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CLASSIFICATION OF IDEALS OF 8-DIMENSIONAL RADFORD HOPF ALGEBRA

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Abstract. Let $H_{m,n}$ be the mn^2-dimensional Radford Hopf algebra over an algebraically closed field of characteristic zero. We give the classification of all ideals of 8-dimensional Radford Hopf algebra $H_{2,2}$ by generators.

Keywords: ideal; Radford Hopf algebra; principal ideal ring

MSC 2020: 16D25, 20G42

1. Introduction

The notion of Radford Hopf algebra is given by Radford in [5] in order to give an example of Hopf algebra whose Jacobson radical is not a Hopf ideal. We recall the definition of Radford Hopf algebra briefly. Let G be a cyclic group of order mn $(n > 1)$ generated by g. Assume that V_i is a 1-dimensional vector space such that the action of g on V_i is the scalar multiplied by ω^i, where ω is a primitive mnth root of unity. Then $\{V_i\colon\ i \in \mathbb{Z}_{mn}\}$ forms a complete set of simple $\Bbbk G$-modules up to isomorphism. Let χ be the $\Bbbk$-linear character of $V_{m(n-1)}$. Namely, $\chi(g) = \omega^{m(n-1)} = \omega^{-m}$. The order of χ is n and the $\Bbbk$-linear character of V_m is χ^{-1}. Let $H_{m,n}$ be generated as an algebra by g and z subject to

$$g^{mn} = 1, \quad zg = \chi(g)gz = \omega^{-m}gz, \quad z^n = g^n - 1. \tag{1.1}$$

We can endow $H_{m,n}$ with a Hopf algebra structure, where the comultiplication Δ, the counit ε, and the antipode S are given, respectively, by

$$\Delta(z) = z \otimes g + 1 \otimes z, \quad \varepsilon(z) = 0, \quad S(z) = -zg^{-1}, \tag{1.2}$$

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$$(1.3) \qquad \Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \quad S(g) = g^{-1},$$

and $H_{m,n}$ is an mn^2-dimensional pointed Hopf algebra with a $\Bbbk$-basis $\{z^i g^j \colon 0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant mn-1\}$. We call $H_{m,n}$ *Radford Hopf algebra* for the given integers m and n. Wang et al. in [6], [7] studied the structure of the Green rings of finite dimensional pointed Hopf algebras of rank one. In particular, they determined the Green rings of Radford Hopf algebras by generators and relations. We showed in [8] that Radford Hopf algebras are principal ideal rings and give the generators of annihilator ideals of indecomposable modules. In this paper we will determine all ideals of 8-dimensional Radford Hopf algebra $H_{2,2}$ by generators.

This paper is organized as follows. In Section 2, we recall from [6], [7] the structure of all indecomposable modules of Radford Hopf algebra $H_{m,n}$. Then we refer to [8] for all ideals and annihilator ideals of indecomposable $H_{m,n}$-modules. In Section 3, we use the generators to classify all 14 different ideals of $H_{2,2}$.

Throughout, we work over an algebraically closed field $\Bbbk$ of characteristic zero. Unless other stated, all algebras, Hopf algebras and modules are vector spaces over $\Bbbk$; all modules are finite dimensional; all maps are $\Bbbk$-linear; $\otimes$ means $\otimes_{\Bbbk}$. We assume that the reader is familiar with the basics of Hopf algebras and representation theory. References [3], [4] are suggested for the former and [1], [2] are suggested for the latter.

2. Preliminaries

In this section, we recall the structure and indecomposable modules of the Radford Hopf algebras. As described in Section 1, for any $h \in \Bbbk G$ with the comultiplication $\Delta(h) = \sum h_1 \otimes h_2$, the $\Bbbk$-linear character χ induces an automorphism σ of $\Bbbk G$ such that $\sigma(h) = \sum \chi(h_1) h_2$. Thus,

$$(2.1) \qquad z^k h = \sigma^k(h) z^k \quad \text{for } 0 \leqslant k \leqslant n-1.$$

Let x be a variable and V an $\Bbbk G$-module. For any natural integer l, let $x^l V$ be a vector space defined by $x^l u + x^l v = x^l(u+v)$ and $\lambda(x^l u) = x^l(\lambda u)$ for all $u, v \in V$ and $\lambda \in \Bbbk$. Then $x^l V$ has a $\Bbbk G$-module structure defined by

$$(2.2) \qquad h \cdot (x^l v) = \chi^{-l}(h) x^l (h \cdot v) \quad \text{for } h \in \Bbbk G,\ v \in V.$$

Denote by Λ_0 the subset of $\mathbb{Z}_{mn}$ consisting of elements divisible by m, and by Λ_1 the complementary subset of Λ_0. Let τ be the permutation of $\mathbb{Z}_{mn}$ determined by $V_{\chi^{-1}} \otimes V_i \cong V_{\tau(i)}$, where $V_{\chi^{-1}}$ is exactly the simple $\Bbbk G$-module V_m with the character χ^{-1}. It is clear that $\tau(i) = m + i$ for any $i \in \mathbb{Z}_{mn}$. Let $\langle \tau \rangle$ be the subgroup of the symmetric group $\mathbf{S}_{mn}$ generated by the permutation τ. Then the group $\langle\tau\rangle$ acts on the index set $\mathbb{Z}_{mn}$, which is divided into m distinct $\langle\tau\rangle$-orbits $[0],[1],\ldots,[m-1]$, where $[i]=\{i,m+i,2m+i,\ldots,(n-1)m+i\}$ for $0\leqslant i\leqslant m-1$. Moreover, $\Lambda_0=[0]$ and $\Lambda_1=[1]\cup[2]\cup\ldots\cup[m-1]$.

Let $M(k,i)=V_i\oplus xV_i\oplus\ldots\oplus x^{k-1}V_i$ for $i\in\Lambda_0$ and $1\leqslant k\leqslant n$. Then $M(k,i)$ is an $H_{m,n}$-module:

$$h\cdot(x^lv)=\chi^{-l}(h)x^l(h\cdot v),\quad 0\leqslant l\leqslant k-1, \tag{2.3}$$

$$z\cdot(x^lv)=\begin{cases}x^{l+1}v, & 0\leqslant l\leqslant k-2,\\ 0, & l=k-1\end{cases} \tag{2.4}$$

for any $h\in\Bbbk G$ and $v\in V_i$.

Let $P_j=V_j\oplus xV_j\oplus\ldots\oplus x^{n-1}V_j$, $j\in\Lambda_1$. Then P_j is an $H_{m,n}$-module:

$$h\cdot(x^lv)=\chi^{-l}(h)x^l(h\cdot v),\quad 0\leqslant l\leqslant n-1, \tag{2.5}$$

$$z\cdot(x^lv)=\begin{cases}x^{l+1}v, & 0\leqslant l\leqslant n-2,\\ (g^n-1)v, & l=n-1\end{cases} \tag{2.6}$$

for any $h\in\Bbbk G$ and $v\in V_j$. For any two $j,j'\in\Lambda_1$, $P_j\cong P_{j'}$ as $H_{m,n}$-modules if and only if $[j]=[j']$. Let $P_{[j]}$ stand for a representative of the isomorphism class $[P_j]$ of P_j.

The following results will be used in the next section and we quote them here for convenience of the reader.

Proposition 2.1 ([6], Theorem 2.5, [7], Proposition 2.4, Proposition 2.8, Theorem 2.9). *The set $\{M(1,i),P_{[j]}\colon i\in\Lambda_0,\ j\in\Lambda_1\}$ forms a complete set of finite dimensional simple $H_{m,n}$-modules up to isomorphism. And the set $\{M(k,i),P_{[j]}\colon i\in\Lambda_0,\ 1\leqslant k\leqslant n,\ j\in\Lambda_1\}$ forms a complete set of finite dimensional indecomposable $H_{m,n}$-modules up to isomorphism.*

Proposition 2.2 ([8], Theorem 3.4). *Let I be any nonzero two-sided ideal of $H_{m,n}$. Then there exist integers $1\leqslant t\leqslant n$, $n-1\geqslant k_1>k_2>\ldots>k_t\geqslant 0$ and polynomials $d_1(y),d_2(y),\ldots,d_t(y)\in\Bbbk[y]$ with $d_1(y)|d_2(y)|\ldots|d_t(y)|y^{mn}-1$ and $0\leqslant\deg(d_1(y))<\deg(d_2(y))<\ldots<\deg(d_t(y))\leqslant mn-1$ such that*

$$I=(z^{k_1}d_1(g),z^{k_2}d_2(g),\ldots,z^{k_t}d_t(g))=(z^{k_1}d_1(g)+z^{k_2}d_2(g)+\ldots+z^{k_t}d_t(g)).$$

We end this section by giving the following lemma, which is useful in the next section.

Lemma 2.3. *We have $(z^l(g-\omega^j))=(z^l)$ for any $j\in\Lambda_1$ and $0\leqslant l\leqslant n-1$.*

P r o o f. Since $\gcd(y-\omega^j, y^n-1)=1$, there exist $u(y), v(y) \in \Bbbk[y]$ such that

$$u(y)(y-\omega^j)+v(y)(y^n-1)=1.$$

Hence, it follows that

$$u(g)(g-\omega^j)+v(g)z^n = u(g)(g-\omega^j)+v(g)(g^n-1)=1.$$

Therefore $(g-\omega^j)+(z^n)=(1)$. Note that $z \in (g-\omega^j)$ by [8], Lemma 3.10. Then we have $(g-\omega^j)=(1)$. Assume that

$$1=\sum_s (z^{n-1}f_{1s}(g)+\ldots+f_{ns}(g))(g-\omega^j)(z^{n-1}h_{1s}(g)+\ldots+h_{ns}(g)),$$

where $f_{ks}(y), h_{ks}(y) \in \Bbbk[y]$ for $1 \leqslant k \leqslant n$. Thus, we have

$$\begin{aligned} z^l &= z^l \sum_s (z^{n-1}f_{1s}(g)+\ldots+f_{ns}(g))(g-\omega^j)(z^{n-1}h_{1s}(g)+\ldots+h_{ns}(g)) \\ &= \sum_s (z^{n-1}\sigma^l(f_{1s}(g))+\ldots+\sigma^l(f_{ns}(g)))z^l(g-\omega^j)(z^{n-1}h_{1s}(g)+\ldots+h_{ns}(g)). \end{aligned}$$

This implies $z^l \in (z^l(g-\omega^j))$. Hence $(z^l(g-\omega^j))=(z^l)$. □

3. Ideals of $H_{2,2}$

In this section, we give explicitly the generators of all ideals of 8-dimensional Radford Hopf algebra. The Radford Hopf algebra $H_{2,2}$ is generated by g and z subject to

$$g^4=1, \quad zg=\omega^{-2}gz=-gz \quad \text{and} \quad z^2=g^2-1,$$

where ω is a primitive 4th root of unity. Let $I=(z^{k_1}d_1(g)+\ldots+z^{k_t}d_t(g))$ be any nonzero ideal of $H_{2,2}$, where $1 \leqslant t \leqslant 2$, $1 \geqslant k_1 > \ldots > k_t \geqslant 0$, $d_1(y)|\ldots|d_t(y)|y^4-1$ and $0 \leqslant \deg(d_1(y)) < \ldots < \deg(d_t(y)) \leqslant 3$. Note that $\Lambda_0=\{0,2\}$, $\Lambda_1=\{1,3\}$. It is clear that $P_1 \cong P_3$ and we denote it by $P_{[1]}$. Since $y^4-1=(y-1)(y-\omega)(y+1)(y+\omega)$, one can obtain that $d_s(y)$ is 1, $y-\omega^i$, $y-\omega^j$, $(y-\omega^i)(y-\omega^j)$, y^2-1, y^2+1, $(y^2-1)(y-\omega^j)$ or $(y^2+1)(y-\omega^i)$ for $i \in \Lambda_0$, $j \in \Lambda_1$, $1 \leqslant s \leqslant t$. We shall prove the following theorem.

Theorem 3.1. *The Radford Hopf algebra $H_{2,2}$ has the following 14 ideals:*

$$\begin{gathered} (0), \quad (1), \quad (g-1), \quad (g+1), \quad (g^2-1), \quad (g^2+1), \\ ((g^2+1)(g-1)), \quad ((g^2+1)(g+1)), \quad (z), \quad (z(g-1)), \\ (z(g+1)), \quad (z(g^2+1)), \quad (z(g^2+1)(g-1)), \quad (z(g^2+1)(g+1)). \end{gathered}$$

In the sequel, we need to show that the ideals in Theorem 3.1 are all different and each ideal of $H_{2,2}$ is one of the ideals in Theorem 3.1.

Lemma 3.2. *We have* $z \in ((g-\omega^i)(g-\omega^j))$ *for* $i \in \Lambda_0$ *and* $j \in \Lambda_1$.

Proof. Note that

$$z(g-\omega^i)(g-\omega^j) - (g-\omega^i)(g-\omega^j)z = -2zg(\omega^i+\omega^j) \in ((g-\omega^i)(g-\omega^j)).$$

Since $\omega^i + \omega^j \neq 0$ and g is invertible, we have $z \in ((g-\omega^i)(g-\omega^j))$. □

Lemma 3.3. *We have* $(z^l(g-\omega^i)(g-\omega^j)) = (z^l(g-\omega^i))$ *for* $0 \leqslant l \leqslant 1$, $i \in \Lambda_0$ *and* $j \in \Lambda_1$.

Proof. It suffices to show that

$$((g-\omega^i)(g-\omega^j)) = (g-\omega^i) \quad \text{and} \quad (z(g-\omega^i)(g-\omega^j)) = (z(g-\omega^i)).$$

Since $(g-\omega^j) = (1)$ by Lemma 2.3, we may assume that

$$1 = \sum_s (zf_{1s}(g) + f_{2s}(g))(g-\omega^j)(zh_{1s}(g) + h_{2s}(g)),$$

where $f_{ks}(y), h_{ks}(y) \in \Bbbk[y]$ for $1 \leqslant k \leqslant 2$. Therefore,

$$\begin{aligned} g-\omega^i &= \sum_s (g-\omega^i)(zf_{1s}(g) + f_{2s}(g))(g-\omega^j)(zh_{1s}(g) + h_{2s}(g)) \\ &= zv + \sum_s (g-\omega^i)f_{2s}(g)(g-\omega^j)h_{2s}(g) \end{aligned}$$

for some $v \in H_{2,2}$. Noting that $z \in ((g-\omega^i)(g-\omega^j))$ by Lemma 3.2, we have

$$g-\omega^i \in ((g-\omega^i)(g-\omega^j)).$$

Hence $((g-\omega^i)(g-\omega^j)) = (g-\omega^i)$. For the second equation, we may assume that

$$g-\omega^i = \sum_s (zp_{1s}(g) + p_{2s}(g))(g-\omega^i)(g-\omega^j)(zr_{1s}(g) + r_{2s}(g)),$$

where $p_{ks}(y), r_{ks}(y) \in \Bbbk[y]$ for $1 \leqslant k \leqslant 2$. Hence,

$$\begin{aligned} z(g-\omega^i) &= z\sum_s (zp_{1s}(g) + p_{2s}(g))(g-\omega^i)(g-\omega^j)(zr_{1s}(g) + r_{2s}(g)) \\ &= \sum_s (z\sigma(p_{1s}(g)) + \sigma(f_{2s}(g)))z(g-\omega^i)(g-\omega^j)(zr_{1s}(g) + r_{2s}(g)). \end{aligned}$$

Thus, we obtain $z(g-\omega^i) \in (z(g-\omega^i)(g-\omega^j))$. Therefore

$$(z(g-\omega^i)) = (z(g-\omega^i)(g-\omega^j)).$$

We finish the proof. □

Lemma 3.4. *If* $(zu_1(g)) = (u_2(g))$, *where* $u_k(y) \in \Bbbk[y]$, $0 \leqslant \deg(u_k(y)) \leqslant 3$, $u_k(y)|y^4-1$, *for* $1 \leqslant k \leqslant 2$, *then* $(zu_1(g)) = (u_2(g)) = (z(g^2-1)) = (g^2-1)$.

P r o o f. It is clear that $(z(g^2-1)) \subseteq (g^2-1)$. Noting that $g^2-1 = \frac{1}{2}zg^2\cdot z(g^2-1)$, we have $(z(g^2-1)) = (g^2-1)$. If $(zu_1(g)) = (u_2(g))$, then we may assume

$$u_2(g) = \sum_s (zf_{1s}(g) + f_{2s}(g))zu_1(g)(zh_{1s}(g) + h_{2s}(g)),$$

where $f_{ks}(y), h_{ks}(y) \in \Bbbk[y]$ for $1 \leqslant k \leqslant 2$. Noticing that $\{z^a g^b\colon 0 \leqslant a \leqslant 1, 0 \leqslant b \leqslant 3\}$ is a $\Bbbk$-basis of $H_{2,2}$, we have

$$u_2(g) = (g^2-1)\sum_s \sigma^{-1}(f_{1s}(g))u_1(g)h_{2s}(g) + f_{2s}(g)\sigma^{-1}(u_1(g))h_{1s}(g).$$

Since the right-hand side annihilates the simple $H_{2,2}$-module $M(1,0)$ and $M(1,2)$, we have $u_2(y) = (y^2-1)c(y)$, where $c(y) \in \Bbbk[y]$ and $0 \leqslant \deg(c(y)) \leqslant 1$. Since $u_2(y)|y^4-1$, we obtain $c(y) = 1$, $y-\omega$ or $y+\omega$. Thus,

$$(zu_1(g)) = (u_2(g)) = ((g^2-1)c(g)) = (g^2-1).$$

□

Lemma 3.5. *For any given* $0 \leqslant l \leqslant 1$, *the following seven ideals are all different:*

$$(z^l),\quad (z^l(g-1)),\quad (z^l(g+1)),\quad (z^l(g^2-1)),$$
$$(z^l(g^2+1)),\quad (z^l(g^2+1)(g-1)),\quad (z^l(g^2+1)(g+1)).$$

P r o o f. We only need to show that if

$$(z^l(g-1)^{p_0}(g^2+1)^{p_1}(g+1)^{p_2}) = (z^l(g-1)^{r_0}(g^2+1)^{r_1}(g+1)^{r_2}),$$

where $0 \leqslant p_k, r_k \leqslant 1$, $0 \leqslant p_0+p_1+p_2$, $r_0+r_1+r_2 \leqslant 2$, then $p_k = r_k$ for $0 \leqslant k \leqslant 2$. We may assume

$$\begin{aligned} &z^l(g-1)^{p_0}(g^2+1)^{p_1}(g+1)^{p_2} \\ &\quad = \sum_s (zf_{1s}(g)+f_{2s}(g))z^l(g-1)^{r_0}(g^2+1)^{r_1}(g+1)^{r_2}(zh_{1s}(g)+h_{2s}(g)), \end{aligned}$$

where $f_{ks}(y), h_{ks}(y) \in \Bbbk[y]$, $1 \leqslant k \leqslant 2$. Since $\{z^a g^b\colon 0 \leqslant a \leqslant 1,\ 0 \leqslant b \leqslant 3\}$ is a $\Bbbk$-basis of $H_{2,2}$, we have

$$\begin{aligned} &(g-1)^{p_0}(g^2+1)^{p_1}(g+1)^{p_2} \\ &\quad = \sum_s \sigma^{-l}(f_{2s}(g))(g-1)^{r_0}(g^2+1)^{r_1}(g+1)^{r_2}h_{2s}(g) \\ &\qquad + \sum_s (g^2-1)\sigma^{-l-1}(f_{1s}(g))(-g-1)^{r_0}(g^2+1)^{r_1}(-g+1)^{r_2}h_{1s}(g). \end{aligned}$$

If $p_k < r_k$, then $p_k = 0$ and $r_k = 1$. Note that the right-hand side annihilates the simple $\Bbbk G$-module V_k and $(g-1)^{p_0}(g^2+1)^{p_1}(g+1)^{p_2} \cdot V_k \neq 0$. It is a contradiction. Hence, $p_k \geqslant r_k$ for $0 \leqslant k \leqslant 2$. In a similar way we conclude $r_k \leqslant p_k$. Then $p_k = r_k$. We finish the proof. □

Lemma 3.6. *We have*

$$(z, g^2-1) = (z), \quad (z(g-1), g^2-1) = (z(g-1)),$$
$$(z, g^2+1) = (1), \quad (z(g+1), g^2-1) = (z(g+1)).$$

P r o o f. By Lemma 3.4, it is easy to see that

$$(z(g-1), g^2-1) = (z(g-1), z(g^2-1)) = (z(g-1)),$$
$$(z(g+1), g^2-1) = (z(g+1), z(g^2-1)) = (z(g+1)),$$
$$(z, g^2-1) = (z, z(g^2-1)) = (z).$$

Note that $2 = g^2+1-z\cdot z \in (z, g^2+1)$. It follows that $(z, g^2+1) = (1)$. □

Lemma 3.7. *We have*

$$(z, (g^2+1)(g+1)) = (g+1), \quad (z(g+1), (g^2+1)(g+1)) = (g+1),$$
$$(z, (g^2+1)(g-1)) = (g-1), \quad (z(g-1), (g^2+1)(g-1)) = (g-1).$$

P r o o f. Since

$$z\cdot z(g+1) - (g^2+1)(g+1) = (g^2-1)(g+1) - (g^2+1)(g+1) = -2(g+1),$$

we have $g+1 \in (z(g+1), (g^2+1)(g+1))$. It is clear that $z(g+1) \in (g+1)$ and $(g^2+1)(g+1) \in (g+1)$. Hence, it follows that

$$(z(g+1), (g^2+1)(g+1)) = (g+1).$$

It is easy to see that

$$(z, (g^2+1)(g+1)) \supseteq (z(g+1), (g^2+1)(g+1)) = (g+1).$$

Since $z \in (g+1)$ by [8], Lemma 3.10 and $(g^2+1)(g+1) \in (g+1)$, it follows that

$$(z, (g^2+1)(g+1)) = (g+1).$$

The proof of $(z(g-1), (g^2+1)(g-1)) = (g-1)$ and $(z, (g^2+1)(g-1)) = (g-1)$ is similar. □

Lemma 3.8. *We have* $z(g^2+1) \in ((g^2+1)(g-1)) \cap ((g^2+1)(g+1))$.

P r o o f. Note that

$$z(g^2+1)(g-1) - (g^2+1)(g-1)z = 2zg(g^2+1) \in ((g^2+1)(g-1)).$$

Since g is invertible, we have $z(g^2+1) \in ((g^2+1)(g-1))$. In a similar way, we conclude that $z(g^2+1) \in ((g^2+1)(g+1))$. Hence, we finish the proof. □

P r o o f of Theorem 3.1. We divide the 14 ideals into two classes:

$$\begin{aligned} \text{Class I:}\quad & (0),\quad (1),\quad (g-1),\quad (g+1),\quad (g^2+1),\\ & (g^2-1),\quad ((g^2+1)(g-1)),\quad ((g^2+1)(g+1));\\ \text{Class II:}\quad & (z),\quad (z(g-1)),\quad (z(g+1)),\quad (z(g^2+1)),\\ & (z(g^2+1)(g-1)),\quad (z(g^2+1)(g+1)). \end{aligned}$$

We have proven that the ideals in each class are all different by Lemma 3.5 and the ideals in different classes are different by Lemma 3.4. Hence, the 14 ideals are all different. Next we will show that each ideal of $H_{2,2}$ must be one of the 14 ideals. Let $I = (z^{k_1}d_1(g) + \ldots + z^{k_t}d_t(g))$ be any nonzero ideal of $H_{2,2}$, where $1 \leqslant t \leqslant 2$, $1 \geqslant k_1 > \ldots > k_t \geqslant 0$, $d_1(y), \ldots, d_t(y) \in \Bbbk[y]$, $d_1(y)|\ldots|d_t(y)|y^4-1$ and $0 \leqslant \deg(d_1) < \ldots < \deg(d_t) \leqslant 3$.

Case 1: $k_1 = 0$. This implies that $I = (d_1(g))$. Since $d_1(y)|y^4-1$, $d_1(y)$ can be taken 1, $y-\omega^i$, $y-\omega^j$, $(y-\omega^i)(y-\omega^j)$, y^2-1, y^2+1, $(y^2-1)(y-\omega^j)$ or $(y^2+1)(y-\omega^i)$ for $i \in \Lambda_0$ and $j \in \Lambda_1$. Then I has to be taken the following ideals:

$$(1),\quad (g-1),\quad (g+1),\quad (g^2-1),\quad (g^2+1),\quad ((g^2+1)(g-1)),\quad ((g^2+1)(g+1)),$$

which are in Class I.

Case 2: $k_1 = 1$. It follows that $I = (zd_1(g))$ or $I = (zd_1(g) + d_2(g))$.

▷ When $I = (zd_1(g))$, I is one of the ideals in Class II or

$$I = (z(g^2-1)) = (g^2-1),$$

which is in Class I.

▷ When $I = (zd_1(g) + d_2(g))$, if $d_2(y) = y-\omega^i$ or $y-\omega^j$ for $i \in \Lambda_0$, $j \in \Lambda_1$, noting that $z \in (g-\omega^i)$ and $z \in (g-\omega^j)$, then $I = (g-\omega^i)$ or (1), which is in Class I. If $d_2(y) = (y-\omega^i)(y-\omega^j)$, then by Lemma 3.3,

$$I = (zd_1(g), (g-\omega^i)(g-\omega^j)) = (zd_1(g), g-\omega^i) = (g-\omega^i),$$

which is in Class I. If $d_2(y) = y^2 - 1$, then $d_1(y) = y - 1$, $y + 1$ or 1. Thus,

$$I = (z(g-1), g^2 - 1), \quad (z(g+1), g^2 - 1) \quad \text{or} \quad (z, g^2 - 1).$$

By Lemma 3.6, $I = (z(g-1))$, $(z(g+1))$ or (z), which is in Class II. If $d_2(y) = y^2+1$, then $d_1(y) = y - \omega$, $y + \omega$ or 1. Therefore,

$$I = (z(g-\omega), g^2 + 1), \quad (z(g+\omega), g^2 + 1) \quad \text{or} \quad (z,\, g^2 + 1).$$

By Lemma 3.6, $I = (1)$, which is in Class I. If $d_2(y) = (y^2 - 1)(y - \omega^j)$ for $j \in \Lambda_1$, then $d_1(y) = 1$, $y - \omega^i$, $y - \omega^j$, $y^2 - 1$ or $(y - \omega^i)(y - \omega^j)$. Therefore,

$$I = (z, g^2 - 1), \quad (z(g-1), g^2 - 1) \quad \text{or} \quad (z(g+1), g^2 - 1).$$

By Lemma 3.6, $I = (z)$, $(z(g-1))$ or $(z(g+1))$, which is in Class II. If $d_2(y) = (y^2+1)(y-1)$, then $d_1(y) = 1$, $y - 1$, $y - \omega^j$, $y^2 + 1$, $(y-1)(y-\omega^j)$ for $j \in \Lambda_1$. Hence,

$$I = (z, (g^2+1)(g-1)),\ (z(g-1), (g^2+1)(g-1)) \text{ or } (z(g^2+1), (g^2+1)(g-1)).$$

By Lemmas 3.7 and 3.8, $I = (g-1)$ or $((g^2+1)(g-1))$, which is in Class I. If $d_2(y) = (y^2+1)(y+1)$, then $d_1(y) = 1$, $y + 1$, $y - \omega^j$, $y^2 + 1$ or $(y+1)(y-\omega^j)$ for $j \in \Lambda_1$. Thus,

$$I = (z, (g^2+1)(g+1)),\ (z(g+1), (g^2+1)(g+1)) \text{ or } (z(g^2+1), (g^2+1)(g+1)).$$

By Lemmas 3.7 and 3.8, $I = (g+1)$ or $((g^2+1)(g+1))$, which is in Class I.

Hence, we finish the proof. □

Remark 3.9. By [8], Propositions 3.5, 3.6 and 3.9, it is clear that the annihilator ideal of $M(1,0)$ is $(g-1)$, the one of $M(1,2)$ is $(g+1)$, and the one of $P_{[1]}$ is (g^2+1). The annihilator ideal of $M(2,0)$ is $(z(g-1))$, and the one of $M(2,2)$ is $(z(g+1))$.

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