Applications of Mathematics

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Applications of Mathematics, Vol. 67 (2022), No. 5, 633–656

Persistent URL: `http://dml.cz/dmlcz/151029`

67 (2022) APPLICATIONS OF MATHEMATICS No. 5, 633–656

A GENERAL DECAY ESTIMATE FOR A FINITE MEMORY THERMOELASTIC BRESSE SYSTEM

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Received August 10, 2020. Published online July 25, 2022.

Abstract. This work considers a Bresse system with viscoelastic damping on the vertical displacement and heat conduction effect on the shear angle displacement. A general stability result with minimal condition on the relaxation function is obtained. The system under investigation, to the best of our knowledge, is new and has not been studied before in the literature. What is more interesting is the fact that our result holds without the imposition of the equal speed of wave propagation condition, and differentiation of the equations of the system, as against the usual practice in the literature.

Keywords: general decay; Bresse system; nonequal speed; viscoelastic; thermoelastic

MSC 2020: 35B35, 35L05, 74D10, 35B40, 35L20

1. Introduction

A viscoelastic material, as the name sounds, has two different properties combined together. The word "viscous" means the material deforms gradually when subjected to an external force. The word "elastic" means that as soon as the deforming force is removed from the material, it will return to its original state. Generally, mechanical properties of materials are often examined in terms of the relationship between stress and strain (or load-deformation) behaviour. Also, the mechanical properties of viscoelastic materials depend on their rate of deformation. The stiffness of the material, for instance, increases according to loading rates. The behaviour of a viscoelastic material also substantially depends on the temperature, since material stiffness can change as temperature changes too. The introduction of a viscoelastic material in a deformable elastic structure changes the Young's modulus, the mass density and the damping coefficients. This damping process presents some new mathematical

The authors appreciate the support of the University of Hafr Al Batin.

DOI: 10.21136/AM.2022.0224-20

challenges which are of great interest to researchers. In this paper, we study the stability of a thermoelastic Bresse system with a viscoelastic effect on it. The Bresse system, which is also known as the circular arch problem, is a model for planar, linear shearable beam with initial curvature involving couplings of longitudinal, vertical and shear motions [4].

In this paper, we consider the following thermoelastic Bresse system with viscoelastic effect:

$$\tag{1.1} \begin{cases} \varrho_1\varphi_{tt} - k_1(\varphi_x+\psi+lw)_x - k_3 l(w_x - l\varphi) \\ \qquad + \displaystyle\int_0^t g(t-s)(\varphi_x+\psi+lw)_x(x,s)\,\mathrm{d}s = 0, \\ \varrho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi+lw) \\ \qquad - \displaystyle\int_0^t g(t-s)(\varphi_x+\psi+lw)(x,s)\,\mathrm{d}s + \gamma\theta_x = 0, \\ \varrho_1 w_{tt} - k_3(w_x - l\varphi)_x + k_1 l(\varphi_x+\psi+lw) \\ \qquad - l\displaystyle\int_0^t g(t-s)(\varphi_x+\psi+lw)(x,s)\,\mathrm{d}s = 0, \\ \varrho_3\theta_t - \beta\theta_{xx} + \gamma\psi_{xt} = 0 \end{cases}$$

for $x\in(0,1)$ and $t>0$, where l, k_1, k_2, k_3, ϱ_1, ϱ_2, ϱ_3, γ, β are physical parameters, which are all positive, and g is a given relaxation function to be specified later. We couple system (1.1) with the following Neumann-Dirichlet-Dirichlet-Neumann boundary conditions:

$$\tag{1.2} \begin{gathered} \varphi_x(0,t)=\varphi_x(1,t)=\psi(0,t)=\psi(1,t)=0, \\ w(0,t)=w(1,t)=\theta_x(0,t)=\theta_x(1,t)=0, \quad t\geqslant 0, \end{gathered}$$

and initial data

$$\tag{1.3} \begin{gathered} \varphi(x,0)=\varphi_0(x),\ \psi(x,0)=\psi_0(x),\ w(x,0)=w_0(x),\ \theta(x,0)=\theta_0, \\ \varphi_t(x,0)=\varphi_1(x),\ \psi_t(x,0)=\psi_1(x),\ w_t(x,0)=w_1(x) \quad \forall\, x\in[0,1]. \end{gathered}$$

The motion for the classical Bresse system is governed by the following system of evolution equations:

$$\tag{1.4} \begin{cases} \varrho_1\varphi_{tt} - S_x - lQ = 0, & x\in(0,1),\ t>0, \\ \varrho_2\psi_{tt} - M_x + S = 0, & x\in(0,1),\ t>0, \\ \varrho_1 w_{tt} - Q_x + lS = 0, & x\in(0,1),\ t>0, \end{cases}$$

where $\psi=\psi(x,t)$ is the shear displacement, $\varphi=\varphi(x,t)$ is the vertical displacement, while $w=w(x,t)$ is the longitudinal angle displacement. The constants $l=R^{-1}$,

$\varrho_1 = \varrho A$, $\varrho_2 = \varrho I$ are physical constants, where I, A, R, ϱ are the second moment of the cross-section, cross sectional curvature, radius of curvature and material density, respectively. The constitutive laws are given by

$$\begin{cases} S = k_1(\varphi_x + \psi + lw), \\ Q = k_3(w_x - l\varphi), \\ M = k_2\psi_x, \end{cases} \tag{1.5}$$

where S, Q and M are respectively the shear force, the axial force and the bending moment. Also, $k_1 = \kappa GA$, $k_2 = EI$, $k_3 = EA$, where G, E, κ are the shear modulus, modulus of elasticity, and shear factor, respectively.

In the present work, we consider a Bresse system with viscoelastic law acting on the vertical displacement and heat conduction effect on the shear angle displacement. If thermoelastic dissipation is effective on the shear angle displacement, we have the evolution equation

$$\varrho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \tag{1.6}$$

where the constants ϱ_3 and γ are the capacity and the diffusivity, respectively, $\theta = \theta(x,t)$ is the temperature difference and $q = q(x,t)$ is the heat flux. If we further couple the evolution equation (1.6) with system (1.4), we arrive at the following system of evolution equations:

$$\begin{cases} \varrho_1\varphi_{tt} - S_x - lQ = 0, & x \in (0,1),\ t > 0, \\ \varrho_2\psi_{tt} - M_x + S = 0, & x \in (0,1),\ t > 0, \\ \varrho_1 w_{tt} - Q_x + lS = 0, & x \in (0,1),\ t > 0, \\ \varrho_3\theta_t + q_x + \gamma\psi_{xt} = 0. \end{cases} \tag{1.7}$$

Also, with viscoelastic law acting on the vertical displacement, the following constitutive laws hold (see Alves et al. [2]):

$$\begin{cases} S = k_1(\varphi_x + \psi + lw) - \displaystyle\int_0^t g(t-s)(\varphi_x + \psi + lw)(x,s)\,\mathrm{d}s, \\ Q = k_3(w_x - l\varphi), \\ M = k_2\psi_x - \gamma\theta, \\ q = -\beta\theta_x, \end{cases} \tag{1.8}$$

where β is the adhesive stiffness. Combining (1.7) and (1.8), we arrive at the thermoelastic Bresse system (1.1).

A most general model, for which the Bresse system is a special case, is the general model for a 3-d nonlinear thermoelastic beam derived by Lagnese et al. [13], where they studied the networks of flexible beams. A yet special case (however more general than the Bresse system), is the model for linear planar, shearable thermoelastic beam, whose motion is described by the following system of equations (see [15] and references therein):

$$(1.9)\quad \begin{cases} \varrho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - k_3 l(w_x - l\varphi) + l\gamma\tilde{\theta} = 0, & x \in (0,1),\ t > 0, \\ \varrho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \gamma\theta_x = 0, & x \in (0,1),\ t > 0, \\ \varrho_1 w_{tt} - k_3(w_x - l\varphi)_x + k_1 l(\varphi_x + \psi + lw) + \gamma\tilde{\theta}_x = 0, & x \in (0,1),\ t > 0, \\ \varrho_3\theta_t - \beta\theta_{xx} + \gamma\psi_{xt} = 0, & x \in (0,1),\ t > 0, \\ \varrho_3\tilde{\theta}_t - \beta\tilde{\theta}_{xx} + \gamma(w_x - l\varphi)_t = 0, & x \in (0,1),\ t > 0, \end{cases}$$

where θ and $\tilde{\theta}$ are temperature deviations from a given reference temperature, along the vertical and longitudinal direction, respectively. As a particular case, when $\theta = \tilde{\theta} = 0$, system (1.9) reduces to the isothermal system of Bresse [4] obtained in 1856, i.e., system (1.4). We could see from (1.9) that both the share angle displacement ψ and the longitudinal displacement w are effectively damped by the thermal energy dissipation produced by θ and $\tilde{\theta}$, respectively, from $(1.9)_4$–$(1.9)_5$. The Bresse system (1.4) is not arbitrarily stable, but there are different damping mechanisms that could be introduced to stabilize the system. However, the effectiveness of a damping mechanism on the stability of the system, still largely depends on the mode of coupling, the wave speed and even the type of imposed boundary conditions. Liu and Rao [15] studied system (1.9) with boundary condition of type Dirichlet-Neumann-Neumann-Dirichlet-Dirichlet on $\varphi, \psi, w, \theta, \tilde{\theta}$, respectively. They showed that the exponential decay rate is preserved when the equal-wave-speed condition holds. Otherwise, only a polynomial decay rate is guaranteed. Other damping mechanisms have also been introduced to stabilize the Bresse system. El Arwadi and Youssef [6] coupled the Bresse system (1.4) with a viscoelastic Kelvin-Voigt damping produced by the following external forces:

$$(1.10)\quad \begin{cases} F_1 = \gamma_1(\varphi_x + \psi + lw)_{xt} + \gamma_0 l(w_x - l\varphi)_t, \\ F_2 = \gamma_2\psi_{xxt} + \gamma_1(\varphi_x + \psi + lw)_t, \\ F_3 = \gamma_0(w_x - l\varphi)_{xt} - \gamma_1 l(\varphi_x + \psi + lw)_t, \end{cases}$$

where γ_0, γ_1, γ_2 are damping coefficients. The authors in [6] imposed a boundary condition of type Dirichlet-Dirichlet-Dirichlet, then established an exponential decay result (without equal-wave-speed condition (1.12)) for the solution energy. They further introduced a numerical scheme using the finite element method, to approximate the solution.

In the recent time, Messaoudi and Hassan [17] investigated a Bresse system with viscoelastic law acting on the bending moment, i.e.,

$$M = k_2\psi_x - \int_0^t g(t-s)\psi_x(\cdot,s)\,\mathrm{d}s,$$

thus obtaining the system

$$(1.11)\qquad \begin{cases} \varrho_1\varphi_{tt} - k_1(\varphi_x+\psi+lw)_x - k_3 l(w_x - l\varphi) = 0, \\ \varrho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi+lw) - \int_0^t g(t-s)\psi_{xx}(x,s)\,\mathrm{d}s = 0, \\ \varrho_1 w_{tt} - k_3(w_x - l\varphi)_x + k_1 l(\varphi_x+\psi+lw) = 0, \end{cases}$$

where $x \in (0,L)$ and $t > 0$. Imposing a Dirichlet-Neumann-Neumann boundary conditions respectively on φ, ψ, w, they proved that the solution energy decays uniformly if and only if there holds the equal-wave speed of propagation

$$(1.12)\qquad \frac{\varrho_1}{k_1} = \frac{\varrho_2}{k_2} \quad \text{and} \quad k_1 = k_3.$$

Moreover, they proved an optimal decay result, but for stronger regular solution (by differentiating equations of system (1.11)), for the case of nonequal-wave-speed, i.e.,

$$(1.13)\qquad \frac{\varrho_1}{k_1} \neq \frac{\varrho_2}{k_2} \quad \text{and} \quad k_1 = k_3.$$

For more results on well-posedness, general stability (polynomial and exponential) of the Bresse system, see [1], [5], [9], [10], [16], [19], [21], [23] and references therein.

The thermoelastic Timoshenko system (setting $w = 0$ in (1.9)) is a special case of the thermoelastic Bresse system (1.9). The resulting thermoelastic Timoshenko system, either with $\theta = 0$ or $\tilde{\theta} = 0$, and with or without viscoelastic damping has been thoroughly studied by different authors in the literature. However, the studies as well showed that even in this special case of (1.9), the effectiveness of a damping mechanism on the stability of the system, still largely depends on the mode of coupling, the wave speed and even the type of imposed boundary conditions. See [7], [8], [11], [12], [3], [18], [20], [22] and references therein for more details.

It is our goal in the present work to investigate the effect of viscoelastic law acting on the shear force and thermoelastic law on the bending moment on the stabilization of the Bresse system (1.4). Indeed, we obtain a general decay rate for the energy functional associated to Problem (1.1)–(1.3). We should mention here that our result is obtain without imposing equal-wave-speed condition (1.12) and without differentiation of the equations of system (1.1). In other words, our result holds even without more regular solution.

This work is organized as follows: In Section 2, we outline few materials which will be of help in proving our main result. In Section 3, we prove some needed important lemmas. In Section 4, we present a general decay rate result for the solution energy of Problem (1.1)–(1.3).

2. Assumptions and functional setting

Throughout this work, we denote by C a positive constant whose value may change from one line to another line, or even within the same line. We denote by $\|\cdot\|_2$ the usual norm in $L^2(0,1)$. On the relaxation function g, we consider the following assumptions:

(A_1) $g\colon [0,\infty) \to (0,\infty)$ is a decreasing C^1-function such that

$$k_1 - \int_0^\infty g(s)\,\mathrm{d}s = l_0 > 0. \tag{2.1}$$

(A_2) There exist a nonincreasing differentiable function $\eta\colon [0,\infty) \to (0,\infty)$ such that

$$g'(t) \leqslant -\eta(t)g(t) \quad \forall\, t \geqslant 0. \tag{2.2}$$

Remark 2.1. Define

$$m(\varphi) = \int_0^1 \varphi(x,t)\,\mathrm{d}x \quad \text{and} \quad m(\theta) = \int_0^1 \theta(x,t)\,\mathrm{d}x. \tag{2.3}$$

Integration of $(1.1)_1$ and $(1.1)_4$ over $(0,1)$ yield

$$\frac{\mathrm{d}^2}{\mathrm{d}t} m(\varphi) + \frac{k_3 l^2}{\varrho_1} m(\varphi) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} m(\theta) = 0, \tag{2.4}$$

respectively. Solving (2.4) keeping in mind initial conditions (1.3), we get

$$m(\varphi) = C_1\cos(a_0 t) + C_2 \sin(a_0 t) \quad \text{and} \quad m(\theta) = m(\theta_0), \tag{2.5}$$

where

$$a_0 = \sqrt{\frac{k_3}{\varrho_1}}l, \quad C_1 = \int_0^1 \varphi_0(x)\,\mathrm{d}x, \quad C_2 = \frac{1}{a_0}\int_0^1 \varphi_1(x)\,\mathrm{d}x.$$

Now, define

$$\bar\varphi(x,t) = \varphi(x,t) - m(u) \quad \text{and} \quad \bar\theta(x,t) = \theta(x,t) - m(\theta).$$

It follows that

$$\int_0^1 \bar\varphi(x,t)\,\mathrm{d}x = 0, \quad \int_0^1 \bar\varphi_t(x,t)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_0^1 \bar\theta(x,t)\,\mathrm{d}x = 0 \quad \forall\, t \geqslant 0. \tag{2.6}$$

Therefore, we have the Poincaré inequalities

$$\|\bar{\varphi}\|_2 \leqslant \|\varphi_x\|_2 \quad \text{and} \quad \|\bar{\theta}\|_2 \leqslant \|\theta_x\|_2. \tag{2.7}$$

Moreover, $(\bar{\varphi}, \psi, w, \bar{\theta})$ satisfies Problem (1.1) with initial data for $\bar{\varphi}$ and $\bar{\theta}$ given as

$$\bar{\varphi}_0 = \varphi_0 - m(\varphi_0), \quad \bar{\varphi}_1 = \varphi_1 - m(\varphi_1) \quad \text{and} \quad \bar{\theta}_0 = \theta_0 - m(\theta).$$

Consequently, in the rest of this paper, we work with $(\bar{\varphi}, \psi, w, \bar{\theta})$. However, for the sake of convenience we write $(\varphi, \psi, w, \theta)$, but keeping in mind (2.6) and (2.7).

We define the following spaces:

$$\begin{aligned} L^2_* &= L^2_*(0,1) = \left\{ u \in L^2(0,1) \colon \int_0^1 u(x)\,\mathrm{d}x = 0 \right\}, \\ H^1_* &= H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1), \\ H^2_* &= H^2_*(0,1) = \{\phi \in H^2(0,1) \colon \phi_x(0) = \phi_x(1) = 0\}. \end{aligned}$$

For the sake of completeness, we state without proof the well-posedness result of problem (1.1)–(1.3).

Theorem 2.1. *Suppose* $(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0) \in H^1_* \times L^2_* \times H^1_0 \times L^2 \times H^1_0 \times L^2 \times H^1_*$ *and condition* (A_1) *holds. Then problem* (1.1)–(1.3) *has a global weak unique solution* $(\varphi, \psi, w, \theta)$ *such that*

$$\begin{aligned} &(\varphi, \psi, w) \in C([0,\infty), H^1_* \times H^1_0 \times H^1_0) \cap C^1([0,\infty), L^2_* \times L^2 \times L^2), \\ &\theta \in C([0,\infty), H^1_*). \end{aligned} \tag{2.8}$$

Furthermore, if

$$(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0) \in H^2_* \cap H^1_* \times H^1_* \times H^2 \cap H^1_0 \times H^1_0 \times H^2 \cap H^1_0 \times H^1_0 \times H^2_* \cap H^1_*,$$

then the unique weak solution of problem (1.1)–(1.3) *possesses more regularity so that*

$$\begin{aligned} &\varphi \in C([0,\infty), H^2_* \cap H^1_*) \cap C^1([0,\infty), H^1_*) \cap C^2((0,\infty), L^2_*), \\ &\psi \in C([0,\infty), H^2 \cap H^1_0) \cap C^1([0,\infty), H^1_0) \cap C^2((0,\infty), L^2), \\ &w \in C((0,\infty), H^2 \cap H^1_0) \cap C^1([0,\infty), H^1_0) \cap C^2((0,\infty), L^2), \\ &\theta \in C((0,\infty), H^2_*) \cap C^1([0,\infty), H^1_*). \end{aligned}$$

The result of Theorem 2.1 is established using the Faedo-Galerkin approximation method, see [14]. We as well state the following basic lemmas which will be repeatedly used throughout this work.

Lemma 2.1. *For any function* $v \in L^2_{\rm loc}([0,\infty), L^2(0,1))$ *we have*

$$\int_0^1 \Big(\int_0^t g(t-s)(v(t)-v(s))\,\mathrm{d}s\Big)^2\,\mathrm{d}x \leqslant (1-l_0)(g\circ v)(t), \tag{2.9}$$

$$\int_0^1 \Big(\int_0^x v(y,t)\,\mathrm{d}y\Big)^2\,\mathrm{d}x \leqslant \|v(t)\|_2^2, \tag{2.10}$$

where

$$(g\circ v)(t) = \int_0^t g(t-s)\|v(t)-v(s)\|_2^2\,\mathrm{d}s.$$

Lemma 2.2. *Let* (φ,ψ,w,θ) *be the solution of problem* (1.1)–(1.3). *Then*

$$\begin{aligned}\int_0^1 \Big(\int_0^x \int_0^t g(t-s)[(\varphi_y+\psi+lw)(y,t)-(\varphi_y+\psi+lw)(y,s)]\,\mathrm{d}s\,\mathrm{d}y\Big)^2\,\mathrm{d}x \\ \leqslant (1-l_0)(g\circ(\varphi_x+\psi+lw))(t),\end{aligned} \tag{2.11}$$

where

$$(g\circ(\varphi_x+\psi+lw))(t) = \int_0^1\int_0^t g(t-s)[(\varphi_x+\psi+lw)(t)-(\varphi_x+\psi+lw)(s)]^2\,\mathrm{d}s\,\mathrm{d}x.$$

Henceforth, for the sake of convenience we will sometimes denote

$$\chi(y,t,s) = (\varphi_y+\psi+lw)(y,t)-(\varphi_y+\psi+lw)(y,s)$$

and

$$\chi(x,t,s) = (\varphi_x+\psi+lw)(t)-(\varphi_x+\psi+lw)(s).$$

3. Useful lemmas

In this section, we will state and prove some lemmas that will be used in obtaining our main result.

Lemma 3.1. *Let* (φ,ψ,w,θ) *be the solution of* (1.1)–(1.3). *The energy functional associated to system* (1.1)–(1.3) *is defined by*

$$\begin{aligned}E(t) &= \frac12(\varrho_1\|\varphi_t\|_2^2+\varrho_2\|\psi_t\|_2^2+\varrho_1\|w_t\|_2^2+\varrho_3\|\theta\|_2^2+k_2\|\psi_x\|_2^2+k_3\|w_x-l\varphi\|_2^2) \\ &\quad+\frac12\Big(k_1-\int_0^t g(s)\,\mathrm{d}s\Big)\|\varphi_x+\psi+lw\|_2^2+\frac{k_1}{2}(g\circ(\varphi_x+\psi+lw))(t),\end{aligned} \tag{3.1}$$

and satisfies for all $t\geqslant 0$,

$$E'(t) = -\frac{k_1}{2}g(t)\|\varphi_x+\psi+lw\|_2^2+\frac{k_1}{2}g'\circ(\varphi_x+\psi+lw)-\beta\|\theta_x\|_2^2\leqslant 0. \tag{3.2}$$

P r o o f. We multiply $(1.1)_1$ by φ_t, $(1.1)_2$ by ψ_t, $(1.1)_3$ by w_t and $(1.1)_4$ by θ. Then integration over $(0,1)$ and addition of the resulting equations give

$$
\begin{aligned}
(3.3)\qquad & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\varrho_1\|\varphi_t\|_2^2+\varrho_2\|\psi_t\|_2^2+\varrho_1\|w_t\|_2^2+\varrho_3\|\theta\|_2^2+k_2\|\psi_x\|_2^2+k_3\|w_x-l\varphi\|_2^2)\\
&+\frac{k_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_x+\psi+lw\|_2^2\\
&=-\beta\|\theta_x\|_2^2+\underbrace{\int_0^1(\varphi_x+\psi+lw)_t\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x}_{J_1(t)}.
\end{aligned}
$$

Estimating the integral J_1 on the right-hand side of (3.3), we have

$$
\begin{aligned}
J_1(t)=&\int_0^1(\varphi_x+\psi+lw)_t(t)\int_0^t g(t-s)(\varphi_x+\psi+lw)(t)\,\mathrm{d}s\,\mathrm{d}x\\
&+\int_0^1\int_0^t g(t-s)[(\varphi_x+\psi+lw)(s)-(\varphi_x+\psi+lw)(t)]\\
&\times(\varphi_x+\psi+lw)_t(t)\,\mathrm{d}s\,\mathrm{d}x\\
=&\frac{1}{2}\Big(\int_0^t g(s)\,\mathrm{d}s\Big)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(\varphi_x+\psi+lw)^2\,\mathrm{d}x\\
&-\frac{1}{2}\int_0^1\int_0^t g(t-s)\frac{\mathrm{d}}{\mathrm{d}t}[(\varphi_x+\psi+lw)(t)-(\varphi_x+\psi+lw)(s)]^2\,\mathrm{d}s\,\mathrm{d}x\\
=&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\Big(\int_0^t g(s)\,\mathrm{d}s\Big)\|\varphi_x+\psi+lw\|^2\Big]-\frac{k_1}{2}g(t)\|\varphi_x+\psi+lw\|^2\\
&-\frac{k_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(g\circ(\varphi_x+\psi+lw))(t)+\frac{k_1}{2}(g'\circ(\varphi_x+\psi+lw))(t).
\end{aligned}
$$

We substitute J_1 into (3.3), then (3.2) is immediately deduced. □

Lemma 3.2. Let the functional I_1 be defined by

$$
I_1(t)=-\varrho_1\int_0^1\varphi_t\int_0^x\int_0^t g(t-s)[(\varphi_y+\psi+lw)(y,t)-(\varphi_y+\psi+lw)(y,s)]\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x.
$$

Then the solution (φ,ψ,w,θ) of Problem 1.1 satisfies for any ε_1, $\varepsilon_2>0$ and any $t_0>0$,

$$
\begin{aligned}
(3.4)\qquad I_1'(t)\leqslant&-\frac{\varrho_1 g_0}{2}\|\varphi_t\|_2^2+\varepsilon_1\|\varphi_x+\psi+lw\|^2+\varepsilon_2\|w_x-l\varphi\|_2^2+C\varrho_1 g_0\|\psi_t\|_2^2\\
&+C\varrho_1 g_0 l^2\|w_t\|_2^2+C\Big(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)(g\circ(\varphi_x+\psi+lw))(t)\\
&-C(g'\circ(\varphi_x+\psi+lw))(t)\quad\forall\,t\geqslant t_0,
\end{aligned}
$$

where $g_0=\int_0^{t_0}g(s)\,\mathrm{d}s>0$.

P r o o f. Remember that

$$\chi(y,t,s) = (\varphi_y + \psi + lw)(y,t) - (\varphi_y + \psi + lw)(y,s).$$

Differentiation of I_1 yields

$$I_1'(t) = \underbrace{-\varrho_1 \int_0^1 \varphi_{tt} \int_0^x \int_0^t g(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x}_{J_2(t)} \underbrace{-\varrho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g'(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x}_{J_3(t)} \underbrace{-\varrho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g(t-s)(\varphi_y + \psi + lw)_t(y,t)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x}_{J_4(t)}. \tag{3.5}$$

Having Lemma 2.2 in mind, J_2, J_3, and J_4 are estimated as follows.

Using $(1.1)_1$, we get

$$\begin{aligned} J_2(t) = &- k_1 \int_0^1 (\varphi_x + \psi + lw)_x \int_0^x \int_0^t g(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x \\ &- k_3 l \int_0^1 (w_x - l\varphi) \int_0^x \int_0^t g(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_0^1 \int_0^t g(t-s)(\varphi_x + \psi + lw)_x(x,s)\,\mathrm{d}s \int_0^x \int_0^t g(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} J_2(t) = &\left(k_1 - \int_0^t g(s)\,\mathrm{d}s\right) \int_0^1 (\varphi_x + \psi + lw) \int_0^t g(t-s)\chi(x,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x \\ &- k_3 l \int_0^1 (w_x - l\varphi) \int_0^x \int_0^t g(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_0^1 \left(\int_0^t g(t-s)\chi(x,t,s)\,\mathrm{d}s\right)^2 \mathrm{d}x. \end{aligned}$$

Then application of Lemma 2.2, Young and Cauchy-Schwarz inequalities give

$$J_2(t) \leqslant \varepsilon_1 \|\varphi_x + \psi + lw\|_2^2 + \varepsilon_2 \|w_x - l\varphi\|_2^2 + C\Big(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\Big)(g \circ (\varphi_x + \psi + lw))(t). \tag{3.6}$$

Next,

$$J_3(t) = -\varrho_1 \int_0^1 \varphi_t \int_0^x \int_0^t g'(t-s)\chi(y,t,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x.$$

We apply Young and Cauchy-Schwarz inequalities:

$$\begin{aligned}(3.7)\qquad J_3(t) &\leqslant \frac{1}{2\varepsilon_3}\int_0^1\Big(\int_0^t \sqrt{-g'(t-s)}\sqrt{-g'(t-s)}\chi(y,t,s)\,\mathrm{d}s\Big)^2\,\mathrm{d}x + \frac{\varepsilon_3}{2}\|\varphi_t\|_2^2\\ &\leqslant \frac{\varepsilon_3}{2}\|\varphi_t\|_2^2 - \frac{g(0)}{2\varepsilon_3}(g'\circ(\varphi_x+\psi+lw))(t).\end{aligned}$$

Again, with integration by parts and making use of (2.6), we have

$$\begin{aligned}J_4(t) = &-\varrho_1\Big(\int_0^t g(s)\,\mathrm{d}s\Big)\Big[\int_0^1\varphi_t\int_0^x\varphi_{yt}(y)\,\mathrm{d}y\,\mathrm{d}x + \int_0^1\varphi_t\int_0^x(\psi+lw)_t(y)\,\mathrm{d}y\,\mathrm{d}x\Big]\\ = &-\varrho_1\Big(\int_0^t g(s)\,\mathrm{d}s\Big)\Big[\int_0^1\varphi_t\int_0^x\psi_t(y)\,\mathrm{d}y\,\mathrm{d}x + l\int_0^1\varphi_t\int_0^x w_t(y)\,\mathrm{d}y\,\mathrm{d}x\Big]\\ &-\varrho_1\Big(\int_0^t g(s)\,\mathrm{d}s\Big)\int_0^1\varphi_t^2\,\mathrm{d}x.\end{aligned}$$

Poincaré and Cauchy-Schwarz inequalities yield

$$(3.8)\qquad J_4(t) \leqslant -\varrho_1 g_0\|\varphi_t\|_2^2 + \frac{\varepsilon_3}{2}\|\varphi_t\|_2^2 + \frac{C\varrho_1^2g_0^2}{\varepsilon_3}\|\psi_t\|_2^2 + \frac{C\varrho_1^2g_0^2l^2}{\varepsilon_3}\|w_t\|_2^2.$$

Now, we add (3.6), (3.7) and (3.8), then (3.5) yields

$$\begin{aligned}I_1'(t) \leqslant &-(\varrho_1g_0-\varepsilon_3)\|\varphi_t\|_2^2 + \varepsilon_1\|\varphi_x+\psi+lw\|_2^2 + \varepsilon_2\|wx-l\varphi\|_2^2 + \frac{C\varrho_1^2g_0^2}{\varepsilon_3}\|\psi_t\|_2^2\\ &+\frac{C\varrho_1^2g_0^2l^2}{\varepsilon_3}\|w_t\|_2^2 + C\Big(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)(g\circ(\varphi_x+\psi+lw))(t)\\ &-\frac{C}{\varepsilon_3}(g'\circ(\varphi_x+\psi+lw))(t).\end{aligned}$$

Choosing $\varepsilon_3 = \varrho_1 g_0/2$, we obtain (3.4). □

Lemma 3.3. Let the functional I_2 be defined by

$$I_2(t) = -\varrho_1\int_0^1(\varphi_x+\psi+lw)\int_0^x\varphi_t(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Then the solution (φ,ψ,w,θ) of Problem 1.1 satisfies for any δ_1, $\delta_2>0$,

(3.9)

$$\begin{aligned}I_2'(t) \leqslant &-\frac{l_0}{2}\|\varphi_x+\psi+lw\|_2^2 + \delta_1\|\psi_t\|^2 + \delta_2\|w_t\|_2^2 + \Big(\varrho_1+\frac{\varrho_1^2}{4\delta_1}+\frac{\varrho_1^2l^2}{4\delta_2}\Big)\|\varphi_t\|_2^2\\ &+\frac{k_3^2l^2}{l_0}\|w_x-l\varphi\|_2^2 + C(g\circ(\varphi_x+\psi+lw))(t)\quad\forall\,t\geqslant 0.\end{aligned}$$

P r o o f. Differentiate I_2 to obtain

$$I_2'(t) = \underbrace{-\varrho_1 \int_0^1 (\varphi_x + \psi + lw)_t \int_0^x \varphi_t(y)\,\mathrm{d}y\,\mathrm{d}x}_{M_1} \underbrace{-\varrho_1 \int_0^1 (\varphi_x + \psi + lw) \int_0^x \varphi_{tt}(y)\,\mathrm{d}y\,\mathrm{d}x}_{M_2}. \tag{3.10}$$

Using integration by parts keeping in mind (2.6), Young and Cauchy-Schwarz inequalities, we estimate M_1 as follows:

$$\begin{aligned} M_1 = &- \varrho\Big[\underbrace{\varphi_t \int_0^x \varphi_t(y)\,\mathrm{d}\Big|_{x=0}^{x=1}}_{\to 0} - \int_0^1 \varphi_t^2\,\mathrm{d}x\Big] \\ &- \varrho_1 \int_0^1 \psi_t \int_0^x \varphi_t(y)\,\mathrm{d}y\,\mathrm{d}x - \varrho_1 l \int_0^1 w_t \int_0^x \varphi_t(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \delta_1\|\psi_t\|_2^2 + \delta_2\|w_t\|_2^2 + \Big(\varrho_1 + \frac{\varrho_1^2}{4\delta_1} + \frac{\varrho_1^2 l^2}{4\delta_2}\Big)\|\varphi_t\|_2^2. \end{aligned}$$

Similarly, using $(1.1)_1$, Young and Cauchy-Schwarz inequalities, M_2 is estimated as follows:

$$\begin{aligned} M_2 = &- \Big(k_1 - \int_0^t g(s)\,\mathrm{d}s\Big)\|\varphi_x + \psi + lw\|_2^2 \\ &- k_3 l \int_0^1 (\varphi_x + \psi + lw) \int_0^x (w_y - l\varphi)(y)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_0^1 (\varphi_x + \psi + lw) \int_0^t g(t-s)[(\varphi_x + \psi + lw)(t) - (\varphi_x + \psi + lw)(s)]\,\mathrm{d}s\,\mathrm{d}x \\ \leqslant &- \Big(k_1 - \int_0^t g(s)\,\mathrm{d}s - \delta_3\Big)\|\varphi_x + \psi + lw\|_2^2 + \frac{l^2 k_3^2}{2\delta_3}\|w_x - l\varphi\|_2^2 \\ &+ \frac{C}{\delta_3}(g \circ (\varphi_x + \psi + lw))(t). \end{aligned}$$

Substituting M_1 and M_2 into (3.10), we arrive at

$$\begin{aligned} I_2'(t) \leqslant &- k_1(l_0 - \delta_3)\|\varphi_x + \psi + lw\|^2 + \delta_1\|\psi_t\|_2^2 + \delta_2\|w_t\|_2^2 \\ &+ \Big(\varrho_1 + \frac{\varrho_1^2}{4\delta_1} + \frac{\varrho_1^2 l^2}{4\delta_2}\Big)\|\varphi_t\|_2^2 + \frac{l^2 k_3^2}{2\delta_3 k_1^2}\|w_x - l\varphi\|_2^2 \\ &+ \frac{C}{\delta_3}(g \circ (\varphi_x + \psi + lw))(t). \end{aligned} \tag{3.11}$$

Choosing $\delta_3 = l_0/2$, we obtain (3.9). □

Lemma 3.4. Let the functional I_3 be defined by

$$I_3(t) = \varrho_3 \int_0^1 \psi_t \int_0^x \theta(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Then the solution $(\varphi, \psi, w, \theta)$ of Problem 1.1 satisfies for any σ_2, $\sigma_3 > 0$ the inequality

$$\begin{aligned} I_3'(t) \leqslant & -\frac{\gamma}{2}\|\psi_t\|_2^2 + \sigma_2\|\psi_x\|^2 + \sigma_3\|\varphi_x + \psi + lw\|_2^2 \\ & + \Big(\frac{\beta^2}{2\gamma} + \frac{\gamma\varrho_3}{\varrho_2} + \frac{k_2^2\varrho_3^2}{4\sigma_2\varrho_2^2} + \frac{\varrho_3^2}{2\sigma_3\varrho_2^2}\Big)\|\theta_x\|_2^2 \\ & + \sigma_3(g \circ (\varphi_x + \psi + lw))(t) \quad \forall\, t \geqslant 0. \end{aligned} \tag{3.12}$$

P r o o f. Differentiation of I_3 gives

$$I_3'(t) = \varrho_3 \int_0^1 \psi_{tt} \int_0^x \theta(y)\,\mathrm{d}y\,\mathrm{d}x + \varrho_3 \int_0^1 \psi_t \int_0^x \theta_t(y)\,\mathrm{d}y\,\mathrm{d}x. \tag{3.13}$$

We substitute $(1.1)_2$, $(1.1)_4$ and integrate by parts keeping in mind (2.6), then apply Young, Cauchy-Schwarz, Poincaré inequalities and Lemma 2.2. Therefore,

$$\begin{aligned} I_3'(t) = & -\frac{k_2\varrho_3}{\varrho_2}\int_0^1 \psi_x\theta\,\mathrm{d}x - \frac{\varrho_3}{\varrho_2}\Big(k_1 - \int_0^t g(s)\,\mathrm{d}s\Big)\int_0^1 (\varphi_x + \psi + lw)\int_0^x \theta(y)\,\mathrm{d}y\,\mathrm{d}x \\ & - \frac{\varrho_3}{\varrho_2}\int_0^1\int_0^t g(t-s)[(\varphi_x + \psi + lw)(t) - (\varphi_x + \psi + lw)(s)]\,\mathrm{d}s \int_0^x \theta(y)\,\mathrm{d}y\,\mathrm{d}x \\ & + \frac{\gamma\varrho_3}{\varrho_2}\int_0^1 \theta^2\,\mathrm{d}x + \beta\int_0^1 \psi_t\theta_x\,\mathrm{d}x - \gamma\int_0^1 \psi_t^2\,\mathrm{d}x \\ \leqslant & -\frac{\gamma}{2}\|\psi_t\|_2^2 + \frac{\beta^2}{2\gamma}\|\theta_x\|_2^2 + \frac{\gamma\varrho_3}{\varrho_2}\|\theta_x\|_2^2 + \sigma_2\|\psi_x\|_2^2 + \frac{k_2^2\varrho_3^2}{4\sigma_2\varrho_2^2}\|\theta_x\|_2^2 \\ & + \sigma_3\|\varphi_x + \psi + lw\|_2^2 + \frac{\varrho_3^2 l_0^2}{4\sigma_3\varrho_2^2}\|\theta_x\|_2^2 + \frac{\varrho_3^2}{4\sigma_3\varrho_2^2}\|\theta_x\|_2^2 \\ & + \sigma_3(g \circ (\varphi_x + \psi + lw))(t). \end{aligned}$$

Thus, (3.12) follows immediately since $l_0^2 \leqslant 1$. □

Lemma 3.5. Let the functional I_4 be defined by

$$I_4(t) = \varrho_2 \int_0^1 \psi\psi_t\,\mathrm{d}x.$$

Then the solution $(\varphi, \psi, w, \theta)$ of Problem 1.1 satisfies

$$\begin{aligned} I_4'(t) \leqslant & -\frac{k_2}{2}\|\psi_x\|_2^2 + \varrho_2\|\psi_t\|_2^2 + C\|\theta_x\|_2^2 + Cl_0^2\|\varphi_x + \psi + lw\|_2^2 \\ & + C(g \circ (\varphi_x + \psi + lw))(t) \quad \forall\, t \geqslant 0. \end{aligned} \tag{3.14}$$

Proof. Differentiation of I_4 and making use of $(1.1)_2$ gives

$$\begin{aligned}I_4'(t) &= \varrho_2\int_0^1 \psi_t^2\,\mathrm{d}x + k_2\int_0^1 \psi\psi_{xx}\,\mathrm{d}x\\&\quad-\Bigl(k_1-\int_0^t g(s)\,\mathrm{d}s\Bigr)\int_0^1 \psi(\varphi_x+\psi+lw)\,\mathrm{d}x\\&\quad-\int_0^1\psi\int_0^t g(t-s)[(\varphi_x+\psi+lw)(t)-(\varphi_x+\psi+lw)(s)]\,\mathrm{d}s\,\mathrm{d}x\\&\quad+\gamma\int_0^1\psi\theta_x\,\mathrm{d}x.\end{aligned}$$

Applying Young, Cauchy-Schwarz and Poincaré inequalities, we get

$$\begin{aligned}I_4'(t)\leqslant&-(k_2-\sigma_1)\|\psi_x\|_2^2+\varrho_2\|\psi_t\|_2^2+\frac{Cl_0^2}{\sigma_1}\|\varphi_x+\psi+lw\|_2^2+\frac{C\gamma^2}{\sigma_1}\|\theta_x\|_2^2\\&+\frac{C}{\sigma_1}(g\circ(\varphi_x+\psi+lw))(t).\end{aligned}$$

Choosing $\sigma_1=k_2/2$, we obtain (3.14). □

Lemma 3.6. Let the functional I_5 be defined by

$$\begin{aligned}I_5(t)=&-\varrho_1\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x-\frac{k_3\varrho_1}{k_1}\int_0^1(w_x-l\varphi)\varphi_t\,\mathrm{d}x\\&+\frac{\varrho_1}{k_1}\int_0^1 w_t\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}\tag{3.15}$$

Then the solution (φ,ψ,w,θ) of Problem 1.1 satisfies the inequality

$$\begin{aligned}I_5'(t)\leqslant&-\frac{k_3^2l}{k_1}\|w_x-l\varphi\|_2^2-\frac{\varrho_1 l}{2}\|w_t\|^2+\frac{k_3\varrho_1 l}{k_1}\|\varphi_t\|_2^2+\frac{2\varrho_1}{l}\|\psi_t\|_2^2\\&+\Bigl[5k_1l+\frac{4\varrho_1g^2(0)}{lk_1^2}\Bigr]\|\varphi_x+\psi+lw\|_2^2+Cl(g\circ(\varphi_x+\psi+lw))(t)\\&-\frac{2\varrho_1g^2(0)}{lk_1^2}(g'\circ(\varphi_x+\psi+lw))(t)\\&-\varrho_1\Bigl(1-\frac{k_3}{k_1}\Bigr)\int_0^1 w_t\varphi_{xt}\,\mathrm{d}x\quad\forall\,t\geqslant 0.\end{aligned}\tag{3.16}$$

Proof. We differentiate I_5, then

$$\begin{aligned}
(3.17)\qquad I_5'(t) = &\underbrace{-\varrho_1\int_0^1 w_t(\varphi_x+\psi+lw)_t\,\mathrm{d}x - \frac{k_3\varrho_1}{k_1}\int_0^1 (w_x-l\varphi)_t\varphi_t\,\mathrm{d}x}_{M_4}\\
&\underbrace{-\varrho_1\int_0^1 w_{tt}(\varphi_x+\psi+lw)\,\mathrm{d}x}_{M_5}\ \underbrace{-\ \frac{k_3\varrho_1}{k_1}\int_0^1 (w_x-l\varphi)\varphi_{tt}\,\mathrm{d}x}_{M_6}\\
&+\underbrace{\frac{\varrho_1}{k_1}\int_0^1 w_{tt}\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x}_{M_7}\\
&+\frac{\varrho_1}{k_1}\int_0^1 w_t\int_0^t g'(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x\\
&+\frac{\varrho_1}{k_1}g(0)\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x.
\end{aligned}$$

We have

$$\begin{aligned}
M_4 = &-\varrho_1 l\int_0^1 w_t^2\,\mathrm{d}x + \frac{k_3\varrho_1 l}{k_1}\int_0^1 \varphi_t^2\,\mathrm{d}x - \varrho_1\int_0^1 w_t\psi_t\,\mathrm{d}x\\
&-\varrho_1\Big(1-\frac{k_3}{k_1}\Big)\int_0^1 w_t\varphi_{xt}\,\mathrm{d}x.
\end{aligned}$$

Next, using $(1.1)_1$ and $(1.1)_3$, we obtain M_5–M_7 as:

$$\begin{aligned}
M_5 = {}& k_3\int_0^1 (w_x-l\varphi)(\varphi_x+\psi+lw)_x\,\mathrm{d}x + k_1 l\int_0^1(\varphi_x+\psi+lw)^2\,\mathrm{d}x\\
&-l\int_0^1(\varphi_x+\psi+lw)\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x,\\
M_6 = &-k_3\int_0^1 (w_x-l\varphi)(\varphi_x+\psi+lw)_x\,\mathrm{d}x - \frac{k_3^2 l}{k_1}\int_0^1 (w_x-l\varphi)^2\,\mathrm{d}x\\
&+\frac{k_3}{k_1}\int_0^1 (w_x-l\varphi)\int_0^t g(t-s)(\varphi_x+\psi+lw)_x(s)\,\mathrm{d}s\,\mathrm{d}x,\\
M_7 = &-\frac{k_3}{k_1}\int_0^1 (w_x-l\varphi)\int_0^t g(t-s)(\varphi_x+\psi+lw)_x(s)\,\mathrm{d}s\,\mathrm{d}x\\
&+\frac{l}{k_1}\int_0^1\Big(\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}x\\
&-\frac{l}{k_1}\int_0^1 (\varphi_x+\psi+lw)\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

Substituting M_4–M_7 into (3.17), we get

$$
\begin{aligned}
(3.18)\qquad I_5'(t) = & -\frac{k_3^2 l}{k_1}\|w_x - l\varphi\|_2^2 - \varrho_1 l\|w_t\|_2^2 + \frac{k_3\varrho_1 l}{k_1}\|\varphi_t\|_2^2 + k_1 l\|\varphi_x + \psi + lw\|_2^2 \\
& - \varrho_1\int_0^1 w_t\psi_t\,\mathrm{d}x + \frac{\varrho_1}{k_1}g(0)\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x \\
& \underbrace{-l\Big(1+\frac{1}{k_1}\Big)\int_0^1(\varphi_x+\psi+lw)\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x}_{\psi_1} \\
& + \underbrace{\frac{l}{k_1}l\int_0^1\Big(\int_0^t g(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}x}_{\psi_2} \\
& + \underbrace{\frac{\varrho_1}{k_1}\int_0^1 w_t\int_0^t g'(t-s)(\varphi_x+\psi+lw)(s)\,\mathrm{d}s\,\mathrm{d}x}_{\psi_3} \\
& - \varrho_1\Big(1-\frac{k_3}{k_1}\Big)\int_0^1 w_t\varphi_{xt}\,\mathrm{d}x.
\end{aligned}
$$

We have the following estimates using Young and Cauchy-Schwarz inequalities:

$$(3.19)\quad -\varrho_1\int_0^1 w_t\psi_t\,\mathrm{d}x \leqslant \varepsilon_0\varrho_1\|w_t\|_2^2 + \frac{\varrho_1}{4\varepsilon_0}\|\psi_t\|_2^2,$$

$$(3.20)\quad \frac{\varrho_1}{k_1}g(0)\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x \leqslant \varepsilon_0\varrho_1\|w_t\|_2^2 + \frac{\varrho_1 g^2(0)}{4\varepsilon_0 k_1^2}\|\varphi_x+\psi+lw\|_2^2,$$

$$
\begin{aligned}
(3.21)\quad \psi_1 \leqslant\ & l\Big(1+\frac{1}{k_1}\Big)\Big(k_1 - \int_0^t g(s)\,\mathrm{d}s\Big)\|\varphi_x+\psi+lw\|_2^2 \\
& + Cl(g\circ(\varphi_x+\psi+lw))(t).
\end{aligned}
$$

Remember that

$$\chi(x,t,s) = (\varphi_x+\psi+lw)(t) - (\varphi_x+\psi+lw)(s).$$

Using Cauchy-Schwarz inequality, we also have

$$
\begin{aligned}
(3.22)\quad \psi_2 &= \frac{l}{k_1}\int_0^1\Big(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}[(\varphi_x+\psi+lw)(t) - \chi(x,t,s)]\,\mathrm{d}s\Big)^2\,\mathrm{d}x \\
&\leqslant \frac{2l}{k_1}(1-l_0)\int_0^1\int_0^t g(t-s)(\chi(x,t,s)^2 + (\varphi_x+\psi+lw)^2(t))\,\mathrm{d}s\,\mathrm{d}x \\
&\leqslant \frac{2l}{k_1}\|\varphi_x+\psi+lw\|_2^2 + 2k_1 l(g\circ(\varphi_x+\psi+lw))(t).
\end{aligned}
$$

Using Young and Cauchy-Schwarz inequalities, we get

$$
\begin{aligned}
(3.23)\qquad \psi_3 &= \frac{\varrho_1}{k_1}\Big(\int_0^t g'(s)\,\mathrm{d}s\Big)\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x \\
&\quad+\frac{\varrho_1}{k_1}\int_0^1 w_t\int_0^t g'(t-s)\chi(x,t,s)\,\mathrm{d}s\,\mathrm{d}x \\
&\leqslant \frac{\varrho_1}{k_1}(g(t)-g(0))\int_0^1 w_t(\varphi_x+\psi+lw)\,\mathrm{d}x+\varepsilon_0\varrho_1\|w_t\|_2^2 \\
&\quad+\frac{\varrho_1}{4\varepsilon_0 k_1^2}\int_0^1\Big(\int_0^t\sqrt{-g'(t-s)}\sqrt{-g'(t-s)}\chi(x,t,s)\,\mathrm{d}s\Big)^2\,\mathrm{d}x \\
&\leqslant 2\varepsilon_0\varrho_1\|w_t\|_2^2+\frac{\varrho_1(g(t)-g(0))^2}{4\varepsilon_0 k_1}\|\varphi_x+\psi+lw\|_2^2 \\
&\quad+\frac{\varrho_1}{4\varepsilon_0 k_1^2}\Big(\int_0^t g'(s)\,\mathrm{d}s\Big)\int_0^1\int_0^t g'(t-s)\chi^2(x,t,s)\,\mathrm{d}s\,\mathrm{d}x \\
&\leqslant 2\varepsilon_0\varrho_1\|w_t\|_2^2+\frac{\varrho_1 g^2(0)}{4\varepsilon_0 k_1^2}\|\varphi_x+\psi+lw\|_2^2 \\
&\quad-\frac{\varrho_1 g^2(0)}{4\varepsilon_0 k_1^2}(g'\circ(\varphi_x+\psi+lw))(t).
\end{aligned}
$$

Substituting (3.19)–(3.23) into (3.18), we get

$$
\begin{aligned}
(3.24)\qquad I_5'(t) = &-\frac{k_3^2 l}{k_1}\|w_x-l\varphi\|_2^2-\varrho_1(l-4\varepsilon_0)\|w_t\|^2+\frac{k_3\varrho_1 l}{k_1}\|\varphi_t\|_2^2+\frac{\varrho_1}{4\varepsilon_0}\|\psi_t\|_2^2 \\
&+\Big[5k_1 l+\frac{\varrho_1 g^2(0)}{2\varepsilon_0 k_1^2}\Big]\|\varphi_x+\psi+lw\|_2^2+lC(g\circ(\varphi_x+\psi+lw))(t) \\
&-\frac{\varrho_1 g^2(0)}{4\varepsilon_0 k_1^2}(g'\circ(\varphi_x+\psi+lw))(t)-\varrho_1\Big(1-\frac{k_3}{k_1}\Big)\int_0^1 w_t\varphi_{xt}\,\mathrm{d}x.
\end{aligned}
$$

Choosing $\varepsilon_0 = l/8$ in (3.24), we obtain (3.16). □

Lemma 3.7. Let the functional I_6 be defined by

$$I_6(t)=\varrho_1\int_0^1(w_x-l\varphi)\int_0^x w_t(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Then the solution (φ,ψ,w,θ) of Problem 1.1 satisfies the inequality

$$
\begin{aligned}
(3.25)\qquad I_6'(t)\leqslant &-\frac{\varrho_1}{2}\|w_t\|_2^2+\frac{\varrho_1 l^2}{2}\|\varphi_t\|_2^2+C\Big(1+l^2+\frac{l^2}{\sigma_0}\Big)\|w_x-l\varphi\|_2^2 \\
&+\sigma_0\|\varphi_x+\psi+lw\|_2^2+C(g\circ(\varphi_x+\psi+lw))(t)\quad\forall\, t\geqslant 0.
\end{aligned}
$$

P r o o f. We differentiate I_6, then use integration by parts to get

$$
\begin{aligned}
I_6' &= \varrho_1 \int_0^1 (w_x - l\varphi)_t \int_0^x w_t(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + \varrho_1 \int_0^1 w(w_x - l\varphi) \int_0^x w_{tt}(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= -\varrho_1 \int_0^1 w_t^2\,\mathrm{d}x \underbrace{- l\varrho_1 \int_0^1 u_t \int_0^x w_t(y)\,\mathrm{d}y\,\mathrm{d}x}_{\psi_4} + k_3 \int_0^1 (w_x - l\varphi)^2\,\mathrm{d}x \\
&\quad \underbrace{-k_1 l \int_0^1 (w_x - l\varphi) \int_0^x (\varphi_y + \psi + lw)(y)\,\mathrm{d}y\,\mathrm{d}x}_{\psi_5} \\
&\quad \underbrace{+l \int_0^1 (w_x - l\varphi) \int_0^x \int_0^t g(t-s)(\varphi_y + \psi + lw)(y,s)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x}_{\psi_6}.
\end{aligned} \tag{3.26}
$$

Using Young and Cauchy-Schwarz inequalities and applying Lemmas 2.1 and 2.2, we estimate $\psi_4 - \psi_6$ as follows:

$$
\begin{aligned}
\psi_4 &\leqslant \delta_0 \|w_t\|_2^2 + \frac{l^2 \varrho_1^2}{\delta_0} \|\varphi_t\|_2^2, \\
\psi_5 &\leqslant \frac{\sigma_0}{2} \|\varphi_x + \psi + lw\|_2^2 + \frac{Cl^2}{\sigma_0} \|w_x - l\varphi\|_2^2
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_6 &= l\Big(\int_0^t g(s)\,\mathrm{d}s\Big) \int_0^1 (w_x - l\varphi) \int_0^x (\varphi_y + \psi + lw)\,\mathrm{d}y\,\mathrm{d}x \\
&\quad + l \int_0^1 (w_x - l\varphi) \int_0^x \int_0^t g(t-s) \\
&\quad \times [(\varphi_y + \psi + lw)(y,t) - (\varphi_y + \psi + lw)(y,s)]\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant \frac{\sigma_0}{2} \|\varphi_x + \psi + lw\|_2^2 + \frac{Cl^2}{\sigma_0} \|w_x - l\varphi\|_2^2 + C(g \circ (\varphi_x + \psi + lw))(t).
\end{aligned}
$$

Substituting $\psi_4 - \psi_6$ into (3.26) and then choosing $\delta_0 = \varrho_1/2$, we arrive at (3.25). $\square$

4. Decay result

We are now set to state and prove our general decay result. First, we will define a Lyapunov functional $\mathcal{L}$. In the next lemma, we prove the equivalence of the energy functional E and the Lyapunov functional $\mathcal{L}$.

Lemma 4.1. *There exist positive constants β_1, β_2 such that for N large enough, the functional $\mathcal{L}$ defined as*

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t) + N_5 I_5(t) + N_6 I_6(t) \tag{4.1}$$

satisfies

$$\beta_1 E(t) \leqslant \mathcal{L}(t) \leqslant \beta_2 E(t) \quad \forall\, t \geqslant 0, \tag{4.2}$$

for some positive constants N, N_1, N_2, N_3, N_4, N_5, and N_6 to be appropriately chosen later.

Proof. We have

$$|\mathcal{L}(t) - NE(t)| \leqslant N_1|I_1(t)| + N_2|I_2(t)| + N_3|I_3(t)| + N_4|I_4(t)| + N_5|I_5(t)| + N_6|I_6(t)|.$$

On account of Young inequality, Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} |\mathcal{L}(t) - NE(t)| &\leqslant N_1|I_1(t)| + N_2|I_2(t)| + N_3|I_3(t)| + N_4|I_4(t)| + N_5|I_5(t)| + N_6|I_6(t)| \\ &\leqslant C(\varrho_1\|\varphi_t\|_2^2 + \varrho_2\|\psi_t\|_2^2 + \varrho_3\|\theta\|_2^2 + b\|\psi_x\|_2^2 + k\|\varphi_x + \psi + lw\|_2^2 + \varrho_1\|w_t\|_2^2) \\ &\quad + Ck_3\|w_x - l\varphi\|_2^2 + C\int_0^1 \Big(\int_0^t g(t-s)(\varphi_x + \psi + lw)(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}x \\ &\quad + C\int_0^1 \Big(\int_0^x \int_0^t g(t-s)[(\varphi_y + \psi + lw)(y,t) - (\varphi_y + \psi + lw)(y,s)]\,\mathrm{d}s\,\mathrm{d}y\Big)^2\,\mathrm{d}x. \end{aligned} \tag{4.3}$$

Like in estimate (3.22), we have

$$\begin{aligned} \int_0^1 \Big(\int_0^t g(t-s)(\varphi_x + \psi + lw)(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}x \\ \leqslant \|\varphi_x + \psi + lw\|_2^2 + (g \circ (\varphi_x + \psi + lw))(t). \end{aligned} \tag{4.4}$$

According to Lemma 2.2, we have

$$\begin{aligned} \int_0^1 \Big(\int_0^x \int_0^t g(t-s)[(\varphi_y + \psi + lw)(y,t) - (\varphi_y + \psi + lw)(y,s)]\,\mathrm{d}s\,\mathrm{d}y\Big)^2\,\mathrm{d}x \\ \leqslant (g \circ (\varphi_x + \psi + lw))(t). \end{aligned} \tag{4.5}$$

Substituting (4.4)–(4.5) into (4.3), it follows that

$$|\mathcal{L}(t) - NE(t)| \leqslant CE(t) \Leftrightarrow (N - C)E(t) \leqslant \mathcal{L}(t) \leqslant (N + C)E(t).$$

Next, we choose N large enough so that

$$N - C > 0. \tag{4.6}$$

Hence, there exist β_1, β_2 positive constants such that (4.2) holds. Therefore $\mathcal{L} \sim E$. $\square$

Theorem 4.1. Let $(\varphi, \psi, w, \theta)$ be the solution of problem (1.1)–(1.3). Assume $k_3 = k_1$ and that (A_1) and (A_2) hold. Then for any $t_0 > 0$, there exist constants α_1, α_2, $\alpha_3 > 0$ such that $l^2 \leqslant \alpha_3 \min\{1/g_0, l_0\}$ and the energy functional E in (3.1) satisfies

$$E(t) \leqslant \alpha_1 \mathrm{e}^{-\alpha_2 \int_{t_0}^t \eta(s)\,\mathrm{d}s} \quad \forall\, t \geqslant t_0. \tag{4.7}$$

P r o o f. Differentiate (4.1) while bearing in mind (3.4), (3.9), (3.12), (3.14), (3.16), and (3.2). Then

$$\begin{aligned}
\mathcal{L}'(t) \leqslant & -\Big[\frac{\varrho_1 g_0}{2} N_1 - \Big(\varrho_1 + \frac{\varrho_1^2}{4\delta_1} + \frac{\varrho_1^2 l^2}{4\delta_2}\Big) N_2 - \frac{k_3 \varrho_1 l}{k_1} N_5 - \frac{l^2 \varrho_1}{2} N_6\Big] \|\varphi_t\|_2^2 \\
& - \Big[\frac{\gamma}{2} N_3 - C\varrho_1 g_0 N_1 - \delta_1 N_2 - \varrho_2 N_4 - \frac{2\varrho_1}{l} N_5\Big] \|\psi_t\|_2^2 \\
& - \Big[\frac{k_3^2 l}{k_1} N_5 - \varepsilon_2 N_1 - \frac{k_3^2 l^2}{l_0} N_2 - C\Big(1 + l^2 + \frac{l^2}{\sigma_0}\Big) N_6\Big] \|w_x - l\varphi\|_2^2 \\
& - \Big[\frac{k_2}{2} N_4 - \sigma_2 N_3\Big] \|\psi_x\|_2^2 - [\beta N - \Gamma_1 N_3 - C N_4] \|\theta_x\|_2^2 \\
& - \Big[\frac{k_1 l_0}{2} N_2 - \varepsilon_1 N_1 - \sigma_3 N_3 - C l_0^2 N_4 - \Gamma_2 N_5 - \sigma_0 N_6\Big] \|\varphi_x + \psi + lw\|_2^2 \\
& - \Big[\frac{\varrho_1 l}{2} N_5 + \frac{\varrho_1}{2} N_6 - \delta_2 N_2 - C\varrho_1 g_0 l^2 N_1\Big] \|w_t\|_2^2 \\
& + C[\Gamma_3 N_1 + N_2 + N_3 + N_4 + l N_5 + N_6](g \circ (\varphi_x + \psi + lw))(t) \\
& + \Big[\frac{k_1}{2} N - C N_1 - \frac{2\varrho_1 g^2(0)}{l k_1^2} N_5\Big] (g' \circ (\varphi_x + \psi + lw))(t) \\
& - \frac{k_1}{2} N g(t) \|\varphi_x + \psi + lw\|_2^2 - \varrho_1 N_5 \Big(1 - \frac{k_3}{k_1}\Big) \int_0^1 w_t \varphi_{xt}\,\mathrm{d}x \quad \forall\, t \geqslant t_0,
\end{aligned}$$

where

$$\Gamma_1 = \Big(\frac{\beta}{(2\gamma)} + \frac{\gamma\varrho_3}{\varrho_2} + \frac{k_2^2 \varrho_3^2}{4\sigma_2 \varrho_2^2} + \frac{\varrho_3^2}{2\sigma_3 \varrho_2^2}\Big), \quad \Gamma_2 = \Big(5k_1 l + \frac{4\varrho_1 g^2(0)}{l k_1^2}\Big)$$

and

$$\Gamma_3 = \Big(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\Big).$$

Set $\delta_1 = \varrho_2/N_2$ and $N_4 = 1$. Therefore,

$$
\begin{aligned}
(4.8)\qquad \mathcal{L}'(t) \leqslant\; & -\Big[\frac{\varrho_1 g_0}{2} N_1 - \Big(\varrho_1 + \frac{\varrho_1^2 N_2}{4\varrho_2} + \frac{\varrho_1^2 l^2}{4\delta_2}\Big) N_2 - \varrho_1 l N_5 - \frac{l^2 \varrho_1}{2} N_6\Big] \|\varphi_t\|_2^2 \\
& - \Big[\frac{\gamma}{2} N_3 - C\varrho_1 g_0 N_1 - 2\varrho_2 - \frac{2\varrho_1}{l} N_5\Big] \|\psi_t\|_2^2 \\
& - \Big[k_3^2 l N_5 - \varepsilon_2 N_1 - \frac{k_3^2 l^2}{l_0} N_2 - C\Big(1 + l^2 + \frac{l^2}{\sigma_0}\Big) N_6\Big] \|w_x - l\varphi\|_2^2 \\
& - \Big[\frac{k_2}{2} - \sigma_2 N_3\Big] \|\psi_x\|_2^2 - [\beta N - \Gamma_1 N_3 - C] \|\theta_x\|_2^2 \\
& - \Big[\frac{l_0}{2} N_2 - \varepsilon_1 N_1 - \sigma_3 N_3 - C l_0^2 - \Gamma_2 N_5 - \sigma_0 N_6\Big] \|\varphi_x + \psi + lw\|_2^2 \\
& - \Big[\frac{\varrho_1 l}{2} N_5 + \frac{\varrho_1}{2} N_6 - \delta_2 N_2 - C\varrho_1 g_0 l^2 N_1\Big] \|w_t\|_2^2 \\
& + C[\Gamma_3 N_1 + N_2 + N_3 + 1 + l N_5 + N_6] (g \circ (\varphi_x + \psi + lw))(t) \\
& + \Big[\frac{k_1}{2} N - C N_1 - \frac{2\varrho_1 g^2(0)}{l k_1^2} N_5\Big] (g' \circ (\varphi_x + \psi + lw))(t) \\
& - \frac{k_1}{2} N g(t) \|\varphi_x + \psi + lw\|_2^2 \quad \forall\, t \geqslant t_0.
\end{aligned}
$$

We choose N_2 large enough so that $\xi_1 = l_0 N_2/2 - C l_0^2 > 0$, then choose σ_0, ε_1, σ_3, N_5 small enough so that

$$
(4.9)\qquad \xi_1 - \varepsilon_1 N_1 - \sigma_3 N_3 - \Big(5k_1 l + \frac{4\varrho_1 g^2(0)}{l k_1^2}\Big) N_5 - \sigma_0 N_6 > 0.
$$

Next, we choose N_1 large enough so that

$$
(4.10)\qquad \xi_2 = \frac{\varrho_1 g_0}{2} N_1 - \Big(\varrho_1 + \frac{\varrho_1^2 N_2}{4\varrho_2} + \frac{\varrho_1^2 l^2}{4\delta_2}\Big) N_2 - \varrho_1 l N_5 > 0,
$$

then choose σ_2 small enough so that

$$
(4.11)\qquad \frac{k_2}{2} - \sigma_2 N_3 > 0.
$$

We further choose N_3 large enough so that

$$
(4.12)\qquad \frac{\gamma}{2} N_3 - C\varrho_1 g_0 N_1 - 2\varrho_2 - \frac{2\varrho_1}{l} N_5 > 0.
$$

Now, we choose N_6 and δ_2 small enough so that

$$
(4.13)\quad \xi_2 - \frac{\varrho_1 l^2}{2} N_6 > 0,\ \xi_3 = k_3^2 l N_5 - C\Big(1 + l^2 + \frac{l^2}{\sigma_0}\Big) N_6 > 0 \text{ and } \frac{\varrho_1 l}{2} N_5 - \delta_2 N_2 > 0.
$$

Then we can choose l and ε_2 small enough so that

$$\frac{\varrho_1}{2}N_6 - C\varrho_1 g_0 l^2 N_1 > 0 \quad \text{and} \quad \xi_3 - \varepsilon_2 N_1 - \frac{l^2}{l_0 k_3^2} N_2 > 0, \tag{4.14}$$

that is, $l^2 \leqslant \alpha_3 \min\{1/g_0, l_0\}$, where $\alpha_3 = \min\{\xi_3/(k_3^2 N_2), N_6/(2CN_1)\}$.

Finally, we choose N very large such that (4.6) holds,

$$\frac{k_1}{2}N - CN_1 - \frac{2\varrho_1 g^2(0)}{lk_1^2} N_5 > 0 \tag{4.15}$$

and

$$\beta N - \Big(\frac{\beta}{2\gamma} + \frac{\gamma\varrho_3}{\varrho_2} + \frac{k_2^2\varrho_3^2}{4\sigma_2\varrho_2^2} + \frac{\varrho_3^2}{2\sigma_3\varrho_2^2}\Big) N_3 - C > 0.$$

Therefore, due to (4.9)–(4.15), we deduce form (4.8) that there exist positive constants λ, α such that

$$\mathcal{L}'(t) \leqslant -\lambda E(t) + \alpha k_1 (g \circ (\varphi_x + \psi + lw))(t) \quad \forall\, t \geqslant t_0. \tag{4.16}$$

Let us multiply (4.16) by $\eta(t)$ and make use of assumption (A_2) and (3.2). Then we obtain

$$\eta(t)\mathcal{L}'(t) \leqslant -\lambda\eta(t)E(t) - 2\alpha E'(t) \quad \forall\, t \geqslant t_0,$$

which implies

$$(\eta(t)\mathcal{L}(t) + 2\alpha E(t))' - \eta'(t)\mathcal{L}(t) \leqslant -\lambda\eta(t)E(t) \quad \forall\, t \geqslant t_0.$$

Since $\eta'(t) \leqslant 0$ for all $t \geqslant 0$, we have

$$(\eta(t)\mathcal{L}(t) + 2\alpha E(t))' \leqslant -\lambda\eta(t)E(t) \quad \forall\, t \geqslant t_0.$$

Knowing that (4.2) holds, we can deduce that

$$\mathcal{F}(t) = \eta(t)\mathcal{L}(t) + 2\alpha E(t) \sim E(t). \tag{4.17}$$

Therefore, there exists α_2 a positive constant such that

$$\mathcal{F}'(t) \leqslant -\alpha_2 \eta(t)\mathcal{F}(t) \quad \forall\, t \geqslant t_0. \tag{4.18}$$

Thus, integration of (4.18) over (t_0, t) gives

$$\mathcal{F}(t) \leqslant \mathcal{F}(t_0)\mathrm{e}^{-\alpha_2 \int_{t_0}^{t} \eta(s)\,\mathrm{d}s} \quad \forall\, t \geqslant t_0. \tag{4.19}$$

Finally, estimate (4.7) follows from (4.17) and (4.19). □

Remark 4.1. Estimate (4.7) is also valid for $t \in [0, t_0]$ due to continuity and boundedness of the energy functional E and the function η. Indeed, since $E(t) \leqslant E(t_0) \leqslant E(0)$ for all $t \geqslant t_0 > 0$, we can find a positive constant α_0 such that

$$E(t) \leqslant \alpha_0 E(0) \mathrm{e}^{\alpha_2 \int_0^{t_0} \eta(s)\,\mathrm{d}s} \mathrm{e}^{-\alpha_2 \int_0^t \eta(s)\,\mathrm{d}s} \quad \forall\, t \geqslant t_0 > 0.$$

Thus, setting $\alpha_1 = \alpha_0 E(0) \mathrm{e}^{\alpha_2 \int_0^{t_0} \eta(s)\,\mathrm{d}s}$, estimate (4.7) holds for all $t \geqslant 0$.

5. Conclusion

In this work, we have established a general decay result for the solution energy of the thermoelastic Bresse system (1.1), for l small enough with $k_3 = k_1$. Consequently, both polynomial and exponential decay rates are special cases of our result. Our result holds without the imposition of the equal speed of wave propagation condition, neither do we require a more regular solution emanating from differentiation of the equations in (1.1).

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