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Commentationes Mathematicae Universitatis Carolinae, Vol. 63 (2022), No. 2, 261–267

Persistent URL: http://dml.cz/dmlcz/151089

Comment.Math.Univ.Carolin. 63,2 (2022) 261–267

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Abstract. Given a coarse space $(X,\mathcal{E})$ with the bornology $\mathcal{B}$ of bounded subsets, we extend the coarse structure $\mathcal{E}$ from $X\times X$ to the natural coarse structure on $(\mathcal{B}\backslash\{\emptyset\})\times(\mathcal{B}\backslash\{\emptyset\})$ and say that a macro-uniform mapping $f\colon(\mathcal{B}\backslash\{\emptyset\})\to X$ (or $f\colon[X]^2\to X$) is a selector (or 2-selector) of $(X,\mathcal{E})$ if $f(A)\in A$ for each $A\in\mathcal{B}\backslash\{\emptyset\}$ ($A\in[X]^2$, respectively). We prove that a discrete coarse space $(X,\mathcal{E})$ admits a selector if and only if $(X,\mathcal{E})$ admits a 2-selector if and only if there exists a linear order "$\le$" on X such that the family of intervals $\{[a,b]\colon a,b\in X,\ a\le b\}$ is a base for the bornology $\mathcal{B}$.

Keywords: bornology; coarse space; selector

Classification: 54C65

1. Introduction

The notion of selectors comes from *topology*. Let X be a topological space, $\exp X$ be the set of all nonempty closed subsets of X endowed with some (initially, the Vietoris) topology, $\mathcal{F}$ be a nonempty subset of $\exp X$. A continuous mapping $f\colon\mathcal{F}\to X$ is called an $\mathcal{F}$-selector of X if $F(A)\in A$ for each $A\in\mathcal{F}$. The question on selectors of topological spaces was studied in a plenty of papers, we mention only [1], [4], [7], [6].

Formally, coarse spaces, introduced independently and simultaneously in [8] and [13], can be considered as asymptotic counterparts of uniform spaces. But actually this notion is rooted in *geometry* and *geometric group theory*, see [13, Chapter 1] and [5, Chapter 4]. At this point, we need some basic definitions.

Given a set X, a family $\mathcal{E}$ of subsets of $X\times X$ is called a *coarse structure* on X if

- ∘ each $E\in\mathcal{E}$ contains the diagonal $\triangle_X:=\{(x,x)\colon x\in X\}$ of X;
- ∘ if $E,\ E'\in\mathcal{E}$ then $E\circ E'\in\mathcal{E}$ and $E^{-1}\in\mathcal{E}$, where $E\circ E'=\{(x,y)\colon \exists\, z\ ((x,z)\in E,\ (z,y)\in E')\}$, $E^{-1}=\{(y,x)\colon(x,y)\in E\}$;
- ∘ if $E\in\mathcal{E}$ and $\triangle_X\subseteq E'\subseteq E$ then $E'\in\mathcal{E}$.

Elements $E\in\mathcal{E}$ of the coarse structure are called *entourages* on X.

DOI 10.14712/1213-7243.2022.012

For $x \in X$ and $E \in \mathcal{E}$ the set $E[x] := \{y \in X \colon (x, y) \in \mathcal{E}\}$ is called the *ball of radius E centered at x*. Since $E = \bigcup_{x \in X} \big(\{x\} \times E[x]\big)$, the entourage E is uniquely determined by the family of balls $\{E[x] \colon x \in X\}$. A subfamily $\mathcal{E}' \subseteq \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if each set $E \in \mathcal{E}$ is contained in some $E' \in \mathcal{E}'$.

The pair $(X, \mathcal{E})$ is called a *coarse space*, see [13], or a *ballean*, see [8], [11].

In this paper, all coarse spaces under consideration are supposed to be *connected*, that is, for any $x, y \in X$, there is $E \in \mathcal{E}$ such $y \in E[x]$. A subset $Y \subseteq X$ is called *bounded* if $Y = E[x]$ for some $E \in \mathcal{E}$, and $x \in X$. The family $\mathcal{B}_X$ of all bounded subsets of X is a bornology on X. We recall that a family $\mathcal{B}$ of subsets of a set X is a *bornology* if $\mathcal{B}$ contains the family $[X]^{<\omega}$ of all finite subsets of X and $\mathcal{B}$ is closed under finite unions and taking subsets. A bornology $\mathcal{B}$ on a set X is called *unbounded* if $X \notin \mathcal{B}$. A subfamily $\mathcal{B}'$ of $\mathcal{B}$ is called a base for $\mathcal{B}$ if for each $B \in \mathcal{B}$, there exists $B' \in \mathcal{B}'$ such that $B \subseteq B'$.

Each subset $Y \subseteq X$ defines a *subspace* $(Y, \mathcal{E}|_Y)$ of $(X, \mathcal{E})$, where $\mathcal{E}|_Y = \{E \cap (Y \times Y) \colon E \in \mathcal{E}\}$. A subspace $(Y, \mathcal{E}|_Y)$ is called *large* if there exists $E \in \mathcal{E}$ such that $X = E[Y]$, where $E[Y] = \bigcup_{y \in Y} E[y]$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f \colon X \to X'$ is called *macro-uniform* if for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $f(E(x)) \subseteq E'(f(x))$ for each $x \in X$. If f is a bijection such that f and f^{-1} are macro-uniform, then f is called an *asymorphism.* If $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ contain large asymorphic subspaces, then they are called *coarsely equivalent.*

For a coarse space $(X, \mathcal{E})$, we denote by $X^\flat$ the set of all nonempty bounded subsets of X, so ($X^\flat = \mathcal{B} \backslash \{\emptyset\}$) and by $\mathcal{E}^\flat$ the coarse structure on $X^\flat$ with the base $\{E^\flat \colon E \in \mathcal{E}\}$, where

$$(A, B) \in E^\flat \Leftrightarrow A \subseteq E[B], \qquad B \subseteq E[A],$$

and say that $(X^\flat, \mathcal{E}^\flat)$ is the *hyperballean* of $(X, \mathcal{E})$. For hyperballeans see [2], [3], [9], [10].

We say that a macro-uniform mapping $f \colon X^\flat \to X$ (or $f \colon [X]^2 \to X$) is a *selector* (or *2-selector*) of $(X, \mathcal{E})$ if $f(A) \in A$ for each $A \in X^\flat$ ($A \in [X]^2$, respectively). We note that a selector is a macro-uniform retraction of $X^\flat$ to $[X]^1$ identified with X.

We recall that a coarse space $(X, \mathcal{E})$ is *discrete* if, for each $E \in \mathcal{E}$, there exists a bounded subset B of $(X, \mathcal{E})$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. Every bornology $\mathcal{B}$ on a set X defines the discrete coarse space $X_{\mathcal{B}} = (X, \mathcal{E}_{\mathcal{B}})$, where $\mathcal{E}_{\mathcal{B}}$ is a coarse structure with the base $\{E_B \colon B \in \mathcal{B}\}$, $E_B[x] = B$ if $x \in B$ and $E_B[x] = \{x\}$ if $x \in X \setminus B$. On the other hand, every discrete coarse space $(X, \mathcal{E})$ coincides with $X_{\mathcal{B}}$, where $\mathcal{B}$ is the bornology of bounded subsets of $(X, \mathcal{E})$.

Our goal is to characterize discrete coarse spaces which admit selectors. After exposition of results, we conclude with some comments and open problems.

2. Results

Let "$\leq$" be a linear order on a set X. We say that $(X, \leq)$ is

- *right (left) well-ordered* if every subset Y of X has the minimal (maximal) element;
- *right (left) bounded* if X has the maximal (minimal) element;
- *bounded* if X is left and right bounded.

Every linear order "$\leq$" on X defines the bornology $\mathcal{B}_{\leq}$ on X such that the family $\{[a,b]\colon a,b \in X,\ a \leq b\}$, where $[a,b] = \{x \in X\colon a \leq x \leq b\}$, is a base for $\mathcal{B}_{\leq}$. Clearly, $X \in \mathcal{B}_{\leq}$ if and only if $(X, \leq)$ is bounded.

We say that a bornology $\mathcal{B}$ on a set X *has an interval base* if there exists a linear order "$\leq$" on X such that $\mathcal{B} = \mathcal{B}_{\leq}$.

Theorem 1. *For a bornology $\mathcal{B}$ on a set X and the discrete coarse space $X_{\mathcal{B}}$, the following statements are equivalent*

(i) *$X_{\mathcal{B}}$ admits a selector;*
(ii) *$X_{\mathcal{B}}$ admits a 2-selector;*
(iii) *$\mathcal{B}$ has an interval base.*

Proof: If $X \in \mathcal{B}$ then we have nothing to prove: every mapping $f\colon \mathcal{B}\setminus\{\emptyset\} \to X$ (or $f\colon [X]^2 \to X$) such that $f(A) \in A$ is a selector (2-selector, respectively) and we take an arbitrary linear order "$\leq$" on X such that $(X, \leq)$ is bounded. In what follows, $X \notin \mathcal{B}$ so $X_{\mathcal{B}}$ is unbounded. The implication (i) $\Rightarrow$ (ii) is evident.

(ii) $\Rightarrow$ (iii) We take a 2-selector f of $X_{\mathcal{B}}$ and define a binary relation "$\prec$" on X as follows: $a \prec b$ if and only if either $a = b$ or $f(\{a,b\}) = a$.

We use the following key observation.

($*$) *For every $B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that if $z \in X \setminus C$ then either $b \prec z$ for each $b \in B$ or $z \prec b$ for each $b \in B$.*

Indeed, we take $C \in \mathcal{B}$ such that $B \subseteq C$ and if A, $A' \in [X]^2$ and $(A, A') \in E^{\flat}_{B}$ then $(f(A), f(A')) \in E_C$.

We take and fix distinct $l, r \in X$ such that $l \prec r$ and use Zorn's lemma to choose a maximal by inclusion subset A of X such that $A = L \cup R$, $L \cap R = \emptyset$, R is right well-ordered by "$\prec$" with the minimal element r, L is left well-ordered by "$\prec$" with the maximal element l and $x \prec y$ for all $x \in L$, $y \in R$.

By the maximality of A and ($*$), A is unbounded in $X_{\mathcal{B}}$. For $a, b \in A$, $a \prec b$, we denote $[a,b]_A = \{x \in A\colon a \prec x \prec b\}$. Applying ($*$) with $B = [a,b]_A$, we see that $[a,b]_A$ is bounded in $X_{\mathcal{B}}$.

We consider three cases.

Case 1: Assume that L and R are unbounded in $X_{\mathcal{B}}$. We define some auxiliary mapping $h\colon X \to A$. For $x \in A$, we put $h(x) = x$. For $x \in X\backslash A$, we use $(*)$ with $B = \{r, x\}$ to find the minimal element $c \in R$ such that $x \prec y$ for each $y \in A$, $c \prec y$. If $c \neq r$ then we put $h(x) = c$. Otherwise, we use $(*)$ to choose the maximal element $d \in L$ such that $y \prec x$ for each $y \in L \cup \{r\}$, $y \prec d$. We put $h(x) = d$.

We take arbitrary $a, b \in A$ such that $a \prec l \prec r \prec b$. If $h(x) \in [a, b]$ then, by the construction of h, we have $a \prec x \prec b$. Applying $(*)$ with $B = [a, b]_A$, we conclude that $h^{-1}([a, b]_A)$ is bounded in $X_{\mathcal{B}}$. In particular, $h^{-1}(c)$ is bounded in $X_{\mathcal{B}}$ for each $c \in A$.

Now we are ready to define the desired linear order "$\leq$" on X. If $h(x) \prec h(y)$ and $h(x) \neq h(y)$ then we put $x < y$. If $c \in R$ then we endow $h^{-1}(c)$ with a right well-order "$\leq$". If $c \in L$ then we endow $h^{-1}(c)$ with a left well-order "$\leq$".

It remains to verify that the family $\{[a, b]\colon a, b \in X,\ a \leq b\}$ is a base for $\mathcal{B}$. Let $a, b \in A$ and $a \leq b$. We have shown that $h^{-1}([a, b]_A) \in \mathcal{B}$, hence $[a, b] \in \mathcal{B}$. If $a, b \in X$ and $a \leq b$ then we take $a' \in A$, $b' \in A$ such that $a' < a$, $b < b'$. Since $[a', b'] \in \mathcal{B}$, we have $[a, b] \in \mathcal{B}$. On the other hand, if Y is a bounded subset of $X_{\mathcal{B}}$ then we apply $(*)$ with $B = Y \cup \{l, r\}$ to find $a \in L$, $b \in R$ such that $h(B) \subseteq [a, b]_A$, hence $B \subseteq [a, b]$.

Case 2: Assume that L is bounded and R is unbounded in $X_{\mathcal{B}}$. Since $L \in \mathcal{B}$, by $(*)$, the set $C = \{x \in X\colon x < y \text{ for each } y \in R\}$ is bounded in $X_{\mathcal{B}}$. We use arguments from Case 1 to define $\leq$ on $X \setminus C$. Then we extend "$\leq$" to X so that $(C, \leq)$ is bounded and $x \prec y$ for all $x \in C$, $y \in X \setminus C$.

Case 3: Assume that L is unbounded and R is bounded in $X_{\mathcal{B}}$. Since $R \in \mathcal{B}$, by $(*)$, the set $D = \{x \in X\colon y \prec x \text{ for each } y \in L\}$ is bounded in $X_{\mathcal{B}}$. We use arguments from Case 1 to define "$\leq$" on $X \setminus D$. Then we extend "$\leq$" to X so that $(D, \leq)$ is bounded and $x \prec y$ for all $x \in X \setminus D$, $y \in D$.

(iii) $\Rightarrow$ (i) We take a linear order "$\prec$" on X witnessing that $\mathcal{B}$ has an interval base. We define a 2-selector $f\colon [X]^2 \to X$ by $f(\{x, y\}) = x$ if and only if $x \prec y$. Then we take the linear order "$\leq$" on X defined in the proof (ii) $\Rightarrow$ (iii). To define a selector s of $X_{\mathcal{B}}$, we denote $X_l = \{x \in X\colon x \leq l\}$, $X_r = \{x \in X\colon r \leq x\}$. By the construction of "$\leq$", X_l is right well-ordered and X_r is left well-ordered. We take an arbitrary $Y \in \mathcal{B} \setminus \{\emptyset\}$. If $Y \cap X_l \neq \emptyset$ then we take the maximal element $a \in Y \cap X_l$ and put $s(Y) = a$. Otherwise, we choose the minimal element $b \in Y \cap X_r$ and put $s(Y) = b$.

To see that s is macro-uniform, we take an interval $[a, b]$ in $(X, \leq)$ and $Y, Z \in \mathcal{B} \setminus \{\emptyset\}$ such that $Y \setminus [a, b] = Z \setminus [a, b]$, $Y \cap [a, b] \neq \emptyset$, $Z \cap [a, b] \neq \emptyset$. If $s(Y) \notin [a, b]$ then $s(Y) = s(Z)$. If $s(Y) \in [a, b]$ then $s(Z) \in [a, b]$. $\square$

An ordinal α endowed with the reverse ordering is called the *antiordinal* of α.

Corollary 2. *If $X_{\mathcal{B}}$ has a selector then $\mathcal{B}$ has an interval base with respect to some linear order "$\leq$" on X such that $(X, \leq)$ is the ordinal sum of an antiordinal and an ordinal.*

Proof: We take the linear order from the proof of Theorem 1 and note that X_l is an antiordinal, X_r is ordinal and $(X, \leq)$ is the ordinal sum of X_l and X_r. $\square$

Corollary 3. *If a bornology $\mathcal{B}$ on a set X has a base linearly ordered by inclusion then the discrete coarse space $X_{\mathcal{B}}$ admits a selector.*

Proof: Since $\mathcal{B}$ has a linearly ordered base, we can choose a base $\{B_\alpha\colon \alpha < \kappa\}$ well-ordered by inclusion. We show that $\mathcal{B}$ has an interval base and apply Theorem 1.

For each $\alpha < \kappa$, let $\mathcal{D}_\alpha = B_{\alpha+1} \setminus B_\alpha$. We endow each $\mathcal{D}_\alpha$ with an arbitrary right well-order "$\leq$". If $x \in \mathcal{D}_\alpha$, $y \in \mathcal{D}_\beta$ and $\alpha < \beta$, we put $x < y$. Then $\mathcal{B} = \mathcal{B}_{\leq}$. $\square$

Remark 4. Let $(X, \leq)$ be the ordinal sum of the antiordinal of ω and the ordinal ω_1. Then the interval bornology $\mathcal{B}_{\leq}$ does not have a linearly ordered base. Indeed, let $X = L \cup R$, $L = \{l_n\colon n < \omega\}$, $l_n < l_m$ if and only if $m < n$, $R = \{r_\alpha\colon \alpha < \omega_1\}$, $r_\alpha < r_\beta$ if and only if $\alpha < \beta$, and $l_n < r_\alpha$ for all n, α. Assuming that $\mathcal{B}_{\leq}$ has a linearly ordered base, we choose a base $\mathcal{B}'$ of $\mathcal{B}_{\leq}$ well-ordered by inclusion and denote $\mathcal{B}'_n = \{A \in \mathcal{B}'\colon \min A = l_n\}$. By the choice of R, there exists $m \in \omega$ such that $\mathcal{B}'_m$ is cofinal in $\mathcal{B}_{\leq}$, but $l_{m+1} \notin A$ for each $A \in \mathcal{B}'_m$ and we get a contradiction.

Theorem 5. *Let $(X, \mathcal{E})$ be a coarse space with the bornology $\mathcal{B}$ of bounded subsets. If f is a 2-selector of $(X, \mathcal{E})$ then f is a 2-selector of $X_{\mathcal{B}}$.*

Proof: Let $B \in \mathcal{B}$, $A, A' \in [X]^2$ and $(A, A') \in E^\flat_B$. Since f is a 2-selector of $(X, \mathcal{E})$, there exists $F \in \mathcal{E}$, $F = F^{-1}$ such that $(f(A), f(A')) \in F$.

If $A \cap B = \emptyset$ then $A = A'$. If $A \subseteq B$ then $A' \in B$, so $(f(A), f(A')) \in E_B$.

Let $A = \{b, a\}$, $A' = \{b', a\}$, $b \in B$, $b' \in B$ and $a \in X \setminus B$. If $a \in F[\{b, b'\}]$ then $f(A), f(A') \in F[\{b, b'\}]$. If $a \notin F[\{b, b'\}]$ then either $f(A) = f(A') = a$ or $f(A)$, $f(A') \in \{b, b'\}$.

In all considered cases, we have $(f(A), f(A')) \in E_{F[B]}$. Hence, f is a 2-selector of $X_{\mathcal{B}}$. $\square$

Remark 6. Every metric space (X, d) has the natural coarse structure $\mathcal{E}_d$ with the base $\{E_r\colon r > 0\}$, $E_r = \{(x, y)\colon d(x, y) \leq r\}$. Let $\mathcal{B}$ denote the bornology of bounded subsets of $(X, \mathcal{E}_d)$. By Corollary 3, the discrete coarse space $X_{\mathcal{B}}$ admits a 2-selector. We show that $(X, \mathcal{E}_d)$ could not admit a 2-selector, so the conversion of Theorem 5 does not hold.

Let $X = \mathbb{Z}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. We suppose that there exists a 2-selector f of $(X, \mathcal{E}_d)$ and choose a natural number n such that if $A, A' \in [X]^2$ and $(A, A') \in E_1^\flat$ then $(f(A), f(A')) \in E_n$, so $d(f(A), f(A')) \le n$. We denote $S_n = \{x \in X \colon d(x, 0) = n\}$. For $x \in S_n$, let $A_x = \{x, -x\}$. Then we can choose $x, y \in S_n$ such that $d(x, y) = 1$, $f(A_x) = x$, $f(A_y) = -y$, but $d(x, -y) > n$.

3. Comments

1. Let $(X, \mathcal{U})$ be a uniform space and let $\mathcal{F}_X$ denote the set of all nonempty closed subsets of X endowed with the Hausdorff–Bourbaki uniformity. Given a subset $\mathcal{F}$ of $\mathcal{F}_X$, a uniformly continuous mapping $f \colon \mathcal{F} \to X$ is called an $\mathcal{F}$-selector if $f(A) \in A$ for each $A \in \mathcal{F}$. If $\mathcal{F} = [X]^2$ then f is called a 2-selector.

In contrast to the topological case, the problem of uniform selections is much less studied. Almost all known results are concentrated around uniformizations of Michael's theorem, for references see [12].

Given a discrete uniform space, how can one detects whether X admits a 2-selector? This question seems very difficult even in the case of a countable discrete metric space X. To demonstrate the obstacles for a simple characterization, we consider the following example.

We take a family $\{C_n \colon n < \omega\}$ of pairwise nonintersecting circles of radius 1 on the Euclidean plane $\mathbb{R}^2$. Then we inscribe a regular n-gon M_n in C_n and denote by X the set of all vertices of $\{M_n \colon n < \omega\}$. It is easy to verify that X does not admit a 2-selector.

2. Given a group G with the identity e, we denote by $\mathcal{E}_G$ a coarse structure on G with the base

$$\{\{(x, y) \in G \times G \colon y \in Fx\} \colon \ F \in [G]^{<\omega}, \ e \in F\}$$

and say that $(G, \mathcal{E}_G)$ is the *finitary coarse space* of G. It should be noticed that finitary coarse spaces of groups (in the form of Cayley graphs) are used in *geometric group theory*, see [5]. We note that the bornology of bounded subsets of $(G, \mathcal{E}_G)$ is the set $[G]^{<\omega}$. Applying Theorem 1 and Theorem 5, we conclude that if $(G, \mathcal{E}_G)$ admits a 2-selector then G must be countable.

Problem 1. Characterize countable groups G such that the finitary coarse space $(G, \mathcal{E}_G)$ admits a 2-selector.

3. Every connected graph $\Gamma[\mathcal{V}]$ with the set of vertices $\mathcal{V}$ can be considered as the metric space $(\mathcal{V}, d)$,

Problem 2. Characterize graphs $\Gamma[\mathcal{V}]$ such that the coarse space of $(\mathcal{V}, d)$, wherever d is the path metric on the set of vertices $\mathcal{V}$, admits a 2-selector.

Acknowledgment. I thank the referee for critical remarks to the initial version of the paper.

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(*Received* January 4, 2021, *revised* February 17, 2021)