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# A FREE BOUNDARY PROBLEM FOR SOME MODIFIED PREDATOR-PREY MODEL IN A HIGHER DIMENSIONAL ENVIRONMENT

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*Abstract.* We focus on the free boundary problems for a Leslie-Gower predator-prey model with radial symmetry in a higher dimensional environment that is initially well populated by the prey. This free boundary problem is used to describe the spreading of a new introduced predator. We first establish that a spreading-vanishing dichotomy holds for this model. Namely, the predator either successfully spreads to the entire space as $t$ goes to infinity and survives in the new environment, or it fails to establish and dies out in the long term. The longterm behavior of the solution and the criteria for spreading and vanishing are also obtained. Moreover, when spreading of the predator happens, we provide some rough estimates of the spreading speed.

*Keywords*: free boundary; predator-prey model; spreading-vanishing dichotomy; spreading speed

*MSC 2020*: 35R35, 35K20, 35J60, 92B05

## 1. Introduction

In this work, we consider the behavior of the solution for the following Leslie-Gower predator-prey model with a free boundary:

$$(1.1)\qquad \begin{aligned} &u_t = d\Big(u_{rr} + \frac{n-1}{r}u_r\Big) + u(1-u) - \beta uv, && t>0,\ r>0,\\ &v_t = \Big(v_{rr} + \frac{n-1}{r}v_r\Big) + v\Big(1-\frac{v}{u+\delta}\Big), && t>0,\ 0<r<h(t),\\ &v(t,r) = 0, && t>0,\ r\geqslant h(t), \end{aligned}$$

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$$\begin{aligned}
&u_r(t,0)=v_r(t,0)=0, \quad h'(t)=-\mu v_r(t,h(t)), && t>0,\\
&h(0)=h_0, \quad v(0,r)=v_0(r), && 0\leqslant r\leqslant h_0,\\
&u(0,r)=u_0(r), && r\geqslant 0,
\end{aligned}$$

where $r=|x|$ ($x\in\mathbb{R}^n, n\geqslant 1$), $d>0$ describes the diffusivity of the prey, $\delta$ denotes the extent to which the environment provides protection to the predator with the intrinsic carrying capacity, and $\beta u$ denotes the functional response to predation. The parameters satisfy the assumption $\beta+\delta\beta<1$. The sphere $\{r=h(t)\}$ is the moving boundary to be determined, $h_0$ and $\mu$ are given positive constants and the initial functions $u_0(r)$ and $v_0(r)$ satisfy

$$(1.2)\qquad \begin{cases} u_0\in C^2([0,\infty))\cap L^\infty((0,\infty)),\ 0\leqslant u_0(x)\leqslant 1 & \text{in } [0,\infty),\\ v_0\in C^2([0,h_0]),\ v_0(h_0)=0 \text{ and } 0\leqslant v_0(x)\leqslant 1,\ v_0\not\equiv 0 & \text{in } [0,h_0). \end{cases}$$

The free boundary problem (1.1) describes the dynamical process of an introduced predator with population density $v(t,|x|)$ invading into the $n$-dimensional habitat of a native prey with population density $u(t,|x|)$. The initial function $v_0(|x|)$, which occupies a ball $\{r<h_0\}$, stands for the population of the predator in the beginning stage of its introduction. We shall consider that the prey population is initially uniformly well distributed and grows in the entire space $\mathbb{R}^n$. The predator population, which initially exists in the ball $\{r<h_0\}$, disperses through random diffusion over an expanding ball $\{r<h(t)\}$, whose boundary $\{r=h(t)\}$ is the spreading front and satisfies the free boundary condition $h'(t)=-\mu v_r(t,h(t))$. This is well-known as the Stefan condition. For the ecological background and derivation of the free boundary problems, one can also refer to [3], [9], [25], [26], [27] and their references. Especially, Mimura et al. in [27] consider the free boundary problems for some reaction-diffusion equations in one-dimensional space.

The problem (1.1) is a variation of the Leslie-Gower predator-prey model, which is considered over a bounded spatial domain with suitable boundary conditions or considered over the entire space $\mathbb{R}^n$ (see [11], [20], [21]). For the initial value problem of the Leslie-Gower predator-prey model, Ducrot in [15] has studied some spreading properties of the modified Leslie-Gower predator-prey reaction-diffusion system. In [5], we have considered the spreading speed properties for the Leslie-Gower predator-prey model with the fractional diffusion term $\Delta^\alpha(\alpha\in(0,1))$. In [6] and [7], we have shown the existence and stability of the Leslie-Gower predator-prey model with nonlocal diffusion. Liu et al. in [24] obtained the asymptotic behavior of two species evolving in a domain with a free boundary in a one-dimensional environment.

If the prey are only food for the predator, that is, $u = C$ ($C$ is a constant), the system reduces to the logistic diffusive equation with a free boundary condition, which has been studied in [12] for the one dimensional case, in [8] for the radially symmetric case and in [9] for the non-radially symmetric case in higher space dimensions. The behavior of the solution for these cases is characterized by a spreading-vanishing dichotomy. Moreover, when spreading occurs, it is shown that $h(t)/t \to k_0 \in (0, 2)$ as $t \to \infty$, and $k_0$ is called the asymptotic spreading speed of $v$. We can find further discussion of the spreading speed and a deduction of the free boundary condition based on ecological assumptions in [3]. Free boundary problems similar to the one in [12] have been studied by many authors, see [10], [14], [19], [28]. They have presented a new approach to describe the front propagation for a population, which is different from the classical method for traveling waves of the diffusive logistic equation on the entire space $\mathbb{R}^n$ with $n \geqslant 1$ (such as in [1], [2], [16]).

Recently, free boundary problems for the Lotka-Volterra competition model have been studied in several earlier papers. For example, many authors have considered the competition model in the one dimensional and homogeneous environment over a bounded spatial interval in [18], [23], over the half spatial line in [17], and over the half spatial line with zero Dirichlet boundary or zero Neuman boundary condition in [29] and with double free boundaries in [30]. For the higher space dimension case, Du et al. [13] studied the diffusive competition model in a homogeneous environment and Zhao et al. deduced the spreading and vanishing properties of the predator-prey model for the heterogeneous environment in [31]. They have all established the spreading-vanishing dichotomy, longterm behavior of the solution and sharp criteria for spreading and vanishing. The main purpose of this work is also to show that the similar results continue to hold for the Leslie-Gower predator-prey model in higher space dimensions and a homogeneous environment which is initially well-populated by the prey.

This paper is organized as follows. In Section 2, we first show the existence and uniqueness of the solution for (1.1). Moreover, we show some rough a priori estimates and some comparison principles. Section 3 is mainly devoted to the proof of the spreading-vanishing dichotomy. In Section 4, we obtain some rough estimates for the spreading speed in the case that spreading of $v$ happens.

## 2. Some preliminaries

In this section, we will prove the existence and uniqueness of the solution to (1.1) for all time $t > 0$. Then, we show some comparison results, which will be used in the following sections.

Here we give the local existence and uniqueness of the solution for system (1.1). The proof can be done by modifying the arguments for a general free boundary problem in [4], [8], [13], [31]. Thus, we omit the details.

**Theorem 2.1.** *For any given* $(u_0, v_0)$ *satisfying* (1.2) *and any* $\alpha \in (0,1)$, *there exists* $T > 0$ *such that problem* (1.1) *admits a unique bounded solution* $(u, v, h) \in (L^\infty(D_T^\infty) \cap C^{(1+\alpha)/2, 1+\alpha}(D_T^\infty)) \times C^{(1+\alpha)/2,1+\alpha}(D_T) \times C^{1+\alpha/2}([0,T])$. *Moreover,*

$$\|u\|_{C^{(1+\alpha)/2,1+\alpha}(D_T^\infty)} + \|v\|_{C^{(1+\alpha)/2,1+\alpha}(D_T)} + \|h\|_{C^{1+\alpha/2}([0,T])} \leqslant C,$$

*where*

$$D_T^\infty = \{(t,r) \in \mathbb{R}^2 \colon\ t \in [0,T], r \in [0,\infty)\},$$
$$D_T = \{(t,r) \in \mathbb{R}^2 \colon\ t \in [0,T],\ r \in [0,h(t)]\},$$

*positive constants* $C$ *and* $T$ *depend on* $h_0$, $\alpha$, $\|u_0\|_{L^\infty([0,\infty))}$, $\|v_0\|_{C^2([0,h_0])}$.

In order to obtain the global existence of the solution in Theorem 2.1, we need the following estimates of the solution $(u, v, h)$ of (1.1).

**Theorem 2.2.** *Let* $(u, v, h)$ *be a solution of* (1.1) *in Theorem* 2.1 *defined on* $[0, T]$ *for some* $T \in (0, \infty]$. *Then there exist positive constants* $M_1$, $M_2$, $M_3$ *independent of* $T$ *such that*

$$\begin{aligned} &0 \leqslant u(t,r) \leqslant M_1 \quad \text{for } 0 < t \leqslant T,\ r \geqslant 0,\\ &0 < v(t,r) \leqslant M_2 \quad \text{for } 0 < t \leqslant T,\ 0 \leqslant r < h(t),\\ &0 < h'(t) \leqslant M_3 \quad \text{for } 0 < t \leqslant T. \end{aligned} \tag{2.1}$$

Proof. By the strong maximum principle, it is easy to see that $u \geqslant 0$ in $(0,T] \times [0,\infty)$ and $v > 0$ in $(0,T] \times [0,h(t))$.

From the comparison principle, we have that $u(t,r) \leqslant \bar{u}(t)$ for $(t,r) \in (0,\infty)^2$, where $\bar{u}(t)$ is the solution for the following problem

$$\begin{cases} \dfrac{\mathrm{d}\bar{u}}{\mathrm{d}t} = \bar{u}(1-\bar{u}),\ t > 0,\\ \bar{u}(0) = \|u_0\|_{L^\infty([0,\infty))}. \end{cases} \tag{2.2}$$

Thus, we obtain that $u(t,r) \leqslant M_1 := \sup\limits_{t\geqslant 0} \bar{u}(t)$. Then $v(t,r)$ satisfies

$$\begin{aligned} &v_t - v_{rr} - \frac{n-1}{r} v_r \leqslant v\Big(1 - \frac{1}{M_1+\delta} v\Big), && t > 0,\ 0 < r < h(t),\\ &v_r(t,0) = v(t,h(t)) = 0, && t > 0,\\ &h'(t) = -\mu v_r(t,h(t)), && t > 0,\\ &v(0,r) = v_0(r) > 0, && 0 \leqslant r \leqslant h_0. \end{aligned}$$

By a similar argument as in [8], we have $v(t,r) \leqslant \max\{\|v_0\|_{C([0,h_0])}, M_1+\delta\} =: M_2$. Using the strong maximum principle, we can obtain $v_r(t,h(t)) < 0$ and $h'(t) > 0$ in $t \in (0,T]$.

It remains to prove that $h'(t) \leqslant M_3$ for $0 < t \leqslant T$ with some $M_3$ independent of $T$. To derive an upper bound of $h'(t)$, we define

$$\Omega_K := \{(t,r)\colon\ 0<t<T,\ h(t)-K^{-1}<r<h(t)\},$$

and construct an auxiliary function

$$w(t,r) = M_2[2K(h(t)-r) - K^2(h(t)-r)^2],$$

where $K$ is a positive constant such that $w(t,r) \geqslant v(t,r)$ holds on $\Omega_K$.

It is easy to show that for $(t,r) \in \Omega_K$, $w_t$ satisfies

$$\begin{aligned} w_t &= 2M_2Kh'(t)[1-K(h(t)-r)] \geqslant 0,\\ -w_r &= 2M_2K[1-K(h(t)-r)] \geqslant 0,\\ -w_{rr} &= 2M_2K^2. \end{aligned}$$

Then, if we choose $K$ satisfying $K^2 \geqslant \frac{1}{2}$, it easily follows that

$$w_t - w_{rr} - \frac{n-1}{r}w_r \geqslant 2M_2K^2 \geqslant v\Big(1-\frac{v}{u+\delta}\Big) \quad \text{in } \Omega_K.$$

Moreover, we have $w(t,h(t)-K^{-1}) = M_2 \geqslant v(t,h(t)-K^{-1})$ and $w(t,h(t)) = v(t,h(t)) = 0$. Since

$$v_0(r) = -\int_r^{h_0} v_0'(s)\,\mathrm{d}s \leqslant (h_0-r)\|v_0'\|_{C([0,h_0])} \quad \text{in } [h_0-K^{-1},h_0],$$

and

$$w(0,r) = M_2[2K(h_0-r) - K^2(h_0-r)^2] \geqslant M_2K(h_0-r) \quad \text{in } [h_0-K^{-1},h_0],$$

we know that if $KM_2 \geqslant \|v_0'\|_{C([0,h_0])}$, then

$$v_0(r) \leqslant (h_0-r)\|v_0'\|_{C([0,h_0])} \leqslant w(0,r) \quad \text{in } [h_0-K^{-1},h_0].$$

Let

$$K = \max\Big\{\sqrt{\frac{1}{2}}, \frac{\|v_0'\|_{C([0,h_0])}}{M_2}\Big\}.$$

By applying the maximum principle to $w-v$ on $\Omega_K$, we can obtain that $v(t,r) \leqslant w(t,r)$ for $(t,r) \in \Omega_K$, which implies that

$$v_r(t,h(t)) \geqslant w_r(t,h(t)) = -2M_2K,\ \ h'(t) = -\mu v_r(t,h(t)) \leqslant 2\mu M_2K =: M_3.$$

This completes the proof. □

**Theorem 2.3.** *If we assume that the conditions of Theorem* 2.1 *hold, then the unique solution obtained in Theorem* 2.1 *can be extended uniquely to all* $t \in [0,\infty)$*; that is, the problem* (1.1) *admits a unique bounded solution* $(u,v,h) \in (L^\infty(D^\infty) \cap C^{(1+\alpha)/2,1+\alpha}(D^\infty)) \times C^{(1+\alpha)/2,1+\alpha}(D_\infty) \times C^{1+\alpha/2}([0,\infty))$*, where* $D^\infty = \{(t,r) \in \mathbb{R}^2\colon\ t \in [0,\infty),\ r \in [0,\infty)\}$*,* $D_\infty = \{(t,r) \in \mathbb{R}^2\colon\ t \in [0,\infty),\ r \in [0,h(t)]\}$*.*

Proof. Let $[0,T_{\max})$ be the maximal time interval. By Theorem 2.1, $T_{\max} > 0$. It remains to show that $T_{\max} = \infty$. On the contrary, we assume $T_{\max} < \infty$. By Theorem 2.2, there exist positive constants $M_1$, $M_2$ and $M_3$ independent of $T_{\max}$ such that

$$\begin{aligned} &0 \leqslant u(t,r) \leqslant M_1 \quad \text{in } [0,T_{\max}) \times [0,\infty),\\ &0 \leqslant v(t,r) \leqslant M_2 \quad \text{in } [0,T_{\max}) \times [0,h(t)],\\ &0 \leqslant h'(t) \leqslant M_3, \quad 0 \leqslant h(t) - h_0 \leqslant M_3 t \quad \text{in } [0,T_{\max}). \end{aligned}$$

By the standard $L^p$ estimates and the Sobolev embedding theorem, we can find a constant $C > 0$ depending on $M_i$ $(i = 1,2,3)$ such that $u$ is continuous for $(t,r) \in [0,T_{\max}) \times [0,\infty)$ and $\|v(t,\cdot)\|_{C^{1+\alpha/2}([0,h(t)])} \leqslant C$. Then it follows from the proof of Theorem 2.1 that there exists a $\tau > 0$ depending on $C$ and $M_i$ $(i = 1,2,3)$ such that the solution of problem (1.1) with initial time $T_{\max} - \tau/2$ can be extended uniquely to the time $T_{\max} - \tau/2 + \tau$. However, this contradicts the assumption. Thus, we get that the solution exists for all $t \in [0,\infty)$.

Due to the positive constants independent of $T$ in Theorem 2.2, the estimates in (2.1) still hold for all $t \in [0,\infty)$. The proof is completed. □

In what follows, we present some comparison principles which will be used in the following sections.

**Lemma 2.1** (Comparison Principle). *Assume that* $T \in (0,\infty)$*,* $\bar h \in C^1([0,T])$*,* $\bar u \in L^\infty(D_T^\infty) \cap C^{1,2}(D_T^\infty)$*,* $\overline{v} \in C(\overline{D}_T^*) \cap C^{1,2}(D_T^*)$ *with* $D_T^* = \{(t,r) \in \mathbb{R}^2\colon\ t \in (0,T], r \in (0,\bar h(t))\}$*. If* $(\bar u, \overline{v}, \bar h)$ *satisfies*

$$\tag{2.3} \begin{cases} \bar u_t \geqslant \Big(\bar u_{rr} + \dfrac{n-1}{r}\bar u_r\Big) + \bar u(1-\bar u), & t > 0,\ r > 0,\\[2mm] \overline{v}_t \geqslant \Big(\overline{v}_{rr} + \dfrac{n-1}{r}\overline{v}_r\Big) + \overline{v}\Big(1 - \dfrac{\overline{v}}{\bar u + \delta}\Big), & t > 0,\ 0 < r < \bar h(t),\\ \bar u_r(t,0) = 0, \quad \overline{v}_r(t,0) = 0, & t > 0,\\ \overline{v}(t,\bar h(t)) = 0, \quad \bar h'(t) \geqslant -\mu \overline{v}_r(t,\bar h(t)), & t > 0, \end{cases}$$

*and*

$$\begin{gathered} \bar h(0) \geqslant h_0 \quad \textit{and} \quad \overline{v}(0,r) \geqslant 0 \quad \textit{in } [0,\bar h(0)],\\ u_0(r) \leqslant \bar u(0,r) \quad \textit{in } [0,\infty), v_0(r) \leqslant \overline{v}(0,r) \quad \textit{in } [0,h_0], \end{gathered}$$

*then the solution* $(u, v, h)$ *of* (1.1) *satisfies*

$$u(t,r) \leqslant \bar{u}(t,r) \quad \text{on } D_T^\infty, \quad v(t,r) \leqslant \overline{v}(t,r) \quad \text{on } D_T, \quad h(t) \leqslant \bar{h}(t) \quad \text{on } [0,T],$$

*where* $D_T^\infty$ *and* $D_T$ *are defined in Theorem* 2.1.

Proof. Assume $h_0 < \bar{h}(0)$. We claim that $h(t) < \bar{h}(t)$ for all $t \in (0,T]$. If not, then there exists $t^* \leqslant T$ such that $h(t) < \bar{h}(t)$ for $t \in (0,t^*)$ and $h(t^*) = \bar{h}(t^*)$. It is easy to see that

$$h'(t^*) \geqslant \bar{h}'(t^*). \tag{2.4}$$

We first show that $u \leqslant \bar{u}$ in $[0,t^*] \times [0,\infty)$. Let $U = \bar{u} - u$. Then $U$ satisfies

$$\begin{aligned} &U_t \geqslant d\Big(U_{rr} + \frac{n-1}{r}U_r\Big) + (1 - \widetilde{M_1})U, && 0 < t \leqslant t^*,\ r > 0,\\ &U_r(t,0) = 0, && 0 < t \leqslant t^*,\\ &U(0,r) \geqslant 0, && r \geqslant 0, \end{aligned}$$

which $\widetilde{M_1}$ is a constant dependent on $\bar{u}$ and $u$ in $[0,T] \times [0,\infty)$. By the maximum principle, we can obtain that $U(t,r) \geqslant 0$ in $[0,t^*] \times [0,\infty)$. That is, $u \leqslant \bar{u}$ in $[0,t^*] \times [0,\infty)$. Then choosing $W = (\overline{v} - v)\mathrm{e}^{-kt}$, it is easy to show that

$$\begin{cases} W_t \geqslant \Big(W_{rr} + \dfrac{n-1}{r}W_r\Big) + (1 - k - \widetilde{M_2})W, & 0 < t \leqslant t^*,\ 0 < r < h(t),\\ W(t,r) = 0, \quad W_r(t,0) = 0, & 0 < t \leqslant t^*,\ r \geqslant \bar{h}(t),\\ W(0,r) \geqslant 0, & r \geqslant 0, \end{cases} \tag{2.5}$$

where $\widetilde{M_2}$ is a constant dependent on $\overline{v}$, $v$ and $\bar{u}$ in $[0,T] \times [0,\infty)$ and $k$ is sufficiently large such that $k \geqslant \widetilde{M_2} + 2$.

Since the first inequality of (2.5) holds only in part of $[0,\infty)$, the maximum principle cannot be used directly. We first prove that for any $l > h(t^*)$,

$$W(t,r) \geqslant -\frac{\widetilde{M_2}(r^2 + 2nt)}{l^2} \quad \text{in } [0,t^*] \times [0,l].$$

We can apply the maximum principle to $\overline{v}$ over the region $\{(t,r)\colon 0 \leqslant t \leqslant T,\ 0 \leqslant r \leqslant \bar{h}(t)\}$ to deduce $\overline{v} \geqslant 0$. Set $\overline{W}(t,r) = W(t,r) + \widetilde{M_2}(r^2 + 2nt)/l^2$. Then $\overline{W}$ satisfies

$$\begin{aligned} &\overline{W}_t \geqslant \Big(\overline{W}_{rr} + \frac{n-1}{r}\overline{W}_r\Big) + (1 - k - \widetilde{M_2})\overline{W}, && 0 < t \leqslant t^*,\ 0 < r < h(t),\\ &\overline{W}(t,r) \geqslant \frac{\widetilde{M_2}(r^2 + 2nt)}{l^2} > 0,\ \overline{W}_r(t,0) = 0, && 0 < t \leqslant t^*,\ h(t) \leqslant r \leqslant l,\\ &\overline{W}(0,r) \geqslant 0, && 0 \leqslant r \leqslant l. \end{aligned}$$

Now we deduce $\min\limits_{[0,t^*]\times[0,l]} \overline{W} := \tau \geqslant 0$. If $\tau < 0$, then there exists $(t_1, r_1) \in \mathbb{R}^2$ with $0 < t_1 \leqslant t^*$ and $0 \leqslant r_1 < h(t_1)$ such that $\overline{W}(t_1, r_1) = \tau < 0$. It is easy to establish $(\overline{W}_t - \overline{W}_{rr} - (n-1)\overline{W}_r/r)(t_1, r_1) \leqslant 0$. However, due to our choice of $k$, we have

$$(1 - k - \widetilde{M}_2)\overline{W}(t_1, r_1) = (1 - k - \widetilde{M}_2)\tau \geqslant -\tau > 0.$$

This is a contradiction. Thus, it is easy to get $\tau \geqslant 0$. That is, $\overline{W} \geqslant 0$ in $[0, t^*] \times [0, l]$, which implies that

$$W(t, r) \geqslant -\frac{\widetilde{M}_2(r^2 + 2nt)}{l^2} \quad \text{for } [0, t^*] \times [0, l].$$

Taking $l \to \infty$ yields that $W(t, r) \geqslant 0$ in $[0, t^*] \times [0, \infty)$, and therefore $v \leqslant \overline{v}$ in $[0, t^*] \times [0, \infty)$. Since $V(t, r) = \overline{v}(t, r) - v(t, r)$ satisfies

$$V_t \geqslant \Big(V_{rr} + \frac{n-1}{r} V_r\Big) + (1 - \widetilde{M}_2)V, \quad 0 < t \leqslant t^*,\ 0 < r < h(t),$$

we can use the strong maximum principle and the Hopf boundary lemma to obtain that $V(t, r) > 0$ in $(0, t^*] \times [0, h(t)]$, and $V_r(t^*, h(t^*)) < 0$. Then we deduce $h'(t^*) < \bar{h}'(t^*)$. This contradicts (2.4). This proves our claim that $h(t) < \bar{h}(t)$ for all $t \in (0, T]$.

Next we apply the above products over $[0, T] \times [0, \infty)$ to conclude that $u \leqslant \bar{u}$ and $v \leqslant \overline{v}$ in $[0, T] \times [0, \infty)$. Moreover, $v \leqslant \overline{v}$ in $[0, T] \times [0, h(t))$.

If $h_0 = \bar{h}(0)$, let $(u_\varepsilon, v_\varepsilon, h_\varepsilon)$ denote the unique solution of (1.1) with $h_0$ replaced by $h_0(1 - \varepsilon)$ for small $\varepsilon > 0$. Since the unique solution of (1.1) depends continuously on the parameters in (1.1) as $\varepsilon > 0$, $(u_\varepsilon, v_\varepsilon, h_\varepsilon)$ converges to $(u, v, h)$. Then the desired results follow as $\varepsilon \to 0$ in the inequalities $u_\varepsilon \leqslant \bar{u}$, $v_\varepsilon \leqslant \overline{v}$ and $h_\varepsilon \leqslant \bar{h}$. □

**Lemma 2.2** (Comparison Principle). *Let $T \in (0, \infty)$, $\underline{h} \in C^1([0, T])$ with $\underline{h} > 0$ for all $t \in [0, T]$, and $\underline{v} \in C(\overline{D}_T^{**}) \cap C^{1,2}(D_T^{**})$ with $D_T^{**} = \{(t, r) \in \mathbb{R}^2 \colon t \in (0, T], r \in (0, \underline{h}(t))\}$. Suppose that $(\underline{v}, \underline{h})$ satisfies*

$$\begin{aligned}
&\underline{v}_t \leqslant \Big(\underline{v}_{rr} + \frac{n-1}{r}\underline{v}_r\Big) + \underline{v}(1 - \delta^{-1}\underline{v}), && t > 0,\ 0 < r < \underline{h}(t),\\
&\underline{v}_r(t, 0) = 0, \quad \underline{v}(t, \underline{h}(t)) = 0, && t > 0,\\
&\underline{h}'(t) \leqslant -\mu \underline{v}_r(t, \underline{h}(t)), && t > 0,
\end{aligned}$$

*and*

$$\underline{h}(0) \leqslant h_0 \quad \text{and} \quad v_0(r) \geqslant \underline{v}(0, r) \quad \text{in } [0, \underline{h}(0)],$$

*then the solution $(u, v, h)$ of (1.1) satisfies*

$$v(t, r) \geqslant \underline{v}(t, r) \quad \text{in } \overline{D}_T^{**},\ h(t) \geqslant \underline{h}(t) \quad \text{in } [0, T].$$

We omit the details of the proof which can be proved as in the process with the above lemma.

## 3. The spreading and vanishing

In this section, we prove the spreading-vanishing dichotomy of the free boundary problem (1.1). Since Theorem 2.2 implies that $r = h(t)$ is monotonic increasing, there exists $h_\infty \in (0, \infty]$ such that $\lim_{t\to\infty} h(t) = h_\infty$.

Let $\lambda_1(a, R)$ be the principal eigenvalue of the problem

$$
\begin{aligned}
-\Delta\varphi &= \lambda a\varphi, & x &\in B_R,\\
\varphi &= 0, & x &\in \partial B_R,
\end{aligned}
$$

where $a > 0$ is a constant and $B_R$ stands for the ball with center at 0 and radius $R$. It is well known in [4] that $\lambda_1(a, R)$ is a strictly decreasing continuous function in $R$ and satisfies

$$\lim_{R\to 0^+} \lambda_1(a, R) = \infty \quad \text{and} \quad \lim_{R\to\infty} \lambda_1(a, R) = 0.$$

Therefore, for $a > 0$, there exists a unique $R^*(a)$ such that $\lambda_1(a, R^*(a)) = 1$, $\lambda_1(a, R) < 1$ for $R > R^*(a)$, and $\lambda_1(a, R) > 1$ for $R < R^*(a)$. Since $\lambda_1(a, R)$ is a strictly decreasing continuous function in $a$ and $R$, we have that $R^*(a)$ is a strictly decreasing continuous function in $a$.

In order to investigate the asymptotic properties of the solution for (1.1), we first recall the following spreading-vanishing dichotomy for the radially symmetric diffusive logistic problem

$$
\begin{cases}
w_t = \left(w_{rr} + \dfrac{n-1}{r} w_r\right) + w(a - bw), & t > 0,\ 0 < r < h(t),\\
w_r(t, 0) = 0, \quad w(t, h(t)) = 0, & t > 0,\\
h'(t) = -\mu w_r(t, h(t)), & t > 0,\\
h(0) = h_0, \quad w(0, r) = w_0(r), & 0 \leqslant r \leqslant h_0.
\end{cases}
\tag{3.1}
$$

**Theorem 3.1** (Spreading-vanishing dichotomy, see Du and Guo [8]). *Assume that $(w(t, r), h(t))$ is the solution of the free boundary problem (3.1). Then one of the following holds.*

(i) *Spreading: $h_\infty = \infty$ and $\lim_{t\to\infty} w(t, r) = a/b$ uniformly in any compact subset of $[0, \infty)$.*

(ii) *Vanishing: $h_\infty \leqslant R^*(a)$ and $\lim_{t\to\infty} \|w(t, \cdot)\|_{C([0,h(t)])} = 0$.*

**Theorem 3.2** (See Du and Guo [8]). *If $h_0 \geqslant R^*(a)$, then spreading always happens. If $h_0 < R^*(a)$, then there exists $\mu^* > 0$ depending on $w_0$ such that vanishing occurs if $\mu \leqslant \mu^*$ and spreading happens if $\mu > \mu^*$.*

In order to investigate the asymptotic properties of solutions to (1.1), we will derive some properties for any global solution.

**Lemma 3.1.** *Let* $(u,v,h)$ *be any solution of* (1.1). *If* $h_\infty=\infty$, *then*

$$0\leqslant \liminf_{t\to\infty} u(t,r)\leqslant \limsup_{t\to\infty} u(t,r)\leqslant 1,\quad 0\leqslant \liminf_{t\to\infty} v(t,r)\leqslant \limsup_{t\to\infty} v(t,r)\leqslant 1.$$

P r o o f. First we recall that the comparison principle gives $0\leqslant u(t,r)\leqslant \bar u(t)$ for $t>0$, $r>0$, where

$$\bar u(t)=\mathrm{e}^t\Big(\mathrm{e}^t-1+\frac{1}{\|u_0\|_{L^\infty([0,\infty))}}\Big)^{-1}$$

is the solution of the problem (2.2). Since $\lim\limits_{t\to\infty}\bar u(t)=1$, we deduce that

$$\limsup_{t\to\infty} u(t,r)\leqslant 1 \text{ uniformly for } r\in[0,\infty).$$

Similarly, it is easy to obtain that

$$\limsup_{t\to\infty} v(t,r)\leqslant 1 \text{ uniformly for } r\in[0,\infty). \tag{3.2}$$

Since $h_\infty=\infty$, there exists $T>0$ such that $h(T)>R^*(1)$. Choose a function $\underline{v}_0(r)$ satisfying $\underline{v}_0\in C^2([0,h(t)])$, $\underline{v}_0(r)\leqslant v(T,r)$ in $[0,h(T)]$, $\underline{v}_0(r)>0$ in $(0,h(T))$ and $\underline{v}_0(h(T))=0$. We consider the following problem:

$$\begin{aligned}
&\underline{v}_t=\Big(\underline{v}_{rr}+\frac{n-1}{r}\underline{v}_r\Big)+\underline{v}\Big(1-\frac{1}{\delta}\underline{v}\Big), && t>T,\ 0<r<\underline{h}(t),\\
&\underline{v}_r(t,0)=\underline{v}(t,\underline{h}(t))=0, && t>T,\\
&\underline{h}(T)=h(T),\quad \underline{h}'(t)=-\mu\underline{v}_r(t,\underline{h}(t)), && t>T,\\
&\underline{v}(T,r)=\underline{v}_0(r), && 0\leqslant r\leqslant h(T).
\end{aligned}$$

By Theorem 2.1 of [8], this problem has a unique solution $(\underline{v},\underline{h})$ for all $t>T$. In view of Lemma 2.2, we have

$$\begin{aligned} v(t,r)&\geqslant \underline{v}(t,r) \quad \text{for } t\geqslant T,\ 0\leqslant r\leqslant \underline{h}(t),\\ h(t)&\geqslant \underline{h}(t) \qquad \text{for } t\geqslant T.\end{aligned} \tag{3.3}$$

Using Theorem 3.1, we can see that

$$\lim_{t\to\infty}\underline{v}(t,r)=\delta \text{ uniformly in any compact subset of } [0,\infty). \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\liminf_{t\to\infty} v(t,r)\geqslant\delta \text{ uniformly in any compact subset of } [0,\infty).$$

This completes the proof. □

**Lemma 3.2.** *If* $h_\infty < \infty$, *then*

$$\lim_{t\to\infty} u(t,r) = 1 \text{ uniformly in any compact subset of } [0,\infty), \tag{3.5}$$

*and*

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,h(t)])} = 0. \tag{3.6}$$

Proof. Define $s = h_0 r/h(t)$, $\phi(t,s) = u(t,r)$, $\psi(t,s) = v(t,r)$. We can get that $(\phi(t,s),\psi(t,s))$ satisfies

(3.7)
$$\begin{cases} \psi_t = \dfrac{h_0^2}{h^2(t)}\Big(\psi_{ss} + \dfrac{n-1}{s}\psi_s\Big) + \dfrac{h'(t)}{h(t)} s\psi_s + \psi\Big(1 - \dfrac{\psi}{\phi+\delta}\Big), & t > 0,\ 0 < s < h_0,\\ \psi_s(t,0) = \psi(t,h_0) = 0, & t > 0,\\ \psi(0,s) = v_0(s), & 0 \leqslant s \leqslant h_0. \end{cases}$$

This is an initial boundary value problem over a fixed ball $\{s < h_0\}$. Due to $h_0 \leqslant h(t) < h_\infty < \infty$, the differential operator is uniformly parabolic. By Theorem 2.2, we have the following estimates

$$\Big\|1 - \frac{\psi}{\phi+\delta}\Big\|_{L^\infty} \leqslant 1 + \frac{1}{\delta}M_2, \quad \Big\|\frac{h'(t)}{h(t)} s\Big\|_{L^\infty} \leqslant M_3.$$

Therefore, we can apply standard $L^p$ theory to get that $\|\psi\|_{W_p^{1,2}([0,2]\times[0,h_0])} \leqslant C_1$ for some constant $C_1$ depending on $\alpha, h_0, \delta, M_2, M_3$ and $\|v_0\|_{C^{1+\alpha}([0,h_0])}$. For each $T \geqslant 1$, we can apply the partial interior-boundary estimate (see Theorem 7.15 in [22]) over $[T,T+2]\times[0,h_0]$ to obtain that $\|\psi\|_{W_p^{1,2}([T,T+2]\times[0,h_0])} \leqslant C_2$ for some constant $C_2$ depending on $\alpha, h_0, \delta, M_2, M_3$ and $\|v_0\|_{C^{1+\alpha}([0,h_0])}$, but independent of $T$. Therefore, we can use the Sobolev embedding theorem ([22]) to show that

$$\|\psi\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times[0,h_0])} \leqslant C_3, \tag{3.8}$$

where $C_3$ is a constant depending on $\alpha, h_0, \delta, M_2, M_3$ and $\|v_0\|_{C^{1+\alpha}([0,h_0])}$.

Similarly, we can use interior estimates of the equation of $\phi$ to deduce that

$$\|\phi\|_{C^{(1+\alpha)/2,1+\alpha}([0,\infty)\times[0,h_0])} \leqslant C_4, \tag{3.9}$$

where $C_4$ is a constant depending on $\alpha, h_0, \delta, M_1, M_3$ and $\|u_0\|_{C^{1+\alpha}([0,h_0])}$.

It follows that there exists a constant $\widetilde{C}$ depending on $\alpha, h_0$, $(u_0,v_0)$ and $h_\infty$ such that

$$\|h\|_{C^{1+\alpha/2}([0,\infty))} \leqslant \widetilde{C}. \tag{3.10}$$

From the above estimate and $h'(t) > 0$, $h_\infty < \infty$, we can deduce that $|h'(t)|$ is uniformly bounded for $t \in [0,\infty)$ and $\lim\limits_{t\to\infty} h'(t) = 0$.

We give the proof of (3.6) for a contrary argument. Assume that there exist $\sigma > 0$ and $\{(t_k, r_k)\}$ with $1 < t_k < \infty$, $0 \leqslant r_k < h(t_k)$ such that $v(t_k, r_k) \geqslant \sigma$ for all $k \in \mathbb{N}$, and $t_k \to \infty$ as $k \to \infty$. Since $v(t, h(t)) = 0$ and $|v_r(t, h(t))|$ is uniformly bounded for $t \in [0, \infty)$, there exists $\varepsilon > 0$ such that $r_k \leqslant h(t_k) - \varepsilon$ for all $k \geqslant 1$. Therefore, a subsequence of $\{r_k\}$ converges to $r_0 \in [0, h_\infty - \varepsilon]$. Without loss of generality, we assume that $r_k \to r_0$ as $k \to \infty$. Correspondingly,

$$s_k := \frac{h_0 r_k}{h(t_k)} \to s_0 := \frac{h_0 r_0}{h_\infty} < h_0.$$

Define

$$\phi_k(t, s) = \phi(t_k + t, s) \quad \text{and} \quad \psi_k(t, s) = \psi(t_k + t, s) \quad \text{for } (t, s) \in [-1, 1] \times [0, h_0].$$

It follows from (3.8) and (3.9) that $\{(\phi_k, \psi_k)\}$ has a subsequence $\{(\phi_{k_i}, \psi_{k_i})\}$ such that

$$\|(\phi_{k_i}, \psi_{k_i}) - (\widetilde{\phi}, \widetilde{\psi})\|_{[C^{(1+\alpha')/2, 1+\alpha'}([-1,1]\times[0,h_0])]^2} \to 0 \text{ as } i \to \infty,$$

where $\alpha' \in (\alpha, 1)$, and $(\widetilde{\phi}, \widetilde{\psi})$ satisfies

$$\begin{cases} \widetilde{\psi}_t = \Big(\dfrac{h_0}{h_\infty}\Big)^2 \Big(\widetilde{\psi}_{ss} + \dfrac{n-1}{s}\widetilde{\psi}_s\Big) + \widetilde{\psi}\Big(1 - \dfrac{\widetilde{\psi}}{\widetilde{\phi} + \delta}\Big), & -1 < t < 1,\ 0 < s < h_0, \\ \widetilde{\psi}_s(t, 0) = \widetilde{\psi}(t, h_0) = 0, & -1 < t < 1. \end{cases}$$

Since $\widetilde{\psi}(0, s_0) \geqslant \sigma$, the maximum principle implies that $\widetilde{\psi} > 0$ in $(-1, 1) \times [0, h_0)$. Thus, we can apply the Hopf boundary lemma to conclude that $\widetilde{\psi}_s(0, h_0) < 0$. It follows that

$$v_r(t_{k_i}, h(t_{k_i})) = \partial_s \psi_{k_i}(0, h_0) \frac{h_0}{h(t_{k_i})} \leqslant \frac{\widetilde{\psi}_s(0, h_0)}{2} \frac{h_0}{h_\infty} < 0 \quad \text{for all large } i.$$

Hence,

$$h'(t_{k_i}) \geqslant \frac{-\mu \widetilde{\psi}_s(0, h_0)}{2} \cdot \frac{h_0}{h_\infty} > 0 \quad \text{for all large } i.$$

This contradics to $h'(t) \to 0$ as $t \to \infty$. This shows that $\lim\limits_{t\to\infty} \|v(t, \cdot)\|_{C([0,h(t)])} = 0$.

By a simple comparison argument. It is easy to see that $\lim\limits_{t\to\infty} u(t, r) = 1$ uniformly in any compact subset of $[0, \infty)$. □

**Lemma 3.3.** *If $h_\infty < \infty$, then $h_\infty \leqslant R^*(1)$.*

P r o o f. We assume $h_\infty > R^*(1)$ to get a contradiction. Note that $R^*(a)$ is a strictly decreasing continuous function in $a$. It is easy to see that for any given $\varepsilon_1 > 0$ sufficiently small, there exist $\tau \gg 1$ such that $h(\tau) > \max\{h_0, R^*(1-\varepsilon_1)\}$.

Let $w = w(t,r)$ be the positive solution of the following initial boundary value problem with a fixed boundary

$$\begin{aligned} &w_t = \Big(w_{rr} + \frac{n-1}{r} w_r\Big) + w\Big(1 - \varepsilon_1 - \frac{w}{\delta}\Big), && t > \tau,\ 0 < r < h(\tau),\\ &w_r(t,0) = 0, \quad w(t,h(\tau)) = 0, && t > \tau,\\ &w(\tau, r) = v(\tau, r), && 0 \leqslant r \leqslant h(\tau). \end{aligned}$$

By the comparison principle, we show that

$$w(t,r) \leqslant v(t,r) \quad \forall\, t \geqslant \tau,\ 0 \leqslant r \leqslant h(\tau).$$

Since $\lambda_1(1-\varepsilon_1, h(\tau)) < \lambda_1(1-\varepsilon_1, R^*(1-\varepsilon_1)) = 1$, we know from [4] that $w(t,r) \to w^*(r)$ as $t \to \infty$ uniformly for $r \in [0, h(\tau)]$, where $w^*$ is the unique positive solution of

$$\begin{aligned} &w^*_{rr} + \frac{n-1}{r} w^*_r + w^*\Big(1 - \varepsilon_1 - \frac{w^*}{\delta}\Big) = 0, \quad r \in (0, h(\tau)),\\ &w^*_r(0) = w^*(h(\tau)) = 0. \end{aligned}$$

Hence, $\liminf\limits_{t\to\infty} v(t,r) \geqslant \lim\limits_{t\to\infty} w(t,r) = w^*(r) > 0$ in $[0, h(\tau))$. This is a contradiction with (3.6). Therefore, we obtain that $h_\infty \leqslant R^*(1)$ holds. □

Combining Lemmas 3.1, 3.2 and 3.3, we can obtain the following dichotomy result.

**Theorem 3.3.** *Suppose that* $(u, v, h)$ *is the unique solution of* (1.1) *with the initial condition* (1.2). *Then the following alternative holds.*

*Either,*

(i) *spreading of* $v$: $h_\infty = \infty$ *and* $\liminf\limits_{t\to\infty} v(t,r) \geqslant \delta$ *uniformly in any compact subset of* $[0,\infty)$;

*or,*

(ii) *vanishing of* $v$: $h_\infty \leqslant R^*(1)$ *and* $\lim\limits_{t\to\infty} \|v(t,\cdot)\|_{C([0,h(t)])} = 0$.

Next we give the criteria for spreading and vanishing.

**Theorem 3.4.** *In Theorem* 3.3, *if* $h_0 \geqslant R^*(1)$, *then spreading of* $v$ *always happens. If* $h_0 < R^*(1)$, *then there exist* $\mu^* \geqslant \mu_* > 0$ *depending on* $(u_0, v_0)$ *such that the spreading of* $v$ *happens exactly when* $\mu > \mu^*$ *and vanishing of* $v$ *occurs exactly when* $\mu \leqslant \mu_*$.

For the case $h_0 \geqslant R^*(1)$, due to $h'(t) > 0$ for $t > 0$, we have $h_\infty > R^*(1)$. Hence, Lemmas 3.3 and 3.1 imply the spreading result. We prove the result for the case $h_0 < R^*(1)$ by the following lemmas.

**Lemma 3.4.** *If $h_0 < R^*(1)$, then there exists $\underline{\mu} > 0$ depending on $(u_0, v_0)$ such that $h_\infty = \infty$ when $\mu \geqslant \underline{\mu}$.*

Proof. Consider the following auxiliary problem:

$$\begin{cases} \underline{v}_t = \Big(\underline{v}_{rr} + \dfrac{n-1}{r}\underline{v}_r\Big) + \underline{v}(1 - \delta^{-1}\underline{v}), & t > 0,\ 0 < r < \underline{h}(t), \\ \underline{v}_r(t,0) = 0, \quad \underline{v}(t, \underline{h}(t)) = 0, & t > 0, \\ \underline{h}'(t) = -\mu \underline{v}_r(t, \underline{h}(t)), & t > 0, \\ \underline{h}(0) = h_0, \quad \underline{v}(0,r) = v_0(r), & 0 \leqslant r \leqslant \underline{h}(0). \end{cases}$$

By Lemma 2.2, we have that

$$\underline{h}(t) \leqslant h(t), \underline{v}(t,r) \leqslant v(t,r) \quad \forall\, t > 0,\ 0 < r < \underline{h}(t).$$

Since $\underline{h}(0) = h(0) < R^*(1)$, by Lemma 2.8 of [8], there exists $\underline{\mu} > 0$ such that $\underline{h}_\infty = \infty$ for $\mu > \underline{\mu}$. Therefore, $h_\infty = \infty$ for $\mu > \underline{\mu}$. The proof is finished. □

**Lemma 3.5.** *Suppose $h_0 < R^*(1)$. Then there exists $\bar{\mu} > 0$ depending on $v_0$ such that $h_\infty < \infty$ if $\mu \leqslant \bar{\mu}$.*

Proof. By Lemma 2.1, we can deduce $u(t,r) \leqslant 1$, $v(t,r) \leqslant \overline{v}(t,r)$, and $h(t) \leqslant \bar{h}(t)$, where $(\overline{v}(t,r), \bar{h}(t))$ is the solution of the following problem:

$$\begin{cases} \overline{v}_t = \Big(\overline{v}_{rr} + \dfrac{n-1}{r}\overline{v}_r\Big) + \overline{v}\Big(1 - \dfrac{\overline{v}}{1+\delta}\Big), & t > 0,\ 0 < r < \bar{h}(t), \\ \overline{v}_r(t,0) = 0, \quad \overline{v}(t, \bar{h}(t)) = 0, & t > 0, \\ \bar{h}'(t) = -\mu \overline{v}_r(t, \bar{h}(t)), & t > 0, \\ \bar{h}(0) = h_0, \quad \overline{v}(0,r) = v_0(r), & 0 \leqslant r \leqslant \bar{h}(0). \end{cases}$$

Since $\bar{h}(0) = h_0 < R^*(1)$, it follows from Theorem 3.2 that there exists $\bar{\mu} > 0$ depending on $v_0$ such that $\bar{h}_\infty < \infty$ if $\mu \leqslant \bar{\mu}$. Therefore, $h_\infty < \infty$ for $\mu \leqslant \bar{\mu}$. The proof is completed. □

**Lemma 3.6.** *Suppose $h_0 < R^*(1)$. Then there exists $\mu^* \geqslant \mu_* > 0$ depending on $(u_0, v_0)$ such that $h_\infty \leqslant R^*(1)$ if $\mu \leqslant \mu_*$ and $h_\infty = \infty$ if $\mu > \mu^*$.*

P r o o f. We will write $(u_\mu, v_\mu, h_\mu)$ in place of $(u, v, h)$ to clarify the dependence of the solution to (1.1) on $\mu$.

Define $\Sigma^* = \{\mu > 0\colon\ h_{\mu,\infty} \leqslant R^*(1)\}$. By Lemmas 3.5 and 3.3, $(0, \bar\mu] \subset \Sigma^*$. In view of Lemma 3.4, $\Sigma^* \cap [\underline{\mu}, \infty) = \emptyset$. Therefore, set $\mu^* := \sup \Sigma^* \in [\bar\mu, \underline{\mu}]$. By the definition and Lemma 3.3, we can find that $h_{\mu,\infty} = \infty$ when $\mu > \mu^*$.

We will show that $\mu^* \in \Sigma^*$. Otherwise, $h_{\mu^*,\infty} = \infty$. We can find $T > 0$ such that $h_{\mu^*}(T) > R^*(1)$. By the continuous dependence of $(u_\mu, v_\mu, h_\mu)$ on $\mu$, there exists $\varepsilon > 0$ such that $h_\mu(T) > R^*(1)$ for $\mu \in [\mu^* - \varepsilon, \mu^* + \varepsilon]$. It follows that for all such $\mu$, $\lim\limits_{t\to\infty} h_\mu(t) > h_\mu(T) > R^*(1)$. This implies that $[\mu^* - \varepsilon, \mu^* + \varepsilon] \cap \Sigma^* = \emptyset$ and $\sup \Sigma^* \leqslant \mu^* - \varepsilon$. This contradicts the definition of $\mu^*$.

Set $\Sigma_* = \{\nu > 0\colon\ \nu \geqslant \bar\mu$ such that $h_{\mu,\infty} \leqslant R^*(1)$ for all $\mu \leqslant \nu\}$, where $\bar\mu$ is given in Lemma 3.5. Then $\mu_* := \sup \Sigma_* \leqslant \mu^*$ and $(0, \mu_*) \subset \Sigma_*$. Similarly as the above argument, it is easy to obtain that $\mu_* \in \Sigma_*$. This completes the proof. □

## 4. Estimates of the spreading speed

In this section, we will give some rough estimates of the spreading speed of $h(t)$ for the case where spreading of $v$ happens. We recall a proposition for a diffusive logistic equation.

**Proposition 4.1** (See Du and Guo [8]). *For any given constants $a > 0$, $b > 0$, $d > 0$ and $k \in [0, 2\sqrt{ad})$, the problem*

$$-dU'' + kU' = aU - bU^2 \quad \text{in } (0, \infty), \quad U(0) = 0, \tag{4.1}$$

*admits a unique positive solution $U = U_k = U_{a,b,k}$, and this solution satisfies $U(r) \to a/b$ as $r \to \infty$. Moreover, $U_k'(r) > 0$ for $r \geqslant 0$, $U_{k_1}'(0) > U_{k_2}'(0)$, $U_{k_1}(r) > U_{k_2}(r)$ for $r > 0$ and $k_1 > k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, a, b) \in (0, \sqrt{2ad})$ such that $\mu U_{k_0}'(0) = k_0$. Furthermore,*

$$\lim_{a\mu/(bd)\to\infty} \frac{k_0}{\sqrt{ad}} = 2, \qquad \lim_{a\mu/(bd)\to 0} \frac{k_0}{\sqrt{ad}} \frac{bd}{a\mu} = \frac{1}{\sqrt{3}}.$$

It was shown in [12] that $k_0(\mu, a, b)$ is increasing in $\mu$ and $a$, and is decreasing in $b$. More precisely,

$$\mu_1 \geqslant \mu_2,\ a_1 \geqslant a_2 \text{ and } b_1 \leqslant b_2 \Rightarrow k_0(\mu_1, a_1, b_1) \geqslant k_0(\mu_2, a_2, b_2),$$

with the strict inequality holding when $(\mu_1, a_1, b_1) \neq (\mu_2, a_2, b_2)$. By using the function $k_0(\mu, a, b)$, we have the following estimates for the spreading speed of $h(t)$.

**Theorem 4.1.** *If $h_\infty = \infty$, then*

$$k_0(\mu, 1, \delta^{-1}) \leqslant \liminf_{t\to\infty} \frac{h(t)}{t} \leqslant \limsup_{t\to\infty} \frac{h(t)}{t} \leqslant k_0(\mu, 1, (M_1+\delta)^{-1}).$$

Proof. Since $(u, v, h)$ satisfies

$$\begin{aligned}
&v_t = \Big(v_{rr} + \frac{n-1}{r} v_r\Big) + v\Big(1 - \frac{v}{u+\delta}\Big) \leqslant v\Big(1 - \frac{1}{M_1+\delta} v\Big), && t>0,\ 0<r<h(t),\\
&v_r(t,0) = v(t,h(t)) = 0, && t>0,\\
&h'(t) = -\mu v_r(t,h(t)) = 0, && t>0,\\
&h(0) = h_0, v(0,r) = v_0(r) > 0, && 0 \leqslant r \leqslant h_0,
\end{aligned}$$

then $(v, h)$ is a lower solution to the following problem:

$$\begin{cases}
\overline{v}_t = \Big(\overline{v}_{rr} + \frac{n-1}{r}\overline{v}_r\Big) + \overline{v}(1-(M_1+\delta)^{-1}\overline{v}), & t>0,\ 0<r<\bar{h}(t),\\
\overline{v}_r(t,0) = 0, \quad \overline{v}(t,\bar{h}(t)) = 0, & t>0,\\
\bar{h}'(t) = -\mu\overline{v}_r(t,\bar{h}(t)), & t>0,\\
\bar{h}(0) = h_0, \quad \overline{v}(0,r) = v_0(r), & 0 \leqslant r \leqslant \bar{h}(0).
\end{cases}$$

It follows that $\bar{h}(t) \geqslant h(t) \to \infty$ as $t \to \infty$. By [8], we can obtain $\lim_{t\to\infty} \bar{h}(t)/t = k_0(\mu, 1, (M_1+\delta)^{-1})$. Thus we have $\limsup_{t\to\infty} h(t)/t \leqslant k_0(\mu, 1, (M_1+\delta)^{-1})$.

Similarly, it is easy to show $\liminf_{t\to\infty} h(t)/t \geqslant k_0(\mu, 1, \delta^{-1})$. □

Remark 4.1. In this theorem, the lower and upper bounds for the spreading speed are dependent on $\mu$ and the bound of the prey. Since the prey species serve as food for the predator, it is reasonable to conclude the results.

Remark 4.2. In [9], the authors consider the non-radially symmetric case in higher space dimensions. Here we consider the spreading of the predator in the prey-predator environment and the radially symmetric solution in higher space dimensions.

## 5. Conclusions

In this paper, we considered a Leslie-Gower predator-prey model in a higher dimensional environment. The model studies the invasive predator that initially occupies the region $[0, h_0]$ and has a tendency to expand its territory. We establish several results in this setting.

(i) Theorem 3.3 provides the asymptotic behavior of the predator when spreading success and spreading failure.

If $h_\infty = \infty$, then $\liminf\limits_{t\to\infty} v(t,r) \geqslant \delta$ uniformly in any compact subset of $[0,\infty)$.

If $h_\infty < \infty$, then $\lim\limits_{t\to\infty} \|v(t,r)\|_{C[0,h(t)]} = 0$.

(ii) By Theorem 3.4, we can establish a spreading-vanishing dichotomy which can be characterized by $R^*(1)$. If $h_\infty > R^*(1)$, the predator will spread successfully, while the predator will vanish eventually when $h_\infty < R^*(1)$. If the size of initial habitat $h_0$ is not less than $R^*(1)$, or $h_0$ is less than $R^*(1)$, but $\mu \geqslant \underline{\mu}$, then the predator will spread successfully, while if the size of initial habitat is less than $R^*$ and $\mu \leqslant \bar{\mu}$, the predator will disappear eventually.

(iii) Finally, Theorem 4.1 reveals that the spreading speed is dependent on the boundary condition and the bound of the prey.

By our discussion, we can show that the invasive predator can knock aquatic ecosystems right out of balance. Studying the spread of an invasive predator, we can give some guidelines, especially ones that encourage the trade of less invasive and aggressive species, or protect the prey as food for the predator.

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