\theta,v,p) + g_{+n}\geqslant 0,\ (\sigma^t_n(\theta,v,p)+g_{+n})v_n = 0 \quad \text{on } \Gamma_{10}, \\
\sigma^t_n(\theta,v,p) - g_{-n}\leqslant 0,\ (\sigma^t_n(\theta,v,p)-g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}.
\end{cases}
$$

Define $a_2(\cdot,\cdot,\cdot)$ by

$$a_2(v,u,w) = \langle \operatorname{rot} v\times u, w\rangle \quad \forall\, v,u,w\in\mathbf{V}.$$

Then taking into account (3.5) and

$$\sigma^t_\tau(\theta,v) = 2\mu(\theta)\varepsilon_{n\tau}(v),\quad \sigma^t_n(\theta,v,p) = -\Big(p+\frac{1}{2}|v|^2\Big) + 2\mu(\theta)\varepsilon_{nn}(v),$$

we introduce the following variational formulation for problem (1.1), (bc_1)–(bc_2), (bct_1)–(bct_{11}).

Problem 2–VE. Find $(v,\theta,\sigma^t_\tau,\sigma^t_n,\sigma^t_{+n},\sigma^t_{-n}) \in K(\Omega)\times W^{1,2}_{\Gamma_D}(\Omega)\times \mathbf{L}^2_\tau(\Gamma_8)\times L^2(\Gamma_9)\times H^{-1/2}(\Gamma_{10})\times H^{-1/2}(\Gamma_{11})$ such that

$$
\begin{cases}
a_0(\theta;v,u) + a_2(v,v,u) - (\sigma^t_\tau,u_\tau)_{\Gamma_8} - (\sigma^t_n,u_n)_{\Gamma_9} - \big\langle \sigma^t_{+n},u_n\big\rangle_{\Gamma_{10}} \\
\qquad - \big\langle\sigma^t_{-n},u_n\big\rangle_{\Gamma_{11}} - \langle f-\alpha_0\theta f,u\rangle = \langle f_1,u\rangle \quad \forall u\in\mathbf{V}, \\
b_0(\theta;\theta,\varphi) - \langle\gamma(\theta)\theta v,\nabla\varphi\rangle = \langle g_1,\varphi\rangle \quad \forall\,\varphi\in W^{1,2}_{\Gamma_D}(\Omega), \\
|\sigma^t_\tau|\leqslant g_\tau,\ \sigma^t_\tau\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\
|\sigma^t_n|\leqslant g_n,\ \sigma^t_n v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\
\sigma^t_{+n} + g_{+n}\geqslant 0,\ \big\langle\sigma^t_{+n}+g_{+n},v_n\big\rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10}, \\
\sigma^t_{-n} - g_{-n}\leqslant 0,\ \big\langle\sigma^t_{-n}-g_{-n},v_n\big\rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11}.
\end{cases}
$$

Then we get Problem 2–VI equivalent to Problem 2–VE, which consists of a variational inequality and a variational equation (cf. Problem II–VI of [9]).

Problem 2–VI. Find $(v,\theta) \in \mathbf{V} \times W^{1,2}_{\Gamma_D}(\Omega)$ such that

$$(3.6)\qquad \begin{cases} a_0(\theta; v, u-v) + a_2(v, v, u-v) + \Phi(u) - \Phi(v) - \langle f - \alpha_0 \theta f, u-v\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad \geqslant \langle f_1, u-v\rangle \quad \forall\, u \in \mathbf{V}, \\ b_0(\theta; \theta, \varphi) - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle = \langle g_1, \varphi\rangle \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega). \end{cases}$$

3.3. Main results. Main results of this paper are the following theorems.

Theorem 3.1. *Under Assumption* 2.1 *assume that* f, ϕ_i, $i = 2, \dots, 7$, g, g_R *are small enough (depending on* α_0*) in the spaces in* (3), (4) *of Assumption* 2.1 *(see* (4.5)*).*

Then there exists a solution (v,θ) *to Problem* I–VI *such that*

$$(3.7)\qquad \|v\|_{\mathbf{V}} \leqslant \frac{\mu_0}{K}, \quad \|\theta\|_{W^{1,2}(\Omega)} \leqslant c(\|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}),$$

where K *is the one in* (4.3) *below.*

If $\mu(\theta)$, $\kappa(\theta)$ *and* $\gamma(\theta)$ *are independent of* θ *and* $\|f\|_{\mathbf{L}^3}$ *is small enough, then the solution satisfying* $\|v\|_{\mathbf{V}} \leqslant c$, $\|\theta\|_{W^{1,2}(\Omega)} \leqslant c$ *for a constant* c *small enough is unique.*

Theorem 3.2. *Under Assumption* 2.1 *there exists a solution* (v,θ) *to Problem* 2–VI *such that*

$$(3.8)\qquad \|v\|_{\mathbf{V}} \leqslant c\Bigg(\|f\|_{\mathbf{L}^3} + \sum_{i=2-7} \|\phi_i\|_{\Gamma_i} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\Bigg),$$
$$\|\theta\|_{W^{1,2}(\Omega)} \leqslant c(\|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}).$$

4. Proof of Theorem 3.2

To prove Theorems 3.1 and 3.2, we use the following proposition.

Proposition 4.1. *Let* $A\colon X \to X^*$ *be an operator on the real reflexive Banach space* X*. Let* A *be coercive and bounded. If for every sequence* $\{x_n\}$ *such that*

$$x_n \rightharpoonup x \quad \text{in } X,$$
$$\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle \leqslant 0$$

there exists a subsequence $\{x_k\}$ *such that*

$$\liminf_{k\to\infty} \langle Ax_k, x_k - v\rangle \geqslant \langle Ax, x - v\rangle \quad \forall\, v \in X,$$

then for any $f \in X^*$ *there exists a solution to*

$$Au = f.$$

(See Proposition 4.4 and Remark 4.1 of [8].)

For every $\varepsilon > 0$, Φ_ε (the Moreau regularization of Φ) is defined by

$$\Phi_\varepsilon(y) = \inf\Big\{\frac{\|y-u\|_{\mathbf{V}}^2}{2\varepsilon} + \Phi(u);\ u \in \mathbf{V}\Big\}, \quad y \in \mathbf{V}.$$

When $\partial\Phi\colon \mathbf{V} \to 2^{\mathbf{V}}$ is the sub-differential of Φ, let $J_\varepsilon = (I + \varepsilon\partial\Phi)^{-1}$ and $(\partial\Phi)_\varepsilon := \varepsilon^{-1}(I - J_\varepsilon)$ (the Yosida approximation of $\partial\phi$) for all $\varepsilon > 0$. Then the functional Φ_ε is convex, continuous, Fréchet differentiable and $\nabla\Phi_\varepsilon = (\partial\Phi)_\varepsilon \equiv \varepsilon^{-1}(I - J_\varepsilon)$ for all $\varepsilon > 0$. Moreover,

$$\Phi_\varepsilon(y) = \frac{\|y - J_\varepsilon y\|_{\mathbf{V}}^2}{2\varepsilon} + \Phi(J_\varepsilon y) \quad \forall\, y \in \mathbf{V}, \tag{4.1}$$

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(y) = \Phi(y), \quad \Phi(J_\varepsilon y) \leqslant \Phi_\varepsilon(y) \leqslant \Phi(y) \quad \forall\, y \in \mathbf{V} \tag{4.2}$$

(cf. Theorem2.9 of [3]). The operator $\nabla\Phi_\varepsilon$ is Lipschitz continuous with the constant ε^{-1} (see Theorem 2.9 and Proposition 2.3 of [3]) and monotone (see Lemma 4.10, Chapter III of [5]).

4.1. Existence of a solution to an auxiliary problem. Taking into account

$$|a_1(v,v,u)| = |((v\cdot\nabla)v, u)| \leqslant K\|v\|_{\mathbf{V}}^2\|u\|_{\mathbf{V}} \quad \forall\, v, u \in \mathbf{V}, \tag{4.3}$$

define $\overline{a}_1(v) \in \mathbf{V}^*$ by

$$\langle \overline{a}_1(v), u\rangle = a_1(v,v,u) \quad \forall\, v, u \in \mathbf{V}.$$

Also, define $\gamma_\varepsilon(t)$ by

$$\gamma_\varepsilon(t) := \frac{\gamma(t)t}{(1+\varepsilon|\gamma(t)|)(1+\varepsilon|t|)}, \quad t \in \mathbb{R},\ \varepsilon > 0.$$

Let us first consider an auxiliary problem for Problem 1–VI:

Problem 1–VIA. Let $\delta > 0$, $\zeta > 0$, $\lambda > 0$ and $\varepsilon > 0$. Find $(v,\theta) \in \mathbf{V} \times W_{\Gamma_D}^{1,2}(\Omega)$ such that

$$\begin{cases} a_0(\theta; v, u) + \dfrac{\delta}{\max\{\delta, \|\overline{a}_1(v)\|_{\mathbf{V}^*}\}} a_1(v,v,u) + \langle \nabla\Phi_\varepsilon(v), u\rangle \\ \qquad -\Big\langle \Big(1 - \dfrac{\lambda}{\max\{\lambda, \|\theta\|_{\mathbf{L}^2}\}}\alpha_0\theta\Big) f, u\Big\rangle = \langle f_1, u\rangle \quad \forall\, u \in \mathbf{V}, \\ b_0(\theta;\theta,\varphi) - \dfrac{\zeta}{\max\{\zeta, \|v\|_{\mathbf{V}}\}}\langle \gamma_\varepsilon(\theta)v, \nabla\varphi\rangle = \langle g_1, \varphi\rangle \quad \forall\, \varphi \in W_{\Gamma_D}^{1,2}(\Omega). \end{cases} \tag{4.4}$$

Theorem 4.1. *There exists a solution* $(v_\varepsilon, \theta_\varepsilon) \in \mathbf{V} \times W_{\Gamma_D}^{1,2}(\Omega)$ *to Problem* 1–VIA.

P r o o f. Let $\mathscr{H} = \mathbf{V} \times W^{1,2}_{\Gamma_D}(\Omega)$ and define an operator $\mathscr{A}\colon \mathscr{H} \to \mathscr{H}^*$ by

$$
\begin{aligned}
\langle \mathscr{A}(v,\theta),(u,\phi)\rangle = a_0(\theta;v,u) &+ \frac{\delta}{\max\{\delta,\|\overline{a}_1(v)\|_{\mathbf{V}^*}\}} a_1(v,v,u) + \langle \nabla\Phi_\varepsilon(v),u\rangle \\
&- \Big\langle \Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda,\|\theta\|_{\mathbf{L}^2}\}}\theta\Big) f, u\Big\rangle + b_0(\theta;\theta,\phi) \\
&- \frac{\zeta}{\max\{\zeta,\|v\|_{\mathbf{V}}\}} \langle \gamma_\varepsilon(\theta)v,\nabla\phi\rangle \quad \forall\,(v,\theta),(u,\phi)\in\mathscr{H},
\end{aligned}
$$

which is well-defined (cf. (4.10)–(4.14) of [9]). Then the existence of a solution to Problem 1–VIA is equivalent to the existence of a solution to

$$\mathscr{A}(v,\theta) = \mathscr{F}, \quad \mathscr{F} = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}.$$

In the same way as in the proof of Theorem 4.5 of [9] we can prove that the operator $\mathscr{A}$ satisfies the conditions of Proposition 4.1. Therefore, by virtue of Proposition 4.1 we come to the conclusion. $\square$

R e m a r k 4.1. Introducing the parameters δ, λ and ζ, we made a truncated problem, Problem 1–VIA. Owing to this truncation, we can define the operator $\mathscr{A}\colon \mathscr{H} \to \mathscr{H}^*$ satisfying the conditions of Proposition 4.1 (cf. the proof of Theorem 4.5 of [9]).

In the proof of Theorem 4.3 below, when the data of the problem are small enough, we will get estimates of the solutions to the truncated problem, which are independent of the parameters. Then, taking the parameters δ, λ and ζ in accordance with the estimates, we will show that the solutions to the truncated problem are solutions to an approximate problem (4.6) below.

4.2. Existence and estimates of solutions to an approximate problem.

Theorem 4.2. *If*

$$\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f+f_1\|_{\mathbf{V}^*} \leqslant \frac{\mu_0^2}{K c_{\alpha_0}}, \tag{4.5}$$

where K is the one in (4.3) *and c_{α_0} is the one in* (4.19) *below, then there exists a solution $(v_\varepsilon,\theta_\varepsilon) \in \mathbf{V}\times W^{1,2}_{\Gamma_D}(\Omega)$ to the problem*

$$
\left\{
\begin{aligned}
& a_0(\theta_\varepsilon;v_\varepsilon,u) + a_1(v_\varepsilon,v_\varepsilon,u) + \langle\nabla\Phi_\varepsilon(v_\varepsilon),u\rangle - \langle(1-\alpha_0\theta_\varepsilon)f,u\rangle \\
& \qquad\qquad\qquad\qquad\qquad\qquad = \langle f_1,u\rangle \quad \forall\,u\in\mathbf{V}, \\
& b_0(\theta_\varepsilon;\theta_\varepsilon,\varphi) - \langle\gamma_\varepsilon(\theta_\varepsilon)v_\varepsilon,\nabla\varphi\rangle = \langle g_1,\varphi\rangle \quad \forall\,\varphi\in W^{1,2}_{\Gamma_D}(\Omega),
\end{aligned}
\right. \tag{4.6}
$$

and the solution satisfies:

$$\|v_\varepsilon\|_{\mathbf{V}} \leqslant \frac{\mu_0}{K}, \quad \|\theta_\varepsilon\|_{W^{1,2}_{\Gamma_D}(\Omega)} \leqslant c\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}. \tag{4.7}$$

P r o o f. Let $(v_\varepsilon, \theta_\varepsilon)$ be a solution to (4.4). Putting $\varphi = \theta_\varepsilon$ in the second equation of (4.4), we have

$$(\kappa(\theta_\varepsilon)\nabla\theta_\varepsilon, \nabla\theta_\varepsilon) + (\beta(x)\theta_\varepsilon, \theta_\varepsilon)_{\Gamma_R} - \frac{\zeta}{\max\{\zeta, \|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\theta_\varepsilon)v_\varepsilon, \nabla\theta_\varepsilon\rangle = \langle g_1, \theta_\varepsilon\rangle. \tag{4.8}$$

Let us prove

$$\langle\gamma_\varepsilon(\theta_\varepsilon)v_\varepsilon, \nabla\theta_\varepsilon\rangle = 0. \tag{4.9}$$

To this end, define

$$\Psi(t) := \int_0^t \gamma_\varepsilon(s)\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

Then $\Psi \in C^1(\mathbb{R})$ and

$$\begin{gathered}\nabla\Psi(\theta) = \gamma_\varepsilon(\theta)\nabla\theta,\ \Psi(\theta) \in W^{1,2}(\Omega) \quad \forall\,\theta \in W^{1,2}(\Omega),\\ \Psi(\theta)|_{\Gamma_D} = 0 \quad \forall\,\theta \in W^{1,2}_{\Gamma_D}(\Omega).\end{gathered} \tag{4.10}$$

Taking into account $v_\varepsilon \cdot n|_{\Gamma_R} = 0$, by (4.10) we have

$$\langle\gamma_\varepsilon(\theta)v_\varepsilon, \nabla\theta_\varepsilon\rangle = \int_\Omega \gamma_\varepsilon(\theta_\varepsilon)v_\varepsilon \cdot \nabla\theta_\varepsilon\,\mathrm{d}x = \int_\Omega v_\varepsilon \cdot \nabla\Psi(\theta_\varepsilon) = 0,$$

which means (4.9). Also,

$$|\langle g_1, \theta_\varepsilon\rangle| \leqslant \frac{\kappa_0}{4}\|\theta_\varepsilon\|^2_{H^1} + c\|g_1\|^2_{(W^{1,2}_{\Gamma_D})^*}, \quad (\beta(x)\theta_\varepsilon, \theta_\varepsilon)_{\Gamma_R} \geqslant 0. \tag{4.11}$$

By (4.8), (4.9) and (4.11), we have

$$\|\theta_\varepsilon\|^2_{W^{1,2}_{\Gamma_D}} \leqslant \frac{2c}{\kappa_0}\|g_1\|^2_{(W^{1,2}_{\Gamma_D})^*}, \tag{4.12}$$

which implies

$$\|\theta_\varepsilon\|_{\mathbf{L}^2} \leqslant c_1\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}. \tag{4.13}$$

Putting

$$\lambda = c_1\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}$$

and taking into account (4.13), we have

$$\frac{\lambda}{\max\{\lambda, \|\theta_\varepsilon\|_{\mathbf{L}^2}\}} = 1 \tag{4.14}$$

and

$$\Bigl|\Bigl\langle\frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|_{\mathbf{L}^2}\}}\theta_\varepsilon f, u\Bigr\rangle\Bigr| \leqslant c_1\alpha_0\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3}\|u\|_{\mathbf{V}}. \tag{4.15}$$

Putting $u = v_\varepsilon$ in the first equation of (4.4), we have

$$a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + \frac{\delta}{\max\{\delta, \|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*}\}} a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\rangle - \Big\langle \Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|_{\mathbf{L}^2}\}}\theta_\varepsilon\Big) f, v_\varepsilon \Big\rangle = \langle f_1, v_\varepsilon\rangle. \tag{4.16}$$

Since Γ_{2j}, Γ_{3j}, Γ_{7j} are convex and the matrix α is positive, by Korn's inequality and Lemma A.3 of [7] we have from (3.3)

$$a_0(\theta; v, v) \geqslant 2\mu_0 \|v\|^2_{\mathbf{V}}. \tag{4.17}$$

Since the operator $\nabla\Phi_\varepsilon$ is monotone and $\nabla\Phi_\varepsilon(0_{\mathbf{V}}) = 0$, we have

$$\langle \nabla\Phi_\varepsilon(v), v\rangle = \langle \nabla\Phi_\varepsilon(v) - \nabla\Phi_\varepsilon(0_{\mathbf{V}}), v - 0_{\mathbf{V}}\rangle \geqslant 0. \tag{4.18}$$

Thus, taking into account (4.3), (4.17), (4.18) and (4.15), we have from (4.16)

$$\begin{aligned} 2\mu_0\|v_\varepsilon\|^2_{\mathbf{V}} &\leqslant a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) \\ &\leqslant \frac{\delta}{\max\{\delta, \|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*}\}} |a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon)| + |\alpha_0\langle\theta_\varepsilon f, v_\varepsilon\rangle| + |\langle f + f_1, v_\varepsilon\rangle| \\ &\leqslant K\|v_\varepsilon\|^3_{\mathbf{V}} + c_{\alpha_0}(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f + f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}}. \end{aligned} \tag{4.19}$$

Note that the estimate above is independent of δ. This implies

$$0 \leqslant K\|v_\varepsilon\|^2_{\mathbf{V}} - 2\mu_0\|v_\varepsilon\|_{\mathbf{V}} + c_{\alpha_0}(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f + f_1\|_{\mathbf{V}^*}). \tag{4.20}$$

Let us consider a quadratic equation for $x > 0$ concerned with the inequality above:

$$Kx^2 - 2\mu_0 x + a = 0.$$

If $0 \leqslant Ka \leqslant \mu_0^2$, then there exists a nonnegative minimum root x_1 ($\leqslant \mu_0/K$) and a maximum root x_2. Thus, we can see from (4.20) that if

$$\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f + f_1\|_{\mathbf{V}^*} \leqslant \frac{\mu_0^2}{Kc_{\alpha_0}}, \tag{4.21}$$

then

$$\|v_\varepsilon\|_{\mathbf{V}} \leqslant \frac{\mu_0}{K} \quad \text{or} \quad \|v_\varepsilon\|_{\mathbf{V}} \geqslant x_2. \tag{4.22}$$

On the other hand, from (4.19) we have another estimation under consideration of δ

$$2\mu_0\|v_\varepsilon\|^2_{\mathbf{V}} \leqslant a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) \leqslant \delta\|v_\varepsilon\|_{\mathbf{V}} + c_{\alpha_0}(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f + f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}},$$

which implies

$$\|v_\varepsilon\|_{\mathbf{V}} \leqslant \frac{1}{2\mu_0}(\delta + c_{\alpha_0}(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f + f_1\|_{\mathbf{V}^*})). \tag{4.23}$$

In view of (4.22), let us take $\delta = K(\mu_0/K)^2 = \mu_0^2/K$.

Thus, in view of (4.5), we have from (4.23)

$$\|v_\varepsilon\|_{\mathbf{V}} \leqslant \frac{\delta}{2\mu_0} + \frac{1}{2\mu_0}\frac{\mu_0^2}{K} = \frac{\mu_0}{K}. \tag{4.24}$$

By (4.24) under condition (4.5) we have that $\|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*} \leqslant K\|v_\varepsilon\|_{\mathbf{V}}^2 \leqslant \mu_0^2/K$ (cf. (4.3)), and so we get

$$\frac{\delta}{\max\{\delta, \|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*}\}} = 1. \tag{4.25}$$

Taking $\zeta = \mu_0/K$, by (4.24) we get

$$\frac{\zeta}{\max\{\zeta, \|v_\varepsilon\|_{\mathbf{V}}\}} = 1. \tag{4.26}$$

By (4.14), (4.25) and (4.26), we see that under condition (4.21), $(v_\varepsilon, \theta_\varepsilon)$ satisfies (4.6). By virtue of (4.22) and (4.12), we get (4.7). □

4.3. Existence and uniqueness of a solution. First, by passing to the limit of solutions in Theorem 4.3, we will prove the existence of a solution to Problem 1–VI. Since estimate (4.7) is independent of the parameter ε, we can extract subsequences, which are denoted as before, such that

$$\begin{aligned} v_\varepsilon &\rightharpoonup v \quad \text{in } \mathbf{V},\\ v_\varepsilon &\to v \quad \text{in } \mathbf{L}^q,\ 1 \leqslant q < 6,\\ \theta_\varepsilon &\rightharpoonup \theta \quad \text{in } W^{1,2}(\Omega),\\ \theta_\varepsilon &\to \theta \quad \text{in } L^q(\Omega),\ 1 \leqslant q < 6, \end{aligned}$$

as $\varepsilon \to 0$.

Subtracting the first formula of (4.6) with $u = v_\varepsilon$ from the first formula of (4.6), we have

$$\begin{aligned} a_0(\theta_\varepsilon; v_\varepsilon, u - v_\varepsilon) + a_1(v_\varepsilon, v_\varepsilon, u - v_\varepsilon) + \langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\rangle \\ - \langle (1 - \alpha_0\theta_\varepsilon) f, u - v_\varepsilon\rangle = \langle f_1, u - v_\varepsilon\rangle \quad \forall\, u \in \mathbf{V}. \end{aligned} \tag{4.27}$$

By Corollaries 4.2 and 4.3 of [8]

$$\begin{gathered} a_0(\theta_\varepsilon; v_\varepsilon, u) \to a_0(\theta; v, u) \quad \text{as } \varepsilon \to 0,\\ \liminf_{\varepsilon\to 0} a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) \geqslant a_0(\theta; v, v), \end{gathered}$$

which imply that

$$\limsup_{\varepsilon\to 0} a_0(\theta_\varepsilon; v_\varepsilon, u - v_\varepsilon) \leqslant a_0(\theta; v, u - v). \tag{4.28}$$

It is easy to prove

$$a_1(v_\varepsilon, v_\varepsilon, u - v_\varepsilon) \to a_1(v, v, u - v) \quad \text{as } \varepsilon \to 0. \tag{4.29}$$

Since Φ_ε is convex, continuous and Fréchet differentiable, we have

$$\Phi_\varepsilon(u) - \Phi_\varepsilon(v_\varepsilon) \geqslant \langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\rangle \quad \forall\, u \in \mathbf{V}, \tag{4.30}$$

which owing to (4.2) implies

$$\Phi_\varepsilon(u) - \Phi(J_\varepsilon v_\varepsilon) \geqslant \langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\rangle \quad \forall\, u \in \mathbf{V}. \tag{4.31}$$

Since $\Phi(0_{\mathbf{V}}) = 0$, by (4.2) $\Phi_\varepsilon(0_{\mathbf{V}}) = 0$, and so from (4.30) we have

$$\Phi_\varepsilon(v_\varepsilon) \leqslant \langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\rangle. \tag{4.32}$$

On the other hand, putting $u = v_\varepsilon$ in the first formula of (4.6), we have

$$a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\rangle = \langle (1-\alpha_0\theta_\varepsilon)f, v_\varepsilon\rangle + \langle f_1, v_\varepsilon\rangle. \tag{4.33}$$

From (4.32) and (4.33) we have

$$a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \Phi_\varepsilon(v_\varepsilon) \leqslant \langle (1-\alpha_0\theta_\varepsilon)f, v_\varepsilon\rangle + \langle f_1, v_\varepsilon\rangle,$$

from which we get

$$|\Phi_\varepsilon(v_\varepsilon)| \leqslant c(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f\|_{\mathbf{L}^3} + \|f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}} + |a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon)|. \tag{4.34}$$

By virtue of (4.1), (4.3), (4.23) and (4.34), we have

$$\|v_\varepsilon - J_\varepsilon v_\varepsilon\|^2_{\mathbf{V}} \leqslant \Big[c(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f\|_{\mathbf{L}^3} + \|f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}})\frac{\mu_0}{K} + \frac{\mu_0^3}{K^2}\Big]2\varepsilon,$$

which shows that since $v_\varepsilon \rightharpoonup v$ in $\mathbf{V}$,

$$J_\varepsilon v_\varepsilon \rightharpoonup v \quad \text{in } \mathbf{V} \quad \text{as } \varepsilon \to 0.$$

Then, by virtue of lower weak semi-continuity of $\Phi(v)$

$$\liminf_{\varepsilon\to 0} \Phi(J_\varepsilon v_\varepsilon) \geqslant \Phi(v). \tag{4.35}$$

By (4.2) we have

$$\Phi_\varepsilon(u) \to \Phi(u) \quad \text{as } \varepsilon \to 0. \tag{4.36}$$

Taking into account (4.35) and (4.36), we have from (4.31)

$$\Phi(u) - \Phi(v) \geqslant \limsup_{\varepsilon\to 0} \langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\rangle \quad \forall\, u \in \mathbf{V}. \tag{4.37}$$

Using

$$\begin{aligned}|\langle \theta_\varepsilon f, v_\varepsilon\rangle - \langle \theta f, v\rangle| &\leqslant |\langle \theta_\varepsilon f, v_\varepsilon\rangle - \langle \theta f, v_\varepsilon\rangle| + |\langle \theta f, v_\varepsilon\rangle - \langle \theta f, v\rangle| \\ &\leqslant \|\theta_\varepsilon - \theta\|_{L^3}\|f\|_{\mathbf{L}^2}\|v_\varepsilon\|_{\mathbf{L}^6} + \|\theta\|_{L^6}\|f\|_{\mathbf{L}^2}\|v_\varepsilon - v\|_{\mathbf{L}^3},\end{aligned}$$

we can prove

$$\langle (1-\alpha_0\theta_\varepsilon) f, u - v_\varepsilon\rangle \to \langle (1-\alpha_0\theta) f, u - v\rangle \quad \text{as } \varepsilon \to 0. \tag{4.38}$$

It is easy to prove

$$\langle f_1, u - v_\varepsilon\rangle \to \langle f_1, u - v\rangle \quad \text{as } \varepsilon \to 0. \tag{4.39}$$

By virtue of (4.28), (4.29), (4.37), (4.38) and (4.39), from (4.27) we get

$$\begin{aligned}a_0(\theta; v, u-v) + a_1(v, v, u-v) + \Phi(u) - \Phi(v)& \\ - \langle (1-\alpha_0\theta) f, u - v\rangle \geqslant \langle f_1, u - v\rangle \quad \forall\, u \in \mathbf{V},&\end{aligned}$$

which is the first formula in (3.4).

We will get the second equation of (3.4). By Corollary 4.2 of [8], we have

$$b_0(\theta_\varepsilon; \theta_\varepsilon, \varphi) \to b_0(\theta; \theta, \varphi) \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{4.40}$$

Let us prove

$$\langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla\phi\rangle \to \langle \gamma(\theta)\theta v, \nabla\varphi\rangle \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{4.41}$$

By Hölder's inequality, we have

$$\begin{aligned}&|\langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla\varphi\rangle - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle| \\ &\quad\leqslant |\langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla\varphi\rangle - \langle \gamma(\theta)\theta v_\varepsilon, \nabla\varphi\rangle| + |\langle \gamma(\theta)\theta v_\varepsilon, \nabla\varphi\rangle - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle| \\ &\quad\leqslant \|\gamma_\varepsilon(\theta_\varepsilon) - \gamma(\theta)\theta\|_{L^3}\|v_\varepsilon\|_{\mathbf{L}^6}\|\nabla\varphi\|_{\mathbf{L}^2} + \|\gamma(\theta)\theta\|_{L^4}\|v_\varepsilon - v\|_{\mathbf{L}^4}\|\nabla\varphi\|_{\mathbf{L}^2}.\end{aligned} \tag{4.42}$$

By the definition of $\gamma_\varepsilon(t)$, we have

$$\begin{aligned}\|\gamma_\varepsilon(\theta_\varepsilon) - \gamma(\theta)\theta\|_{L^3} &\leqslant \Big\| \frac{\gamma(\theta_\varepsilon)\theta_\varepsilon}{(1+\varepsilon|\gamma(\theta_\varepsilon)|)(1+\varepsilon|\theta_\varepsilon|)} - \gamma(\theta)\theta\Big\|_{L^3} \\ &\leqslant \|\gamma(\theta_\varepsilon)\theta_\varepsilon - \gamma(\theta)\theta\|_{L^3} + \varepsilon\|\gamma(\theta)\theta(|\gamma(\theta_\varepsilon)| + |\theta_\varepsilon| + \varepsilon|\gamma(\theta_\varepsilon)||\theta_\varepsilon|)\|_{L^3}.\end{aligned} \tag{4.43}$$

Then we see that $\gamma(\theta_k)$ converges to $\gamma(\theta)$ in space $L^p(\Omega)$ for any $p \in (1,\infty)$ as ε goes to zero (see Lemma 4.1 of [8]). Thus, from (4.42), (4.43) we get (4.41).

By virtue of (4.40) and (4.41), from the second formula of (4.6) we get the second formula of (3.4). Estimates (3.7) follow from (4.7).

Next, let us prove the uniqueness of the solution. Suppose that there are two solutions (v_1, θ_1) and (v_2, θ_2). Since μ is independent of θ, denoting $a_0(\cdot; v, u)$ by $a_0(v, u)$ we have

$$\begin{aligned} a_0(v_1, v_2 - v_1) + a_1(v_1, v_1, v_2 - v_1) + \Phi(v_2) - \Phi(v_1) - \langle f - \alpha_0 \theta_1 f, v_2 - v_1\rangle \\ \geqslant \langle f_1, v_2 - v_1\rangle, \\ a_0(v_2, v_1 - v_2) + a_1(v_2, v_2, v_1 - v_2) + \Phi(v_1) - \Phi(v_2) - \langle f - \alpha_0 \theta_2 f, v_1 - v_2\rangle \\ \geqslant \langle f_1, v_1 - v_2\rangle, \end{aligned}$$

which imply

$$\begin{aligned} a_0(v_1 - v_2, v_1 - v_2) \leqslant |\alpha_0| |(\theta_1 - \theta_2) f, v_1 - v_2)| \\ + |a_1(v_1, v_1, v_1 - v_2) - a_1(v_2, v_2, v_1 - v_2)|. \end{aligned} \tag{4.44}$$

By virtue of (4.44), we have

$$\begin{aligned} 2\mu \|v_1 - v_2\|_{\mathbf{V}}^2 &\leqslant \frac{\mu}{2} \|v_1 - v_2\|_{\mathbf{V}}^2 + c\|f\|_{\mathbf{L}^3}^2 \|\theta_1 - \theta_2\|^2 \\ &\quad + |a_1(v_1 - v_2, v_1, v_1 - v_2) + a_1(v_2, v_1 - v_2, v_1 - v_2)| \\ &\leqslant \frac{\mu}{2} \|v_1 - v_2\|_{\mathbf{V}}^2 + c\|f\|_{\mathbf{L}^3}^2 \|\theta_1 - \theta_2\|^2 + c_1(\|v_1\|_{\mathbf{V}} + \|v_2\|_{\mathbf{V}}) \|v_1 - v_2\|_{\mathbf{V}}^2 \end{aligned}$$

and so

$$\frac{3\mu}{2} \|v_1 - v_2\|_{\mathbf{V}}^2 \leqslant c\|f\|_{\mathbf{L}^3}^2 \|\theta_1 - \theta_2\|^2 + c_1(\|v_1\|_{\mathbf{V}} + \|v_2\|_{\mathbf{V}}) \|v_1 - v_2\|_{\mathbf{V}}^2. \tag{4.45}$$

Since $\kappa(\theta), \gamma(\theta)$ are independent of θ, put $\kappa(\theta) = \kappa, \gamma(\theta) = c_v$. Then from

$$\begin{aligned} (\kappa \nabla \theta_1, \nabla \varphi) + (\beta(x)\theta_1, \varphi)_{\Gamma_R} - c_v \langle v_1 \theta_1, \nabla \varphi_1\rangle = \langle g_1, \varphi\rangle, \\ (\kappa \nabla \theta_2, \nabla \varphi) + (\beta(x)\theta_2, \varphi)_{\Gamma_R} - c_v \langle v_2 \theta_2, \nabla \varphi\rangle = \langle g_1, \varphi\rangle \end{aligned}$$

we have

$$\begin{aligned} \kappa(\nabla\theta_1 - \nabla\theta_2, \nabla\theta_1 - \nabla\theta_2) + (\beta(x)(\theta_1 - \theta_2), \theta_1 - \theta_2)_{\Gamma_R} \\ - c_v \langle v_1(\theta_1 - \theta_2), \nabla(\theta_1 - \theta_2)\rangle - c_v \langle (v_1 - v_2)\theta_2, \nabla(\theta_1 - \theta_2)\rangle = 0. \end{aligned} \tag{4.46}$$

Taking into account $c_v \langle v_1(\theta_1 - \theta_2), \nabla(\theta_1 - \theta_2)\rangle = 0$ (see (3.1) with $\gamma(\theta)$=const), we have from (4.46)

$$\kappa \|\nabla\theta_1 - \nabla\theta_2\|_{\mathbf{L}^2}^2 \leqslant \frac{c_v c}{\kappa} \|v_1 - v_2\|_{\mathbf{V}}^2 \|\theta_2\|_{W_{\Gamma_D}^{1,2}}^2 + \frac{\kappa}{2} \|\nabla\theta_1 - \nabla\theta_2\|_{\mathbf{L}^2}^2,$$

and so

$$\frac{\kappa}{2} \|\theta_1 - \theta_2\|_{W^{1,2}}^2 \leqslant \frac{c_v c}{\kappa} \|v_1 - v_2\|_{\mathbf{V}}^2 \|\theta_2\|_{W_{\Gamma_D}^{1,2}}^2. \tag{4.47}$$

Therefore, summing (4.45) and (4.47), we get

$$\begin{aligned}(4.48)\quad \min\Big\{\frac{3\mu}{2},\frac{\kappa}{2}\Big\}&(\|v_1-v_2\|^2_{\mathbf{V}}+\|\theta_1-\theta_2\|^2_{W^{1,2}})\\ &\leqslant c\|f\|^2_{\mathbf{L}^3}\|\theta_1-\theta_2\|^2_{\mathbf{L}^2}+c_1(\|v_1\|_{\mathbf{V}}+\|v_2\|_{\mathbf{V}})\|v_1-v_2\|^2_{\mathbf{V}}\\ &\quad+\frac{c_v c}{\kappa}\|v_1-v_2\|^2_{\mathbf{V}}\|\theta_2\|^2_{W^{1,2}_{\Gamma_D}}.\end{aligned}$$

Thus, if $\|v_i\|_{\mathbf{V}}, \|\theta_2\|_{W^{1,2}_{\Gamma_D}}$ and $\|f\|_{\mathbf{L}^3}$ are small, then we have from (4.48) that $v_1=v_2$, $\theta_1=\theta_2$.

5. Proof of Theorem 3.2

First, we look for solutions to the auxiliary problem:

Problem 2–VIA. Let $\zeta>0$, $\lambda>0$ and $\varepsilon>0$. Find $(v,\theta)\in\mathbf{V}\times W^{1,2}_{\Gamma_D}(\Omega)$ such that

$$(5.1)\quad \begin{cases} a_0(\theta;v,u)+a_2(v,v,u)+\langle\nabla\Phi_\varepsilon(v),u\rangle-\Big\langle\Big(1-\dfrac{\alpha_0\lambda}{\max\{\lambda,\|\theta\|_{\mathbf{L}^2}\}}\theta\Big)f,u\Big\rangle\\ \qquad\qquad\qquad=\langle f_1,u\rangle\quad\forall\,u\in\mathbf{V},\\ b_0(\theta;\theta,\varphi)-\dfrac{\zeta}{\max\{\zeta,\|v_\varepsilon\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\theta)v,\nabla\varphi\rangle=\langle g_1,\varphi\rangle\quad\forall\,\varphi\in W^{1,2}_{\Gamma_D}(\Omega).\end{cases}$$

Theorem 5.1. *There exists a solution $(v_\varepsilon,\theta_\varepsilon)\in\mathbf{V}\times W^{1,2}(\Omega)$ to Problem* 2–VIA.

Proof. Let $\mathscr{H}=\mathbf{V}\times W^{1,2}_{\Gamma_D}(\Omega)$. Define an operator $\mathscr{A}\colon\mathscr{H}\to\mathscr{H}^*$ by

$$\begin{aligned}\langle\mathscr{A}(v,\eta),(u,\phi)\rangle&=a_0(\eta;v,u)+a_2(v,v,u)+\langle\nabla\Phi_\varepsilon(v),u\rangle\\ &\quad-\Big\langle\Big(1-\frac{\alpha_0\lambda}{\max\{\lambda,\|\theta\|_{\mathbf{L}^2}\}}\theta\Big)f,u\Big\rangle+b_0(\theta;\theta,\varphi)\\ &\quad-\frac{\zeta}{\max\{\zeta,\|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\eta)v,\nabla\varphi\rangle\quad\forall\,(v,\eta),(u,\phi)\in\mathscr{H}.\end{aligned}$$

Using

$$a_2(v,v,v)=0,\quad |a_2(v,v,u)|\leqslant K\|v\|^2_{\mathbf{V}}\|u\|_{\mathbf{V}},$$
$$|a_2(v_\varepsilon,v_\varepsilon,v_\varepsilon-u)|\leqslant c\|\nabla v_\varepsilon\|_{\mathbf{L}^2}\|v_\varepsilon\|_{\mathbf{L}^4}\|v_\varepsilon-u\|_{\mathbf{L}^4},$$

we can verify that the proof of Theorem 4.2 for Problem 1–VIA is valid for Problem 2–VIA. Thus, we come to the asserted conclusion (cf. Proof of Theorem 5.1 of [9]). □

Theorem 5.2. *There exists a solution* $(v_\varepsilon, \theta_\varepsilon) \in \mathbf{V} \times W^{1,2}_{\Gamma_D}(\Omega)$ *to the approximate problem*

$$(5.2)\qquad \begin{cases} a_0(\theta_\varepsilon; v_\varepsilon, u) + a_2(v_\varepsilon, v_\varepsilon, u) + \langle \nabla\Phi_\varepsilon(v_\varepsilon), u\rangle - \langle (1-\alpha_0\theta_\varepsilon) f, u\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad = \langle f_1, u\rangle \quad \forall\, u \in \mathbf{V}, \\ b_0(\theta_\varepsilon; \theta_\varepsilon, \varphi) - \langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla\varphi\rangle = \langle g_1, \varphi\rangle \quad \forall\, \varphi \in W^{1,2}_{\Gamma_D}(\Omega) \end{cases}$$

and the solution satisfies:

$$(5.3)\qquad \begin{gathered} \|v_\varepsilon\|_{\mathbf{V}} \leqslant \frac{c}{2\mu_0}(\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f\|_{\mathbf{V}^*} + \|f_1\|_{\mathbf{V}^*}), \\ \|\theta_\varepsilon\|_{W^{1,2}_{\Gamma_D}(\Omega)} \leqslant c\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}. \end{gathered}$$

P r o o f. Let $(v_\varepsilon, \theta_\varepsilon)$ be a solution to (5.1). In the same way as in (4.8)–(4.12) we have

$$(5.4)\qquad \|\theta_\varepsilon\|^2_{W^{1,2}_{\Gamma_D}} \leqslant \frac{2c}{\kappa_0}\|g_1\|^2_{(W^{1,2}_{\Gamma_D})^*}$$

which implies

$$\|\theta_\varepsilon(x)\|_{\mathbf{L}^2} \leqslant c_1\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}.$$

Then putting $\lambda = c_1\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}$, we have from the first equation of (5.1)

$$(5.5)\quad a_0(\theta_\varepsilon; v_\varepsilon, u) + a_2(v_\varepsilon, v_\varepsilon, u) + \langle \nabla\Phi_\varepsilon(v_\varepsilon), u\rangle - \langle (1-\alpha_0\theta_\varepsilon) f, v_\varepsilon\rangle = \langle f_1, u\rangle \quad \forall\, u \in \mathbf{V}.$$

Putting $u = v_\varepsilon$ in (5.5), we have

$$(5.6)\quad a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + a_2(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\rangle - \langle (1-\alpha_0\theta_\varepsilon) f, v_\varepsilon\rangle = \langle f_1, v_\varepsilon\rangle.$$

Taking into account $a_2(v_\varepsilon, v_\varepsilon, v_\varepsilon) = 0$, (4.17), (4.18) and (5.4), we have from (5.6)

$$2\mu_0\|v_\varepsilon\|^2_{\mathbf{V}} \leqslant a_0(\theta; v, v) \leqslant c(\alpha_0 c_1\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f\|_{\mathbf{V}^*} + \|f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}}$$

which implies

$$(5.7)\qquad \|v_\varepsilon\|_{\mathbf{V}} \leqslant \frac{c}{2\mu_0}(\alpha_0 c_1\|g_1\|_{(W^{1,2}_{\Gamma_D})^*}\|f\|_{\mathbf{L}^3} + \|f\|_{\mathbf{V}^*} + \|f_1\|_{\mathbf{V}^*}).$$

Taking the right-hand side of (5.7) as ζ in (5.1), we get the second equation of (5.2). By (5.4) and (5.7), we get (5.3). □

Now repeating the arguments in Subsection 4.3 with the solutions of Theorem 5.2, we complete the proof of Theorem 3.2. □

References

[1] *P. Acevedo, C. Amrouche, C. Conca*: Boussinesq system with non-homogeneous boundary conditions. Appl. Math. Lett. *53* (2016), 39–44. zbl MR doi

[2] *G. V. Alekseev, A. B. Smishliaev*: Solvability of the boundary-value problems for Boussinesq equations with inhomogeneous boundary conditions. J. Math. Fluid Mech. *3* (2001), 18–39. zbl MR doi

[3] *V. Barbu*: Nonlinear Differential Equations of Monotone Types in Banach Spaces. Springer Monographs in Mathematics. Springer, Berlin, 2010. zbl MR doi

[4] *A. Y. Chebotarev*: Variational inequalities for Navier-Stokes type operators and one-sided problems for equations of viscous heat-conducting fluids. Math. Notes *70* (2001), 264–274. zbl MR doi

[5] *H. Gajewski, K. Gröger, K. Zacharias*: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Mathematische Lehrbücher und Monographien. II. Abteilung 38. Academie, Berlin, 1974. (In German.) zbl MR

[6] *H. Kim*: The existence and uniqueness of very weak solutions of the stationary Boussinesq system. Nonlinear Anal., Theory Methods Appl., Ser. A *75* (2012), 317–330. zbl MR doi

[7] *T. Kim, D. Cao*: Some properties on the surfaces of vector fields and its application to the Stokes and Navier-Stokes problems with mixed boundary conditions. Nonlinear Anal., Theory Methods Appl., Ser. A *113* (2015), 94–114. zbl MR doi

[8] *T. Kim, D. Cao*: The steady Navier-Stokes and Stokes systems with mixed boundary conditions including one-sided leaks and pressure. Methods Appl. Anal. *23* (2016), 329–364. zbl MR doi

[9] *T. Kim, D. Cao*: Mixed boundary value problems of the system for steady flow of heat-conducting incompressible viscous fluids with dissipative heating. Methods Appl. Anal. *27* (2020), 87–124. zbl MR doi

[10] *D. A. Kovtunov*: Solvability of the stationary heat convection problem for a high-viscosity fluid. Differ. Equ. *45* (2009), 73–85. zbl MR doi

[11] *H. Morimoto*: On the existence of weak solutions of equations of natural convection. J. Fac. Sci., Univ. Tokyo, Sect. I A *36* (1989), 87–102. zbl MR

[12] *H. Morimoto*: On the existence and uniqueness of the stationary solution to the equation of natural convection. Tokyo J. Math. *14* (1991), 217–226. zbl MR doi

[13] *H. Morimoto*: Heat convection equation with nonhomogeneous boundary conditions. Funkc. Ekvacioj, Ser. Int. *53* (2010), 213–229. zbl MR doi

[14] *J. Naumann, M. Pokorný, J. Wolf*: On the existence of weak solutions to the equations of steady flow of heat-conducting fluids with dissipative heating. Nonlinear Anal., Real World Appl. *13* (2012), 1600–1620. zbl MR doi

[15] *E. J. Villamizar-Roa, M. A. Rodríguez-Bellido, M. A. Rojas-Medar*: The Boussinesq system with mixed nonsmooth boundary data. C. R., Math., Acad. Sci. Paris *343* (2006), 191–196. zbl MR doi

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