Lahiri Indrajit; Sujoy Majumder

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A POWER OF A MEROMORPHIC FUNCTION SHARING TWO SMALL FUNCTIONS WITH A DERIVATIVE OF THE POWER

Lahiri Indrajit, Kalyani, Sujoy Majumder, Raiganj

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Abstract. In connection to a conjecture of W. Lü, Q. Li and C. Yang (2014), we prove a result on small function sharing by a power of a meromorphic function with few poles with a derivative of the power. Our results improve a number of known results.

Keywords: meromorphic function; derivative; small function

MSC 2020: 30D35

1. Introduction definitions and results

In this paper a meromorphic function means a function that is meromorphic in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna theory; e.g., $N(r,f)$, $m(r,f)$, $T(r,f)$, $N(r,a;f)$, $\overline{N}(r,a;f)$, $m(r,a;f)$ etc., see [7]. We denote by $S(r,f)$ a quantity, not necessarily the same at each of its occurrences, that satisfies the condition $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ except possibly a set of finite linear measure.

A meromorphic function $a = a(z)$ is called a small function of a meromorphic function f if $T(r,a) = S(r,f)$. Let us denote by $S(f)$ the class of all small functions of f. Clearly $\mathbb{C} \subset S(f)$ and if f is a transcendental function then every polynomial is a member of $S(f)$.

Let f and g be two non-constant meromorphic functions and $a \in S(f) \cap S(g)$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the small function a CM (counting multiplicities) and if we do not consider the multiplicities, then we say that f and g share the small function a IM (ignoring multiplicities).

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Let k be a positive integer and $a \in S(f)$. We use $N_{k)}(r,a;f)$ to denote the counting function of zeros of $f-a$ with multiplicity not greater than k, $N_{(k+1}(r,a;f)$ to denote the counting function of zeros of $f-a$ with multiplicity greater than k. Similarly we use $\overline{N}_{k)}(r,a;f)$ and $\overline{N}_{(k+1}(r,a;f)$ to denote their respective reduced functions.

In 1996, Brück studied the relation between f and f' if an entire function f shares only one finite value CM with its derivative f' (see [1]). In this direction an interesting conjecture was proposed by Brück (see [1]), which is still open in its full generality.

Conjecture A. *Let f be a non-constant entire function. Suppose*

$$\varrho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

the hyper-order of f, is not a positive integer or infinity. If f and f' share a finite value a CM, then

$$\frac{f'-a}{f-a} = c \tag{1.1}$$

for some nonzero constant c.

The conjecture for the special cases
(1) $a = 0$,
(2) $N(r,0;f') = S(r,f)$
had been established by Brück, see [1]. From the differential equations

$$\frac{f'-a}{f-a} = \mathrm{e}^{z^n}, \quad \frac{f'-a}{f-a} = \mathrm{e}^{\mathrm{e}^z}, \tag{1.2}$$

we see that when $\varrho_1(f)$ is a positive integer or infinity the conjecture does not hold.

The conjecture for the case where f is of finite order had been proved by Gundersen and Yang (see [6]), and the case where f is of infinite order with $\varrho_1(f) < \frac{1}{2}$ had been proved by Chen and Shon (see [3]). Recently Cao in [2] proved that the Brück conjecture is also true when f is of infinite order with $\varrho_1(f) = \frac{1}{2}$. But the case $\varrho_1(f) > \frac{1}{2}$ is still open. However, the corresponding conjecture for meromorphic functions fails in general (see [6]). For example, if

$$f(z) = \frac{2\mathrm{e}^z + z + 1}{\mathrm{e}^z + 1},$$

then f and f' share 1 CM, but (1.1) does not hold.

It is interesting to ask what happens if f is replaced by a power of itself, say, f^n in Brück's conjecture. From (1.2) we see that the conjecture does not hold without any restriction on the hyper-order when $n = 1$. So we only need to focus on the problem when $n \geqslant 2$.

Perhaps Yang and Zhang (see [15]) were the first to consider the uniqueness of a power of an entire function $F = f^n$ and its derivative F' when they share a certain value and that leads to a specific form of the function f.

Yang and Zhang in [15] proved that the Brück conjecture holds for the function f^n and the order restriction on f is not needed if n is relatively large. In fact, they proved the following result.

Theorem A ([15]). *Let f be a non-constant entire function, $n\,(\geqslant 7)$ be an integer and let $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$, and f assumes the form $f(z) = ce^{z/n}$, where c is a nonzero constant.*

Improving all the results obtained in [15], Zhang in [16] proved the following theorem.

Theorem B ([16]). *Let f be a non-constant entire function, and let n, k be positive integers and $a\,(\not\equiv 0, \infty)$ be a meromorphic small function of f. If $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM and $n \geqslant k + 5$, then $f^n \equiv (f^n)^{(k)}$, and f assumes the form $f(z) = ce^{z\lambda/n}$, where c is a nonzero constant and $\lambda^k = 1$.*

In 2009, Zhang and Yang (see [17]) further improved the above result in the following manner.

Theorem C ([17]). *Let f be a non-constant entire function, and let n, k be positive integers and $a\,(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM and $n \geqslant k + 2$. Then conclusion of Theorem* B *holds.*

In 2010, Zhang and Yang (see [18]) further improved the above result in the following manner.

Theorem D ([18]). *Let f be a non-constant entire function, and let n, k be positive integers. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geqslant k+1$. Then conclusion of Theorem* B *holds.*

In 2011, Lü and Yi (see [11]) proved the following extension of Theorem D.

Theorem E ([11]). *Let f be a transcendental entire function, and let n, k be two integers with $n \geqslant k + 1$, and let $F = f^n$ and $Q \not\equiv 0$ be polynomials. If $F - Q$ and $F^{(k)} - Q$ share 0 CM, then $F \equiv F^{(k)}$ and $f(z) = ce^{wz/n}$, where c and w are nonzero constants such that $w^k = 1$.*

R e m a r k 1.1. It is easy to see that the condition $n \geqslant k + 1$ in Theorem E is sharp by the following example.

E x a m p l e 1.1. Let $f(z) = \mathrm{e}^{\mathrm{e}^z} \int_0^z \mathrm{e}^{-\mathrm{e}^t}(1-\mathrm{e}^t)t\,\mathrm{d}t$ and $n = 1$, $k = 1$. Then

$$\frac{f'(z)-z}{f(z)-z} = \mathrm{e}^z$$

and $f'(z) - z$ and $f(z) - z$ share 0 CM, but $f' \not\equiv f$.

In [12], Lü, Li and Yang asked the question of considering two shared polynomials in Theorem E instead of a single shared polynomial. They answered the question for the first derivative of the power of a transcendental entire function and further proposed the following conjecture:

Conjecture B. *Let f be a transcendental entire function, and let n be a positive integer. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geqslant k+1$, then $(f^n)^{(k)} = f^n Q_2/Q_1$, where Q_1 and Q_2 are polynomials with $Q_1 Q_2 \not\equiv 0$. If, further, $Q_1 \equiv Q_2$, then $f = ce^{\omega z/n}$, where c and ω are nonzero constants such that $\omega^k = 1$.*

Recently the second author (see [13]) fully resolved Conjecture B; thus giving rise to a further investigation of the possibility of replacing in Conjecture B the shared polynomials by shared small functions. Here we on one hand solve this problem and also on the other hand we try to relax the nature of sharing of small functions, thereby improving a number of known results including that in [13].

Extending the idea of weighted sharing (see [8], [9]), Lin and Lin in [10] introduced the notion of weakly weighted sharing which is defined as follows.

Definition 1.1 ([10])**.** Let f and g be two non-constant meromorphic functions sharing a "IM", for $a \in S(f) \cap S(g)$, and let k be a positive integer or ∞.

(i) $\overline{N}^E_{k)}(r,a)$ denotes the counting function of those zeros of $f-a$ whose multiplicities are equal to the corresponding zeros of $g-a$, where both of their multiplicities are not greater than k, and where each zero is counted only once.

(ii) $\overline{N}^0_{(k}(r,a)$ denotes the counting function of those zeros of $f-a$ which are zeros of $g-a$, where both of their multiplicities are not less than k, and where each zero is counted only once.

Definition 1.2 ([10])**.** For $a \in S(f) \cap S(g)$, if k is a positive integer or ∞ and

$$\overline{N}_{k)}(r,a;f) - \overline{N}^E_{k)}(r,a) = S(r,f), \quad \overline{N}_{k)}(r,a;g) - \overline{N}^E_{k)}(r,a) = S(r,g);$$
$$\overline{N}_{(k+1}(r,a;f) - \overline{N}^0_{(k+1}(r,a) = S(r,f), \quad \overline{N}_{(k+1}(r,a;g) - \overline{N}^0_{(k+1}(r,a) = S(r,g);$$

or if $k = 0$ and

$$\overline{N}(r,a;f) - \overline{N}_0(r,a) = S(r,f), \quad \overline{N}(r,a;g) - \overline{N}_0(r,a) = S(r,g),$$

then we say f and g weakly share a with weight k. Here we write f, g share "(a,k)" to mean that f, g weakly share a with weight k.

Obviously, if f and g share "(a,k)", then f and g share "(a,p)" for any p, $0 \leqslant p \leqslant k$. Also we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)", respectively (for the definitions of "IM" and "CM" see pages 225–226 in [14]).

We note that a rational function f with $\overline{N}(r,\infty;f) = S(r,f)$ must be a polynomial. Also a small function of a polynomial must be a constant. Since $k \geqslant 1$, clearly if f is a polynomial, then the relation $(f^n)^{(k)} = cf^n$ does not hold for any nonzero constant c and $n \geqslant k$. Therefore in the following theorems we assume f to be transcendental.

Theorem 1.1. *Let f be a transcendental meromorphic function such that $N(r,\infty;f) = S(r,f)$ and let $a_i = a_i(z)\,(\not\equiv 0,\infty)$ be small functions of f, where $i = 1,2$. Let n and k be two positive integers such that $n \geqslant k+1$. If $f^n - a_1$ and $(f^n)^{(k)} - a_2$ share "$(0,1)$", then $(f^n)^{(k)} \equiv f^n a_2 a_1^{-1}$. Furthermore, if $a_1 \equiv a_2$, then $f(z) = c\mathrm{e}^{z\lambda/n}$, where c and λ are nonzero constants such that $\lambda^k = 1$.*

Theorem 1.2. *Let f be a transcendental meromorphic function such that $\overline{N}(r,\infty;f) = S(r,f)$ and let $a_i = a_i(z)\,(\not\equiv 0,\infty)$ be small functions of f, where $i = 1,2$. Let n and k be two positive integers such that $n \geqslant k$. If $f^n - a_1$ and $(f^n)^{(k)} - a_2$ share "$(0,0)$" and $\overline{N}_{2)}(r,0;f) = S(r,f)$, then $(f^n)^{(k)} \equiv a_2 a_1^{-1} f^n$. Furthermore, if $a_1 \equiv a_2$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form $f(z) = c\mathrm{e}^{z\lambda/n}$, where c is a nonzero constant and $\lambda^k = 1$.*

N o t e 1.1. If $k \geqslant 2$, then in Theorem 1.2 instead of $\overline{N}_{2)}(r,0;f) = S(r,f)$ we can assume $N_{1)}(r,0;f) = S(r,f)$.

R e m a r k 1.2. It is easy to see that the condition $n \geqslant k+1$ in Theorem 1.1 is sharp by the following examples.

E x a m p l e 1.2. Let $f(z) = \mathrm{e}^{2z} + z$. Then $f - a_1$ and $f' - a_2$ share 0 CM and $N(r,\infty;f) = 0$, but $f' \not\equiv a_2 a_1^{-1} f$, where $a_1(z) = z+1$ and $a_2(z) = 3$.

E x a m p l e 1.3. Let $f(z) = \mathrm{e}^{2z} + z^2 + z$. Then $f - a_1$ and $f' - a_2$ share 0 CM and $N(r,\infty;f) = 0$, but $f' \not\equiv a_2 a_1^{-1} f$, where $a_1(z) = z^2 + z + 1$ and $a_2(z) = 2z + 3$.

E x a m p l e 1.4. Let

$$f(z) = \mathrm{e}^{\mathrm{e}^{z^2}} + 1, \quad a_1(z) = \frac{1}{1+\mathrm{e}^{-z^2}}, \quad a_2(z) = -\frac{2z}{1+\mathrm{e}^{-z^2}}.$$

We note that

$$f(z) - a_1(z) = \frac{1}{\mathrm{e}^{z^2}+1}\big((\mathrm{e}^{z^2}+1)\mathrm{e}^{\mathrm{e}^{z^2}} + 1\big)$$

and

$$f'(z) - a_2(z) = \frac{2z}{1+\mathrm{e}^{-z^2}}\big((\mathrm{e}^{z^2}+1)\mathrm{e}^{\mathrm{e}^{z^2}} + 1\big).$$

Then $f - a_1$ and $f' - a_2$ share "$(0,\infty)$" and $N(r,\infty;f) = 0$, but $f \not\equiv a_2 a_1^{-1} f'$.

Example 1.5. Let

$$f(z) = 1 - 5(z+1) + z\mathrm{e}^z$$

and $a_1(z) = a_2(z) = -(4 + 4z + 5z^2)$. We note that

$$f(z) - a_1(z) = z(\mathrm{e}^z + 5z - 1)$$

and

$$f'(z) - a_2(z) = (z+1)(\mathrm{e}^z + 5z - 1).$$

Then $f - a_1$ and $f' - a_2$ share "$(0, \infty)$" and $N(r, \infty; f) = 0$, but $f \not\equiv f'$.

Remark 1.3. It is easy to see that the conditions $\overline{N}_{2)}(r, 0; f) = S(r, f)$ and $\overline{N}(r, \infty; f) = S(r, f)$ in Theorem 1.2 are essential by the following examples.

Example 1.6. Let

$$f(z) = z^2 + \frac{1}{2}\mathrm{e}^{(z-1)^2}, \quad a_1(z) = z^2 + \frac{1}{2} \quad \text{and} \quad a_2(z) = 3z - 1.$$

We note that

$$f(z) - \left(z^2 + \frac{1}{2}\right) = \frac{1}{2}(\mathrm{e}^{(z-1)^2} - 1)$$

and

$$f'(z) - (3z - 1) = (z-1)(\mathrm{e}^{(z-1)^2} - 1).$$

Obviously $f - a_1$ and $f' - a_2$ share 0 IM, and $\overline{N}_{2)}(r, 0; f) \neq S(r, f)$ and $\overline{N}(r, \infty; f) = 0$, but $f' \not\equiv a_2 a_1^{-1} f$.

Example 1.7. Let

$$f(z) = \frac{2}{1 - \mathrm{e}^{-2z}}.$$

Clearly $f'(z) = -4\mathrm{e}^{-2z}(1 - \mathrm{e}^{-2z})^{-2}$. We note that

$$f(z) - 1 = \frac{1 + \mathrm{e}^{-2z}}{1 - \mathrm{e}^{-2z}} \quad \text{and} \quad f'(z) - 1 = -\frac{(1 + \mathrm{e}^{-2z})^2}{(1 - \mathrm{e}^{-2z})^2}.$$

Obviously f and f' share 1 IM, $\overline{N}(r, \infty; f) \neq S(r, f)$ and $\overline{N}_{2)}(r, 0; f) = 0$, but $f' \not\equiv f$.

Example 1.8. Let $f(z) = 1 + \tan z$. Since $\tan z$ does not assume the values $\pm\mathrm{i}$, it follows that $f(z)$ does not assume the values $1 \pm \mathrm{i}$. So by the second fundamental theorem, $\overline{N}(r, 0; f) = \overline{N}_{2)}(r, 0; f) = T(r, f) + S(r, f)$ and $\overline{N}(r, \infty; f) = T(r, f) + S(r, f)$. Also we see that $f'(z) - 1 = (f(z) - 1)^2$ and so f and f' share the value 1 IM, but $f \not\equiv f'$.

2. Lemmas

In this section we present the lemmas which will be needed in the sequel.

Lemma 2.1 ([4]). *Suppose that f is a transcendental meromorphic function and that*

$$f^n(z)P(f(z)) = Q(f(z)),$$

where $P(f(z))$ and $Q(f(z))$ are differential polynomials in f with functions of small proximity related to f as the coefficients and suppose that the degree of $Q(f(z))$ is at most n. Then $m(r, P) = S(r, f)$.

Lemma 2.2 ([7]). *Let f be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leqslant \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 2.3 ([5]). *Let $f(z)$ be a non-constant entire function and $k\,(\geqslant 2)$ be an integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = \mathrm{e}^{az+b}$, where $a\,(\neq 0)$, b are constants.*

Lemma 2.4. *Let f be a non-constant meromorphic function such that*

$$(f^n)^{(k)} \equiv f^n, \tag{2.1}$$

where $k, n \in \mathbb{N}$. If $n \geqslant k$, then f assumes the form $f(z) = c\mathrm{e}^{z\lambda/n}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

Proof. First we suppose (2.1) holds. We claim that f does not have any pole. In fact, if z_0 is a pole of f with multiplicity p, then z_0 is a pole of f^n with multiplicity np and a pole of $(f^n)^{(k)}$ with multiplicity $np + k$, which is impossible by (2.1). Hence f is a non-constant entire function. From (2.1), it is clear that f cannot be a polynomial. Therefore f is a transcendental entire function. We now consider the following two cases.

Case 1. Let $n > k$. If z_1 is a zero of f with multiplicity q, then z_1 is a zero of f^n with multiplicity nq and a zero of $(f^n)^{(k)}$ with multiplicity $nq - k$, which is impossible by (2.1). Therefore from (2.1), we conclude that $f^n(z)(f^n(z))^{(k)} \neq 0$. If $k \geqslant 2$, then by Lemma 2.3 we have $f(z) = c\mathrm{e}^{z\lambda/n}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$.

Next we suppose $k = 1$. Since $f(z) \neq 0, \infty$, it follows that $f(z) = \mathrm{e}^{\alpha(z)}$, where $\alpha(z)$ is a non-constant entire function. Now from (2.1) we have $\alpha'(z) = n^{-1}$, i.e., $\alpha(z) = zn^{-1} + c_0$, where $c_0 \in \mathbb{C}$. Consequently $f(z) = c\mathrm{e}^{z/n}$, where $c = \mathrm{e}^{c_0}$.

Case 2. Let $n = k$. First we suppose $n = k = 1$. Then from (2.1) we have $f(z) \equiv f'(z)$ and so $f(z) = ce^z$, where $c \in \mathbb{C} \setminus \{0\}$.

Next we suppose $n = k \geqslant 2$. Let $F = f^n$. Then we have

$$
\begin{aligned}
(2.2)\qquad F^{(k)} &= \frac{\mathrm{d}^k}{\mathrm{d}z^k}(f^k) = \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}(kf^{k-1}f') = k\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}((k-1)f^{k-2}(f')^2 + f^{k-1}f'') \\
&= k(k-1)\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}(f^{k-2}(f')^2) + k\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}(f^{k-1}f'') \\
&= k(k-1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}((k-2)f^{k-3}(f')^3) + k(k-1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(2f^{k-2}f'f'') \\
&\quad + k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}((k-1)f^{k-2}f'f'') + k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-1}f'') \\
&= k(k-1)(k-2)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-3}(f')^3) + 2k(k-1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-2}f'f'') \\
&\quad + k(k-1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-2}f'f'') + k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-1}f'') \\
&= \ldots = k!(f')^k + R(f),
\end{aligned}
$$

where $R(f)$ is a differential polynomial in f such that each term of $R(f)$ contains f^m for some m $(1 \leqslant m \leqslant n-1)$ as a factor.

From (2.1), we observe that f cannot have any multiple zero. Let z_2 be a simple zero of f. Clearly z_2 is a zero of F of multiplicity k. From (2.1), it is clear that z_2 is also a zero of $F^{(k)}$. On the other hand z_2 is a zero of $R(f)$. Now from (2.2), we observe that z_2 is a zero of f', which is impossible. Therefore f cannot have any simple zero. Hence f does not have any zero. Since from (2.1) we see that $(f^n(z))^{(k)} f^n(z) \neq 0$, by Lemma 2.3 we have $f(z) = ce^{z\lambda/n}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$. This completes the proof. □

3. Proofs of the theorems

P r o o f of Theorem 1.1. Let

$$
(3.1)\qquad F = f^n.
$$

Since $S(r, f^n) = S(r, f)$, from Lemma 2.2 we see that

$$
nT(r, f) \leqslant \overline{N}(r, 0; F) + \overline{N}(r, a_1; F) + S(r, f^n) = \overline{N}(r, 0; f) + \overline{N}(r, a_1; F) + S(r, f).
$$

Since $n \geqslant k+1$, it follows that $\overline{N}(r, a_1; F) \neq S(r, f)$. As $F - a_1$ and $F^{(k)} - a_2$ share "$(0, 1)$", it follows that $\overline{N}(r, a_2; F^{(k)}) \neq S(r, f)$.

Let z_0 be a common zero of $F-a_1$ and $F^{(k)}-a_2$ such that $a_i(z_0)\neq 0,\infty$ (otherwise the reduced counting functions of those zeros of $F-a_1$ and $F^{(k)}-a_2$ which are the zeros or poles of $a_1(z)$ and $a_2(z)$, respectively, are equal to $S(r,f)$), where $i=1,2$. Clearly $F(z_0), F^{(k)}(z_0)\neq 0$. Suppose z_0 is a zero of $F-a_1$ of multiplicity p_0. Since $F-a_1$ and $F^{(k)}-a_2$ share "$(0,1)$", it follows that z_0 must be a zero of $F^{(k)}-a_2$ of multiplicity q_0. Then in some neighbourhood of z_0, we get by Taylor's expansion

$$F(z)=a_{10}+a_{1r_0}(z-z_0)^{r_0}+a_{1r_0+1}(z-z_0)^{r_0+1}+\ldots,\quad a_{10}\neq 0,$$
$$a_1(z)=b_{10}+b_{1s_0}(z-z_0)^{s_0}+b_{1s_0+1}(z-z_0)^{s_0+1}+\ldots,\quad b_{10}\neq 0.$$

Since z_0 is a zero of $F-a_1$ of multiplicity p_0, it follows that $a_{10}=b_{10}$ and $p_0\geqslant \min\{r_0,s_0\}$, when $p_0=r_0=s_0$. Let us assume that

$$F(z)-a_1(z)=c_{1p_0}(z-z_0)^{p_0}+c_{1p_0+1}(z-z_0)^{p_0+1}+\ldots,\quad c_{1p_0}\neq 0.$$

Therefore

$$\frac{F(z)-a_1(z)}{a_1(z)}=O((z-z_0)^{p_0})\quad\text{and so}\quad \frac{F(z)}{a_1(z)}-1=O((z-z_0)^{p_0}).$$

Similarly

$$\frac{(F^{(k)}(z)-a_2(z))}{a_2(z)}=O((z-z_0)^{q_0})\quad\text{and}\quad \frac{F^{(k)}(z)}{a_2(z)}-1=O((z-z_0)^{q_0}).$$

Finally we conclude that $F-a_1$ and $F^{(k)}-a_2$ share "$(0,1)$" if and only if Fa_1^{-1} and $F^{(k)}a_2^{-1}$ share "$(1,1)$" except for the zeros and poles of $a_1(z)$ and $a_2(z)$, respectively.

Let $F_1=f^na_1^{-1}$ and $G_1=(f^n)^{(k)}a_2^{-1}$. Clearly F_1 and G_1 share "$(1,1)$" except for the zeros and poles of $a_1(z)$ and $a_2(z)$, respectively, and so $\overline{N}(r,1;F_1)=\overline{N}(r,1;G_1)+S(r,f)$. Let

$$\Phi=\frac{F_1'(F_1-G_1)}{F_1(F_1-1)}=\frac{F_1'}{F_1-1}\Big(1-\frac{G_1}{F_1}\Big)=\frac{F_1'}{F_1-1}\Big(1-\frac{a_1}{a_2}\cdot\frac{F^{(k)}}{F}\Big).\tag{3.2}$$

We now consider the following two cases.

Case 1. Let $\Phi\not\equiv 0$. Then clearly $G_1\not\equiv F_1$, i.e., $(f^n)^{(k)}\not\equiv a_2a_1^{-1}f^n$. Now from (3.2) we get $m(r,\infty;\Phi)=S(r,f)$.

Let z_1 be a zero of f of multiplicity p such that $a_i(z_1)\neq 0,\infty$, where $i=1,2$. Then z_1 will be a zero of F_1 and G_1 of multiplicities np and $np-k$, respectively, and so from (3.2) we get

$$\Phi(z)=O((z-z_1)^{np-k-1}).\tag{3.3}$$

Since $n\geqslant k+1$, it follows that Φ is holomorphic at z_1.

Let z_2 be a common zero of $F_1 - 1$ and $G_1 - 1$ such that $a_i(z_2) \neq 0, \infty$, where $i = 1, 2$. Suppose z_2 is a zero of $F_1 - 1$ of multiplicity q. Since F_1 and G_1 share "$(1, 1)$" except for the zeros and poles of $a_1(z)$ and $a_2(z)$, respectively, it follows that z_2 must be a zero of $G_1 - 1$ of multiplicity r. Then in some neighbourhood of z_2, we get by Taylor's expansion

$$F_1(z) - 1 = b_q(z - z_2)^q + b_{q+1}(z - z_2)^{q+1} + \ldots, \quad b_q \neq 0,$$
$$G_1(z) - 1 = c_r(z - z_2)^r + c_{r+1}(z - z_2)^{r+1} + \ldots, \quad c_r \neq 0.$$

Clearly

$$F_1'(z) = qb_q(z - z_2)^{q-1} + (q + 1)b_{q+1}(z - z_2)^q + \ldots$$

Note that

$$F_1(z) - G_1(z) = \begin{cases} b_q(z - z_2)^q + \ldots & \text{if } q < r, \\ -c_r(z - z_2)^r - \ldots & \text{if } q > r, \\ (b_q - c_q)(z - z_2)^q + \ldots & \text{if } q = r. \end{cases}$$

Clearly from (3.2) we get

$$\Phi(z) = O((z - z_2)^{t-1}), \tag{3.4}$$

where $t \geqslant \min\{q, r\}$. Now from (3.4), it follows that Φ is holomorphic at z_2.

We note from (3.2) that if z_* is a zero of $F_1 - 1$ that is also a zero of a_2 with multiplicity p_1, then z_* is a possible pole of Φ with multiplicity at most $1 + p_1$. Again if z^* is a zero of f that is also a zero of a_2 with multiplicity p_2, then z^* is a possible pole of Φ with multiplicity at most $k + p_2$. So from (3.2), the above discussion and the hypothesis of Theorem 1.1 we note that

$$\begin{aligned} N(r, \infty; \Phi) &\leqslant (k + 1)N\Big(r, \frac{a_1}{a_2}\Big) + (k + 1)N(r, 0; a_1) + (k + 1)N(r, 0; a_2) \\ &\quad + (k + 1)\overline{N}(r, F_1) + (k + 1)\overline{N}(r, f) \\ &= (k + 1)\overline{N}(r, F_1) + S(r, f) = S(r, f). \end{aligned}$$

Consequently $T(r, \Phi) = S(r, f)$.

Let $q \geqslant 2$. Since F_1 and G_1 share "$(1, 1)$" except for the zeros and poles of $a_1(z)$ and $a_2(z)$, it follows that $r \geqslant 2$. Therefore from (3.4) we see that

$$\overline{N}_{(2}(r, 1; F_1) \leqslant N(r, 0; \Phi) + S(r, f) \leqslant T(r, \Phi) + S(r, f) = S(r, f).$$

Since F_1 and G_1 share "$(1, 1)$" except for the zeros and poles of $a_1(z)$ and $a_2(z)$, it follows that $\overline{N}_{(2}(r, 1; G_1) = S(r, f)$. Again from (3.2) we get

$$\frac{1}{F_1} = \frac{1}{\Phi}\Big(\frac{F_1'}{F_1 - 1} - \frac{F_1'}{F_1}\Big)\Big(1 - \frac{a_1}{a_2}\frac{(f^n)^{(k)}}{f^n}\Big)$$

and so $m(r, 1/F_1) = S(r, f)$. Hence

$$m(r, 0; f) = m\Big(r, \frac{1}{f}\Big) = S(r, f). \tag{3.5}$$

We consider the following two sub-cases.

Sub-case 1.1. Let $n > k + 1$. From (3.3) we see that $N(r, 0; f) \leqslant N(r, 0; \Phi) \leqslant T(r, \Phi) + O(1) = S(r, f)$. Then from (3.5) we get $T(r, f) = S(r, f)$, which is a contradiction.

Sub-case 1.2. Let $n = k+1$. Since for $p \geqslant 2$, we have $np-k-1 = (k+1)p-k-1 \geqslant p$, from (3.3) we see that

$$N_{(2}(r, 0; f) \leqslant N(r, 0; \Phi) \leqslant T(r, \Phi) + O(1) = S(r, f).$$

Then (3.5) gives

$$T(r, f) = N_{1)}(r, 0; f) + S(r, f). \tag{3.6}$$

Note that $\overline{N}_{(2}(r, a_1; F) = \overline{N}_{(2}(r, 1; F_1) + S(r, f) = S(r, f)$, $\overline{N}_{(2}(r, a_2; F^{(k)}) = \overline{N}_{(2}(r, 1; G_1) + S(r, f) = S(r, f)$ and $\overline{N}(r, \infty; F) = S(r, f)$. Let

$$\beta = \frac{F^{(k)} - a_2}{F - a_1}, \quad \text{i.e., } F^{(k)} - a_2 = \beta(F - a_1). \tag{3.7}$$

We claim that $\beta \not\equiv 0$. If not, suppose $\beta \equiv 0$. Then from (3.7) we have $(f^n)^{(k)} \equiv a_2$. Since $n = k + 1$, we immediately have $N_{1)}(r, 0, f) = S(r, f)$ and so from (3.6) we arrive at a contradiction. Hence $\beta \not\equiv 0$. We now consider following two sub-cases.

Sub-case 1.2.1. Suppose $T(r, \beta) \neq S(r, f)$. Let z_{11} be a zero of $F - a_1$ such that $F^{(k)}(z_{11}) - a_2(z_{11}) \neq 0$. Then obviously β has a pole at z_{11}. Let z_{12} be a zero of $F^{(k)} - a_2$ such that $F(z_{12}) - a_1(z_{12}) \neq 0$. In that case β has a zero at z_{12}. Let z_{13} be a common zero of $F - a_1$ and $F^{(k)} - a_2$. Since $F - a_1$ and $F^{(k)} - a_2$ share "$(0, 1)$", it follows that β has a zero at z_{13} if z_{13} is a zero of $F - a_1$ and $F^{(k)} - a_2$ with multiplicities $p_{13}\,(\geqslant 2)$ and $q_{13}\,(\geqslant 2)$, respectively, such that $p_{13} < q_{13}$ and β has a pole at z_{13} if $q_{13} < p_{13}$. Therefore

$$\overline{N}(r, 0; \beta) \leqslant \overline{N}_{(2}(r, a_2; F^{(k)}) + S(r, f) = S(r, f)$$

and

$$\overline{N}(r, \infty; \beta) \leqslant \overline{N}_{(2}(r, a_1; F) + S(r, f) = S(r, f).$$

Let $\xi = \beta'/\beta$. Clearly

$$\begin{aligned} T(r, \xi) &= N\Big(r, \infty; \frac{\beta'}{\beta}\Big) + m\Big(r, \frac{\beta'}{\beta}\Big) \\ &= \overline{N}(r, 0; \beta) + \overline{N}(r, \infty; \beta) + S(r, \beta) = S(r, f) + S(r, \beta). \end{aligned}$$

Note that

$$T(r,\beta) \leqslant T(r,F^{(k)}-a_2)+T(r,F-a_1) \leqslant T(r,F^{(k)})+T(r,F)+S(r,F)+S(r,G)$$
$$\leqslant (k+1)T(r,f^n)+nT(r,f)+S(r,f) = n(k+2)T(r,f)+S(r,f),$$

which implies that $S(r,\beta)$ can be replaced by $S(r,f)$. Consequently $T(r,\xi)=S(r,f)$. By logarithmic differentiation we get from (3.7)

$$F^{(k+1)}F-\xi F^{(k)}F-F^{(k)}F' = a_1F^{(k+1)}-(\xi a_1+a_1')F^{(k)}-a_2F' + (a_2'-\xi a_2)F+\xi a_1a_2+a_2a_1'-a_1a_2'. \tag{3.8}$$

We deduce from (3.1) that

(3.9)

$$\begin{aligned}
F^{(k)} &= \frac{\mathrm{d}^k}{\mathrm{d}z^k}(f^{k+1}) = \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}((k+1)f^kf') = (k+1)\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}(kf^{k-1}(f')^2+f^kf'')\\
&= (k+1)k\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}(f^{k-1}(f')^2)+(k+1)\frac{\mathrm{d}^{k-2}}{\mathrm{d}z^{k-2}}(f^kf'')\\
&= (k+1)k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}((k-1)f^{k-2}(f')^3)+(k+1)k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(2f^{k-1}f'f'')\\
&\quad+(k+1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(kf^{k-1}f'f'')+(k+1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^kf''')\\
&= (k+1)k(k-1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-2}(f')^3)+2(k+1)k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-1}f'f'')\\
&\quad+(k+1)k\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^{k-1}f'f'')+(k+1)\frac{\mathrm{d}^{k-3}}{\mathrm{d}z^{k-3}}(f^kf''')\\
&= \ldots = (k+1)!f(f')^k+\frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f''+\ldots+(k+1)f^kf^{(k)}.
\end{aligned}$$

Therefore

$$\frac{f'}{f}F^{(k)} = (k+1)!(f')^{k+1}+\frac{k(k-1)}{4}(k+1)!f(f')^{k-1}f'' + \ldots+(k+1)f^{k-1}f'f^{(k)} \tag{3.10}$$

and

$$F^{(k+1)} = (k+1)!(f')^{k+1}+\frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f'' + \ldots+(k+1)f^kf^{(k+1)}. \tag{3.11}$$

Substituting (3.1), (3.9), (3.10) and (3.11) into (3.8), we have

$$f^n(z)P(z)=Q(z), \tag{3.12}$$

where $Q(z)$ is a differential polynomial in f of degree n and

$$
\begin{aligned}
(3.13)\qquad P(z) &= F^{(k+1)} - \xi F^{(k)} - n\frac{f'}{f}F^{(k)} \\
&= -k(k+1)!(f')^{k+1} - (k+1)!\xi f(f')^k \\
&\quad + \frac{k(k+1)(3-k)(k+1)!}{4} f(f')^{k-1}f'' \\
&\quad + \ldots + (k+1)f^k f^{(k+1)} - (k+1)\xi f^k f^{(k)} - (k+1)^2 f^{k-1}f'f^{(k)} \\
&= -k(k+1)!(f')^{k+1} + R_1(f)
\end{aligned}
$$

is a differential polynomial in f of degree $k+1$, where $R_1(f)$ is a differential polynomial in f such that each term of $R_1(f)$ contains f^m for some m $(1 \leqslant m \leqslant n-1)$ as a factor.

We suppose that $P \equiv 0$. Then from (3.13) we get $F^{(k+1)} - \xi F^{(k)} - nF^{(k)}f'/f \equiv 0$ and so $F^{(k+1)}/F^{(k)} = \xi + nf'/f = \beta'/\beta + F'/F$. By integration we have $F^{(k)} = D\beta F$, where $D \in \mathbb{C} \setminus \{0\}$. Since $n = k+1$ and $\overline{N}(r,\infty;\beta) = S(r,f)$, it follows that $\overline{N}(r,0;f) = S(r,f)$. Then from (3.6) we have $T(r,f) = S(r,f)$, which is a contradiction. So $P \not\equiv 0$. Then by Lemma 2.1 we get $m(r,P) = S(r,f)$. Since $N(r,f) = S(r,f)$ we have

$$
(3.14)\qquad T(r,P) = S(r,f) \quad \text{and} \quad T(r,P') = S(r,f).
$$

Note that from (3.13) we get

$$
(3.15)\qquad P'(z) = A_1(f')^k f'' + B_1(f')^{k+1} + S_1(f),
$$

which is a differential polynomial in f, where $A_1 = -\frac{1}{4}k(k+1)^2(k+1)!$, $B_1 = -(k+1)!\xi$ and $S_1(f)$ is a differential polynomial in f such that each term of $S_1(f)$ contains f^m for some m $(1 \leqslant m \leqslant n-1)$ as a factor.

Let z_3 be a simple zero of f such that $\xi(z_3) \neq 0, \infty$. Then from (3.13) and (3.15) we have

$$
P(z_3) = -k(k+1)!(f'(z_3))^{k+1}, \quad P'(z_3) = A_1(f'(z_3))^k f''(z_3) + B_1(z_3)(f'(z_3))^{k+1}.
$$

This shows that z_3 is a zero of $Pf'' - (K_1P' - K_2P)f'$, where $K_1 = -k(k+1)!/A_1$ and $K_2 = B_1/A_1$. Also $T(r,K_1) = S(r,f)$ and $T(r,K_2) = S(r,f)$. Let

$$
(3.16)\qquad \Phi_1 = \frac{Pf'' - (K_1P' - K_2P)f'}{f}.
$$

Then clearly $m(r,\Phi_1) = S(r,f)$ and since $N_{(2}(r,0;f) + N(r,f) = S(r,f)$, we have $T(r,\Phi_1) = S(r,f)$. From (3.16) we obtain

$$
(3.17)\qquad f''(z) = \alpha_1(z)f(z) + \beta_1(z)f'(z),
$$

where

$$\alpha_1 = \frac{\Phi_1}{P} \quad \text{and} \quad \beta_1 = K_1\frac{P'}{P} - K_2. \tag{3.18}$$

Differentiating (3.17) and using it repeatedly we have

$$f^{(i)}(z) = \alpha_{i-1}(z)f(z) + \beta_{i-1}(z)f'(z), \tag{3.19}$$

where $i \geqslant 2$ and $T(r, \alpha_{i-1}) = S(r, f)$, $T(r, \beta_{i-1}) = S(r, f)$.

Also (3.18) yields

$$P' = \Big(\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\Big)P \quad \text{and} \quad \beta_1 = K_1\frac{P'}{P} - K_2 = \frac{-k(k+1)!}{A_1}\frac{P'}{P} - \frac{B_1}{A_1}, \tag{3.20}$$

so that

$$A_1\beta_1 + B_1 + k(k+1)!\frac{P'}{P} = 0. \tag{3.21}$$

Now we consider following two sub-cases.

Sub-case 1.2.1.1. Let $k = 1$. Now from (3.13) and (3.17) we have

$$P = -2(f')^2 - 2\xi ff' + 2ff'' = -2(f')^2 + (2\beta_1 - 2\xi)ff' + 2\alpha_1 f^2$$

and so

$$P' = (-2\beta_1 - 2\xi)(f')^2 + (2\beta_1' - 2\xi' + 2\beta_1^2 - 2\beta_1\xi)ff' + (2\alpha_1\beta_1 - 2\alpha_1\xi + 2\alpha_1')f^2.$$

Note that $K_1 = 1$ and $K_2 = \xi$ and so from (3.20) we have

$$(\beta_1' - \xi' - \beta_1\xi + \xi^2)f' + (-2\alpha_1\xi + \alpha_1')f \equiv 0. \tag{3.22}$$

If $-2\alpha_1\xi + \alpha_1' \equiv 0$, then from (3.22) we get, because $ff' \not\equiv 0$,

$$\beta_1' - \xi' - \beta_1\xi + \xi^2 \equiv 0. \tag{3.23}$$

Let $\beta_1 \equiv \xi$. Then a simple calculation gives $2\beta'\beta^{-1} = P'P^{-1}$ and so on integration we get $\beta^2 = d_0P$, where $d_0 \in \mathbb{C}\setminus\{0\}$. This contradicts the fact that $T(r, \beta) \neq S(r, f)$. So $\beta_1 \not\equiv \xi$. Now from (3.23) we get $(\beta_1' - \xi')(\beta_1 - \xi)^{-1} = \xi = \beta'\beta^{-1}$. So on integration we get $\beta = d_1(\beta_1 - \xi)$, where $d_1 \in \mathbb{C}\setminus\{0\}$. This contradicts the fact that $T(r, \beta) \neq S(r, f)$. So we conclude that $-2\alpha_1\xi + \alpha_1' \not\equiv 0$. Then from (3.22) we see that if z_4 is a simple zero of f, then z_4 is either a pole of $-2\alpha_1\xi + \alpha_1'$ or a zero of $\beta_1' - \xi' - \beta_1\xi + \xi^2$. Hence

$$N_{1)}(r, 0; f) \leqslant N(r, \infty; -2\alpha_1\xi + \alpha_1') + N(r, 0; \beta_1' - \xi' - \beta_1\xi + \xi^2) = S(r, f).$$

So we arrive at a contradiction by (3.6).

Sub-case 1.2.1.2. Let $k \geqslant 2$. From (3.9) and (3.11) we have $F^{(k)} = T_1(f)$, $F^{(k+1)} = (k+1)!(f')^{k+1} + T_2(f)$ and $F^{(k+2)} = \frac{1}{2}(k+1)(k+2)(k+1)!(f')^k f'' + T_3(f)$, where $T_1(f)$, $T_2(f)$ and $T_3(f)$ are differential polynomials in f such that each term of $T_1(f)$, $T_2(f)$ and $T_3(f)$ contain f as a factor.

Comparing (3.8) and (3.12) and noting that $F = f^n = f^{k+1}$ we have

$$\begin{aligned} Q &= a_1F^{(k+1)} - (\xi a_1 + a_1')F^{(k)} - a_2F' + (a_2' - \xi a_2)F + \gamma \\ &= a_1((k+1)!(f')^{k+1} + T_2(f)) - (\xi a_1 + a_1')T_1(f) \\ &\quad - (k+1)a_2f^kf' + (a_2' - \xi a_2)f^{k+1} + \gamma, \end{aligned} \tag{3.24}$$

where $\gamma = \xi a_1a_2 + a_2a_1' - a_1a_2'$.

Now suppose $\gamma(z) \equiv 0$. Then by integration we obtain $\beta = d_2a_2a_1^{-1}$, where $d_2 \in \mathbb{C} \setminus \{0\}$ and so $T(r,\beta) = S(r,f)$, which is a contradiction. Consequently $\gamma(z) \not\equiv 0$. Similarly we can verify that $\xi a_1 + a_1' \not\equiv 0$ and $a_2' - \xi a_2 \not\equiv 0$. We further note that $T(r,\gamma) = S(r,f)$. Differentiating (3.24) we have

$$\begin{aligned} Q' &= a_1'F^{(k+1)} + a_1F^{(k+2)} - (\xi a_1 + a_1')F^{(k+1)} - (\xi a_1 + a_1')'F^{(k)} \\ &\quad - a_2'F' - a_2F'' + (a_2' - \xi a_2)'F + (a_2' - \xi a_2)F' + \gamma' \\ &= a_1'((k+1)!(f')^{k+1} + T_2(f)) - (\xi a_1 + a_1')((k+1)!(f')^{k+1} + T_2(f)) \\ &\quad + a_1\Big(\frac{(k+1)(k+2)}{2}(k+1)!(f')^kf'' + T_3(f)\Big) - (\xi a_1 + a_1')'T_1(f) \\ &\quad - (k+1)a_2'f^kf' - a_2(k(k+1)f^{k-1}(f')^2 + (k+1)f^kf'') \\ &\quad + (a_2' - \xi a_2)'f^{k+1} + (k+1)(a_2' - \xi a_2)f^kf' + \gamma'. \end{aligned} \tag{3.25}$$

Let z_5 be a simple zero of $f(z)$ such that z_5 is not a zero or a pole of a_1, a_2 and ξ. Then from (3.12), (3.24) and (3.25) we have

$$\gamma(z_5) = A(z_5)(f'(z_5))^{k+1}, \quad \gamma'(z_5) = A_2(z_5)(f'(z_5))^kf''(z_5) + B_2(z_5)(f'(z_5))^{k+1},$$

where $A(z) = -(k+1)!a_1(z)$, $A_2(z) = -\frac{1}{2}(k+1)(k+2)(k+1)!a_1(z)$ and $B_2(z) = (k+1)!\xi(z)a_1(z)$. This shows that z_5 is a zero of $\gamma f'' - (K_3\gamma' - K_4\gamma)f'$, where $K_3 = AA_2^{-1}$ and $K_4 = B_2A_2^{-1}$. Also $T(r,K_3) = S(r,f)$ and $T(r,K_4) = S(r,f)$.

Let

$$\Phi_2 = \frac{\gamma f'' - (K_3\gamma' - K_4\gamma)f'}{f}. \tag{3.26}$$

Then clearly $T(r,\Phi_2) = S(r,f)$. From (3.26) we obtain

$$f'' = \varphi_1 f + \psi_1 f', \tag{3.27}$$

where

$$\varphi_1 = \frac{\Phi_2}{\gamma} \quad \text{and} \quad \psi_1 = K_3\frac{\gamma'}{\gamma} - K_4. \tag{3.28}$$

Now we show that $\psi_1 \not\equiv \beta_1$. If $\psi_1 \equiv \beta_1$ then from (3.18) and (3.28) we have

$$\frac{2}{(k+1)(k+2)}\frac{\gamma'}{\gamma} + \frac{2}{(k+1)(k+2)}\xi \equiv \frac{4}{(k+1)^2}\frac{P'}{P} - \frac{4}{k(k+1)^2}\xi,$$

i.e.,

$$2k(k+2)\frac{P'}{P} - k(k+1)\frac{\gamma'}{\gamma} \equiv (k^2+3k+4)\frac{\beta'}{\beta}.$$

On integration we have

$$\beta^{k^2+3k+4} \equiv \frac{d_3 P^{2k(k+2)}}{\gamma^{k(k+1)}},$$

where $d_3 \in \mathbb{C} \setminus \{0\}$ and so from (3.14) we have $T(r,\beta) = S(r,f)$, a contradiction.

Now from (3.27) we have

$$f^{(i)} = \varphi_{i-1} f + \psi_{i-1} f', \tag{3.29}$$

where $i \geqslant 2$ and $T(r,\varphi_{i-1}) = S(r,f)$, $T(r,\psi_{i-1}) = S(r,f)$.

Also from (3.13), (3.15) and (3.29) we have, respectively,

$$P = -k(k+1)!(f')^{k+1} + \sum_{j=1}^{k+1} T_j f^j (f')^{k+1-j}, \tag{3.30}$$

$$P' = (A_1\psi_1 + B_1)(f')^{k+1} + \sum_{j=1}^{k+1} S_j f^j (f')^{k+1-j}, \tag{3.31}$$

where $T(r,T_j) = S(r,f)$ and $T(r,S_j) = S(r,f)$.

Multiplying (3.30) by P' and (3.31) by P and then subtracting we get

$$H_0(f')^{k+1} + H_1 f(f')^k + \ldots + H_{k+1} f^{k+1} \equiv 0, \tag{3.32}$$

where

(3.33)

$$H_0 = P\Big(A_1\psi_1 + B_1 + k(k+1)!\frac{P'}{P}\Big) \quad \text{and } H_j = PS_j - P'T_j \text{ for } j = 1,2,\ldots,k+1.$$

Since $\beta_1 \not\equiv \psi_1$ and $P \not\equiv 0$, it follows from (3.21) and (3.33) that $H_0 \not\equiv 0$. Again since $H_0(f')^{k+1} \not\equiv 0$, from (3.32) we conclude that $H_i \not\equiv 0$ for at least one $i \in \{1,2,\ldots,k+1\}$. Let $S = \{1,2,\ldots,k+1\}$ and $S_1 = \{i \in S\colon\ H_i \not\equiv 0\}$. Note that $T(r,H_0) = S(r,f)$ and $T(r,H_j) = S(r,f)$ for $j \in S_1$.

Now from (3.32) we see that a simple zero of f must be either a zero of H_0 or a pole of at least one H_i's, where $i \in S_1$. Therefore

$$N_{1)}(r,0;f) \leqslant N(r,0;H_0) + \sum_{j,\, j\in S_1} N(r,\infty;H_j) + S(r,f) = S(r,f).$$

So we arrive at a contradiction by (3.6).

Sub-case 1.2.2. Suppose $T(r,\beta) = S(r,f)$. Then from (3.7) we have

$$F^{(k)} - \beta F \equiv a_2 - \beta a_1. \tag{3.34}$$

If $a_2 - \beta a_1 \equiv 0$, then from (3.34) we get $(f^n)^{(k)} \equiv a_2 a_1^{-1} f^n$, which contadicts the fact that $\Phi \not\equiv 0$. So we suppose that $a_2 - \beta a_1 \not\equiv 0$. Let z_6 be a simple zero of f. If z_6 is not a pole of β, then from (3.34) we see that z_6 is a zero of $a_2 - a_1\beta$. Therefore

$$N_{1)}(r,0;f) \leqslant N(r,0;a_2 - a_1\beta) + N(r,\infty;\beta) = S(r,f).$$

So by (3.6) we arrive at a contradiction.

Case 2. Let $\Phi \equiv 0$. Now from (3.2) we get $F_1 \equiv G_1$, i.e., $(f^n)^{(k)} \equiv a_2 a_1^{-1} f^n$.

Furthermore if $a_1 \equiv a_2$, then $f^n \equiv (f^n)^{(k)}$, and by Lemma 2.4, f assumes the form $f(z) = ce^{z\lambda/n}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$. □

P r o o f of Theorem 1.2. Let $F_1 = f^n a_1^{-1}$ and $G_1 = (f^n)^{(k)} a_2^{-1}$. Clearly F_1 and G_1 share "$(1,0)$" except for the zeros and poles of $a_1(z)$ and $a_2(z)$ and so $\overline{N}(r,1;F_1) = \overline{N}(r,1;G_1) + S(r,f)$. We now consider following two cases.

Case 1. Let $F_1 \not\equiv G_1$. Then

$$\begin{aligned}
\overline{N}(r,1;F_1) &\leqslant \overline{N}(r,0;G_1 - F_1 \mid F_1 \neq 0) + S(r,f) \\
&\leqslant \overline{N}\Big(r,0;\frac{G_1 - F_1}{F_1}\Big) + S(r,f) \leqslant T\Big(r,\frac{G_1 - F_1}{F_1}\Big) + S(r,f) \\
&\leqslant T\Big(r,\frac{G_1}{F_1}\Big) + S(r,f) \leqslant N\Big(r,\infty;\frac{G_1}{F_1}\Big) + m\Big(r,\infty;\frac{G_1}{F_1}\Big) + S(r,f) \\
&= N\Big(r,\infty;\frac{a_1}{a_2}\frac{(f^n)^{(k)}}{f^n}\Big) + m\Big(r,\infty;\frac{a_1}{a_2}\frac{(f^n)^{(k)}}{f^n}\Big) + S(r,f) \\
&\leqslant k\overline{N}(r,\infty;f) + k\overline{N}(r,0;f^n) + S(r,f) = k\overline{N}(r,0;f) + S(r,f).
\end{aligned} \tag{3.35}$$

Now using (3.35) and $\overline{N}_{2)}(r,0;f) = S(r,f)$, we get from the second fundamental theorem that

$$\begin{aligned}
nT(r,f) = T(r,f^n) + S(r,f) &\leqslant T(r,F_1) + S(r,f) \\
&\leqslant \overline{N}(r,\infty;F_1) + \overline{N}(r,0;F_1) + \overline{N}(r,1;F_1) + S(r,F) \\
&\leqslant \overline{N}(r,\infty;f) + \overline{N}(r,0;f^n) + \overline{N}(r,1;F_1) + S(r,f) \\
&\leqslant (k+1)\overline{N}(r,0;f) + S(r,f) \\
&\leqslant \frac{k+1}{3}N(r,0;f) + S(r,f) \leqslant \frac{k+1}{3}T(r,f) + S(r,f).
\end{aligned} \tag{3.36}$$

Since $n \geqslant k$, (3.36) leads to a contradiction.

Case 2. $F_1 \equiv G_1$. Then $(f^n)^{(k)} \equiv a_2 a_1^{-1} f^n$. Furthermore if $a_1 \equiv a_2$, then $f^n \equiv (f^n)^{(k)}$, and, by Lemma 2.4, f assumes the form $f(z) = ce^{z\lambda/n}$, where $c \in \mathbb{C} \setminus \{0\}$ and $\lambda^k = 1$. □

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Authors' addresses: *Indrajit Lahiri*, Department of Mathematics, University of Kalyani, Kalyani, West Bengal 741235, India, e-mail: `ilahiri@hotmail.com`; *Sujoy Majumder*, Department of Mathematics, Raiganj University, Raiganj, West Bengal 733134, India, e-mail: `sm05math@gmail.com`, `sujoy.katwa@gmail.com`.