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ISOMORPHISMS BETWEEN GRADED FROBENIUS ALGEBRAS CONSTRUCTED FROM TWISTED SUPERPOTENTIALS

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Abstract. In order to distinguish the connected graded Frobenius algebras determined by different twisted superpotentials, we introduce the nondegeneracy of twisted superpotentials. We give the sufficient and necessary condition for connected graded Frobenius algebras determined by two nondegenerate twisted superpotentials to be isomorphic. As an application, we classify the connected $\mathbb{Z}$-graded Frobenius algebra of length 3, whose dimension of the degree 1 is 2.

Keywords: graded Frobenius algebra; coalgebra; twisted superpotential

MSC 2020: 16W50, 16W55

1. Introduction

Frobenius algebras, as a class of algebras with some symmetric properties, have been applied to many branches of mathematics, cf. [5], [6]. Consequently, it is significant to explore the way of constructing Frobenius algebras. It is proved in [8] that we can derive a graded Frobenius algebra with two specifical properties from an admissible system. Conversely, the authors declared that the basic graded Frobenius algebra, possessing the two properties, is isomorphic to an algebra grew out from an admissible system. The structure of graded Frobenius algebras stemmed from different admissible systems over K has been studied in the paper as well. In the previous paper (see [2]), the authors showed that for any connected $\mathbb{Z}$-graded Frobenius algebra $A = K \oplus A_1 \oplus \ldots \oplus A_n$ over a field K, that is, when A is isomorphic as a graded right A-module to the shift $A^*(n)$ of the dual A^* of the left A-graded module A, if A

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is generated by A_1, there exists a twisted superpotential w in $(A_1^*)^{\otimes n}$ with respect to a linear automorphism of $(A_1)^*$ induced by the Nakayama automorphism of A such that A is isomorphic to the dual algebra of the subcoalgebra generated by w inside $T((A_1)^*)$. Conversely, they proved that if V is a finite dimensional vector space and w is a twisted superpotential in $(V^*)^{\otimes n}$ corresponding to a linear automorphism ν of V^*, one can construct a graded algebra A as above, whose Nakayama automorphism is induced by ν. However, the authors did not investigate when the Frobenius algebras established by different twisted superpotentials are isomorphic.

In this note, we focus on the isomorphisms between the connected graded Frobenius algebras corresponding to different twisted superpotentials. Let V be a finite-dimensional K-vector space, which has a basis $\{x_1, \dots, x_m\}$. Let $U = V^*$, $y_i = x_i^*$, $1 \leqslant i \leqslant m$, then $\{y_1, \dots, y_m\}$ is the dual basis of U. We know that there is a bijection from the group of all K-linear bijective transformations of U to the group of all invertible $m \times m$ matrices, where m is the dimension of U. For each linear automorphism ν of U, M is the matrix of ν under the basis $\{y_1, \dots, y_m\}$. Assume that $w_1 \in (U)^{\otimes n}$ is a nondegenerate ν_1- twisted superpotential of degree n, $w_2 \in (U)^{\otimes n}$ is a nondegenerate ν_2-twisted superpotential of degree n. Let M_1, M_2 be the matrix corresponding to ν_1, ν_2, respectively. Let $C = \langle w_1 \rangle$, $D = \langle w_2 \rangle$ be the subcoalgebra of $T(U)$ generated by w_1, w_2, respectively. The relevant definition can be viewed in Preliminaries.

In Section 3, we prove that C and D are isomorphic as graded K-coalgebras if and only if $w_2 = w_1 \circ s^{\otimes n}$ and hence $\nu_2 = s^* \nu_1 (s^*)^{-1}$ for some $s \in \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ is the group of all K-linear bijective transformations of V. Naturally, we can obtain that C and D are isomorphic as graded K-coalgebras if and only if M_1 and M_2 are similar matrices. Specifically, if w_1 and w_2 are the different ν-twisted superpotential of degree n, we can deduce when the corresponding Frobenius algebras of w_1 and w_2 are isomorphic.

In Section 4, we focus on the nondegenerate twisted superpotentials of degree 3 and calculate all the corresponding connected graded Frobenius algebras of length 3, whose dimension of the degree 1 is two.

2. Preliminaries

Throughout this paper, K is a field. Let A be a finite dimensional algebra over K. Then K-dual space $A^* = \mathrm{Hom}(A, K)$ has a natural coalgebra structure. Let C be a finite dimensional coalgebra over K. Then the K-dual space $C^* = \mathrm{Hom}(C, K)$ has a natural algebra structure. Recall that an algebra A is called $\mathbb{Z}$-*graded* if there exists subspace $(A_i)_{i \in \mathbb{Z}}$ such that

$$A = \bigoplus_{i \in \mathbb{Z}} A_i \quad \text{and} \quad A_i \cdot A_j \subseteq A_{i+j}$$

for all $i, j \in \mathbb{Z}$. The elements in A_i are said to be *homogeneous* of degree i. Dually, a coalgebra C is called $\mathbb{Z}$-*graded* if there exists subspace $(C_i)_{i\in\mathbb{Z}}$ such that

$$C = \bigoplus_{i\in\mathbb{Z}} C_i \quad \text{and} \quad \Delta(C_n) \subseteq \bigoplus_{i+j=n} C_i \otimes C_j$$

for all $n \in \mathbb{Z}$. We say that two $\mathbb{Z}$-graded algebras A and B are isomorphic if there is an algebra isomorphism $f\colon A \to B$ which preserves the degrees of homogeneous elements. Similarly, we may define two isomorphic $\mathbb{Z}$-graded coalgebras. In this paper, by a graded algebra (coalgebra) we always mean a $\mathbb{Z}$-graded algebra (coalgebra). If $A = \bigoplus_{i\in\mathbb{Z}} A_i$ is a nonnegatively graded algebra with $A_0 = K$, then A is called a *connected graded algebra*.

Since A is a finite dimensional algebra, A^* has a natural left and right A-module structure. Then A is a *Frobenius algebra* if $A \cong A^*$ as left A-modules. This is equivalent to $A \cong A^*$ as right A-modules. Therefore, recall from [7] that A is a Frobenius algebra if and only if there is a K-bilinear form $\sigma\colon A\times A \to K$ satisfying the following conditions. For all $x, y, z \in A$,

(i) $\sigma(xy, z) = \sigma(x, yz)$,

(ii) the map σ is nondegenerate, that is, $\sigma(x, y) = 0$ for all x only if $y = 0$.

The map σ is called the *Frobenius form* of the algebra. For each Frobenius algebra, the K-algebra automorphism $\mu\colon A \to A$ given by

$$\sigma(\mu(a), b) = \sigma(b, a) \tag{2.1}$$

for all $a, b \in A$, is usually called *Nakayama automorphism*, cf. [4].

If C^* is a Frobenius algebra, then C is called *coFrobenius coalgebra*. Assume that $A = A_0 \oplus A_1 \oplus \ldots \oplus A_n$ is a finite dimensional graded algebra. Then A is a *graded Frobenius algebra* of length n if A is a Frobenius algebra and the Frobenius form σ satisfies that $\sigma(a, b) = 0$ for all $a \in A_i$, $b \in A_j$ such that $i + j \neq n$. The graded Frobenius algebra considered in this paper is indeed the n-graded Frobenius algebra introduced in [1].

Let V be a finite-dimensional K-vector space. Let $T(V) = K \oplus V \oplus V \otimes V \oplus \ldots$ be the tensor algebra, which is typically an $\mathbb{N}$-graded algebra. Let W, V be finite dimensional K-vector spaces. Then the map $\lambda\colon W^* \otimes V^* \to (W \otimes V)^*$ defined by

$$\lambda(f \otimes g)(w \otimes v) = f(w) \cdot g(v)$$

for all $f \in W^*$, $g \in V^*$, $w \in W$, $v \in V$, is an isomorphism. Then we can obtain that there is an isomorphism

$$\lambda_n\colon (V^*)^{\otimes n} \to (V^{\otimes n})^*$$

for each $n \in \mathbb{N}$. We identify $(V^*)^{\otimes n}$ with $(V^{\otimes n})^*$ through the isomorphism λ_n. For details about the above, the reader can refer to [3].

Assume that $\{x_1, \ldots, x_n\}$ is the basis of V. Let $U = V^*$, and $y_i = x_i^*\colon V \to K$, $1 \leqslant i \leqslant n$ defined by $x_i^*\Big(\sum\limits_{j=1}^{n} k_j x_j\Big) = k_i$. Then $\{y_1, \ldots, y_m\}$ form the basis of U, which is usually called the *dual basis* of U. The tensor space $T(U)$ can be seen as the locally finite dual space of $T(V)$, hence, it has the coalgebra structure induced by the algebra structure of $T(V)$. The coproduct and counit of $T(U)$ are determined by

$$\begin{aligned}
\Delta(y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n}) &= 1 \otimes (y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n}) \\
&\quad + \sum_{j=1}^{n-1} (y_{i_1} \otimes \ldots \otimes y_{i_j}) \otimes (y_{i_{j+1}} \otimes \ldots \otimes y_{i_n}) \\
&\quad + (y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n}) \otimes 1, \\
\varepsilon(y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n}) &= 0 \quad \text{for } n > 0.
\end{aligned} \tag{2.2}$$

For each $w \in T(U)$, the minimal subcoalgebra of $T(U)$ containing w is denoted by $\langle w \rangle$.

For a homogeneous element $\varpi \in U^{\otimes n}$ $(n \geqslant 1)$, recall the partial derivations of ϖ as follows, see [2]:

$$\partial^l_{y_i}(\varpi) = (y_i^* \otimes 1^{\otimes n-1})(\varpi) \quad \text{for each } 1 \leqslant i \leqslant m, \tag{2.3}$$

$$\partial^r_{y_i}(\varpi) = (1^{\otimes n-1} \otimes y_i^*)(\varpi) \quad \text{for each } 1 \leqslant i \leqslant m. \tag{2.4}$$

Let ν be a linear automorphism of U. An element $w \in U^{\otimes n}$ is called a *ν-twisted superpotential* of degree n if $\mathfrak{c}_n(w) = (\nu \otimes 1^{\otimes n-1})(w)$, where $\mathfrak{c}_n\colon U^{\otimes n} \to U^{\otimes n}$, $y_{i_1} \otimes \ldots \otimes y_{i_n} \mapsto y_{i_n} \otimes y_{i_1} \otimes \ldots \otimes y_{i_{n-1}}$, cf. [2].

Let A^* be the dual graded space of A. It means that A^* is a graded vector space concentrated in degrees $-n, \ldots, 0$ with $-i$th component A_i^* for each $0 \leqslant i \leqslant n$. Let $A^*(n)$ be the nth shift of A^*, that is, the ith component of $A^*(n)$ is equal to A^*_{n-i} for $i = 0, \ldots, n$.

The next lemma was proved in [7], Lemma 3.2, or more generally in [1], Theorem 4.1.

Lemma 2.1. *Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_n$ be a graded Frobenius algebra of length n. Then $A_i \cong A^*_{n-i}$ as a K-vector space for each i. Moreover, $A \cong A^*(n)$ as graded right A-modules.*

3. Twisted superpotentials

Throughout this section, let V be a finite-dimensional K-vector space, which has a basis $\{x_1, \ldots, x_m\}$. Let $U = V^*$, $y_i = x_i^*$, $1 \leqslant i \leqslant m$. Then $\{y_1, \ldots, y_m\}$ is the dual basis of U. Let $T(U)$ be the tensor coalgebra with the same coproduct defined as in (2.2). We assume that the degree of elements in U is -1. So, $T(U)$ is a graded coalgebra concentrated in nonpositive degrees. Let $\mathrm{GL}(V)$ denote the group of all invertible K-linear transformations of V.

Assume that w is a ν-twisted superpotential of degree n, let C be the subcoalgebra of $T(U)$ generated by w. Since w is a homogeneous element, C is a graded subcoalgebra. If $C_{-1} = U$, we call w *nondegenerate*. The result of the following lemma is evident.

Lemma 3.1. *Let w be a ν-twisted superpotential of degree n. Then for each $0 \neq k \in K$, kw is ν-twisted superpotential, and $\langle w\rangle = \langle kw\rangle$.*

Lemma 3.2. *Let w be a ν-twisted superpotential of degree n. Then for $s \in \mathrm{GL}(V)$, $w\circ s^{\otimes n}$ is $s^*\nu(s^*)^{-1}$-twisted superpotential. Moreover, if w is nondegenerate, then so is $w \circ s^{\otimes n}$.*

Proof. For each $a_1 \otimes a_2 \otimes \ldots \otimes a_n \in V^{\otimes n}$, since $\mathfrak{c}_n(w) = (\nu \otimes 1^{\otimes n-1})(w)$, it can be obtained that

$$\begin{aligned}
\mathfrak{c}_n(w \circ s^{\otimes n})(a_1 \otimes a_2 \otimes \ldots \otimes a_n) &= (w \circ s^{\otimes n})(a_2 \otimes a_3 \otimes \ldots \otimes a_1)\\
&= w(s(a_2) \otimes s(a_3) \otimes \ldots \otimes s(a_1))\\
&= \mathfrak{c}_n(w)(s(a_1) \otimes s(a_2) \ldots \otimes s(a_n))\\
&= w(\nu^* s(a_1) \otimes s(a_2) \otimes \ldots \otimes s(a_n)),\\
(s^*\nu(s^*)^{-1} \otimes 1^{\otimes n-1})(w \circ s^{\otimes n})(a_1 \otimes \ldots \otimes a_n) &= (w \circ s^{\otimes n})(s^{-1}\nu^* s(a_1) \otimes \ldots \otimes a_n)\\
&= w(\nu^* s(a_1) \otimes \ldots \otimes s(a_n)).
\end{aligned}$$

Then we have $\mathfrak{c}_n(w \circ s^{\otimes n}) = (\nu \otimes 1^{\otimes n-1})(w \circ s^{\otimes n})$, and $w \circ s^{\otimes n}$ is $s^*\nu(s^*)^{-1}$-twisted superpotential.

Let $C = \langle w\rangle$ and $D = \langle w \circ s^{\otimes n}\rangle$ be the graded subcoalgebras of $T(U)$ generated by w and $w \circ s^{\otimes n}$, respectively. According to [2], Lemma 4.8

$$\begin{aligned}
C_{-1} &= \mathrm{span}\{\partial^r_{y_{i_{n-1}}} \ldots \partial^r_{y_{i_1}}(w)\colon\ 1 \leqslant i_1, \ldots, i_{n-1} \leqslant m\},\\
D_{-1} &= \mathrm{span}\{\partial^r_{y_{i_{n-1}}} \ldots \partial^r_{y_{i_1}}(w \circ s^{\otimes n})\colon\ 1 \leqslant i_1, \ldots,\ i_{n-1} \leqslant m\}.
\end{aligned}$$

For any $1 \leqslant i_1, \ldots, i_{n-1} \leqslant m$ we have

$$\partial^r_{y_{i_{n-1}}} \ldots \partial^r_{y_{i_1}}(w \circ s^{\otimes n}) = \partial^r_{y_{i_{n-1}}} \ldots \partial^r_{y_{i_1}}((s^*)^{\otimes n}(w)).$$

Assume that M is the matrix of the linear automorphism $(s^*)^{\otimes n}$ of $U^{\otimes n}$ on the basis $\{y_{i_1} \otimes \ldots \otimes y_{i_n-1} \colon 1 \leqslant i_1, \ldots, i_n \leqslant m\}$ with lexicographic order. Then $\partial^r_{y_{i_{n-1}}} \ldots \partial^r_{y_{i_1}}((s^*)^{\otimes n}(w))$ is a linear combination of $\partial^r_{y_{i_{n-1}}} \ldots \partial^r_{y_{i_1}}(w)$, whose coefficients are those of $i_1 \ldots i_{n-1}$th row of M in the lexicographic order. We have $C_{-1} = D_{-1}$ since M is invertible. Then we can obtain that if w is nondegenerate, then so is $w \circ s^{\otimes n}$. □

Lemma 3.3. *Assume that w is a ν-twisted superpotential of degree n and a ν'-twisted superpotential of degree n. If w is nondegenerate, then $\nu = \nu'$.*

P r o o f. Let $w_1 \colon V \to (V^{\otimes n-1})^*$ be the k-linear map defined by $w_1(y)(z) = w(z \otimes y)$ for each $y \in V$, $z \in V^{\otimes n-1}$. Similarly, the k-linear map $\widehat{w_1} \colon V \to (V^{\otimes n-1})^*$ is defined by $\widehat{w_1}(y)(z) = w(y \otimes z)$. The fact that w is a ν-twisted superpotential and ν'-twisted superpotential implies the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "w_1"] \arrow[d, "\nu^*"'] & (V^{\otimes n-1})^* \arrow[d, "\mathrm{id}"] \\ V \arrow[r, "\widehat{w_1}"] & (V^{\otimes n-1})^*. \end{tikzcd}$$

We have $\widehat{w_1} \circ \nu^* = w_1$. By the same reason, we have $\widehat{w_1} \circ (\nu')^* = w_1$, and hence $\widehat{w_1} \circ \nu^* = \widehat{w_1} \circ (\nu')^*$. For w being nondegenerate, $\widehat{w_1}$ is injective and it follows that $\nu^* = (\nu')^*$. As a result, we have $\nu = \nu'$. □

Theorem 3.4. *Assume that w_1 is a nondegenerate ν_1-twisted superpotential of degree n, w_2 is a nondegenerate ν_2-twisted superpotential of degree n. Let $C = \langle w_1 \rangle$ and $D = \langle w_2 \rangle$. Then C and D are isomorphic as graded K-coalgebras if and only if $w_2 = w_1 \circ s^{\otimes n}$ and hence $\nu_2 = s^* \nu_1 (s^*)^{-1}$ for some $s \in \mathrm{GL}(V)$.*

P r o o f. Suppose

$$s(x_i) = \sum_{j=1}^{m} a_{ij} x_j, \quad 1 \leqslant i \leqslant m.$$

Then s^* acts on U as follows

$$s^*(y_i) = \sum_{j=1}^{m} a_{ji} y_j, \quad 1 \leqslant i \leqslant m.$$

Assume that

$$w_1 = \sum_{1 \leqslant i_1, i_2, \ldots, i_n \leqslant m} s_{i_1 i_2 \ldots i_n} y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n}.$$

If $w_2 = w_1 \circ s^{\otimes n}$, then,

$$w_2 = \sum_{1 \leqslant i_1, i_2, \ldots, i_n \leqslant m} \left(\sum_{1 \leqslant j_1, j_2, \ldots, j_n \leqslant m} s_{j_1 j_2 \ldots j_n} a_{i_1 j_1} \ldots a_{i_n j_n} \right) y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n}.$$

For each $1 \leqslant t \leqslant m$,

$$\begin{aligned} w_1 &= \sum_{1 \leqslant i_1, i_2, \ldots, i_n \leqslant m} s_{i_1 i_2 \ldots i_n} y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_n} \\ &= \sum_{1 \leqslant i_1, \ldots, i_t \leqslant m} y_{i_1} \otimes y_{i_2} \otimes \ldots \otimes y_{i_t} \\ &\quad \otimes \left(\sum_{1 \leqslant i_{t+1}, i_{t+2}, \ldots, i_n \leqslant m} s_{i_1 i_2 \ldots i_n} y_{i_{t+1}} \otimes y_{i_{t+2}} \otimes \ldots \otimes y_{i_n} \right). \end{aligned}$$

According to [2], Theorems 4.4 and 4.5, we have

$$C_{t-n} = \operatorname{span} \left\{ \sum_{1 \leqslant i_{t+1}, i_{t+2}, \ldots, i_n \leqslant m} s_{i_1 i_2 \ldots i_n} y_{i_{t+1}} \otimes y_{i_{t+2}} \otimes \ldots \otimes y_{i_n} : \ 1 \leqslant i_1, \ldots, i_t \leqslant m \right\}.$$

Similarly,

$$\begin{aligned} w_2 &= \sum_{i_1, i_2, \ldots, i_n = 1}^{m} \left(\sum_{j_1, j_2, \ldots, j_n = 1}^{m} s_{j_1 j_2 \ldots j_n} a_{i_1 j_1} \ldots a_{i_n j_n} \right) y_{i_1} \otimes \ldots \otimes y_{i_n} \\ &= \sum_{i_1, \ldots, i_t = 1}^{m} y_{i_1} \otimes \ldots \otimes y_{i_t} \\ &\quad \otimes \left(\sum_{i_{t+1}, i_{t+2}, \ldots, i_n = 1}^{m} \left(\sum_{j_1, \ldots, j_n = 1}^{m} s_{j_1 j_2 \ldots j_n} a_{i_1 j_1} \ldots a_{i_n j_n} \right) y_{i_{t+1}} \otimes \ldots \otimes y_{i_n} \right) \\ &= \sum_{i_1, \ldots, i_t = 1}^{m} y_{i_1} \otimes \ldots \otimes y_{i_t} \otimes \left(\sum_{j_1, \ldots, j_t = 1}^{m} a_{i_1 j_1} \ldots a_{i_t j_t} \right. \\ &\quad \left. \sum_{i_{t+1}, \ldots, i_n = 1}^{m} \sum_{j_{t+1}, \ldots, j_n = 1}^{m} s_{j_1 j_2 \ldots j_n} a_{i_{t+1} j_{t+1}} \ldots a_{i_n j_n} y_{i_{t+1}} \otimes \ldots \otimes y_{i_n} \right). \end{aligned}$$

Let

$$\begin{aligned} \alpha_{i_1 i_2 \ldots i_t} = &\sum_{j_1, \ldots, j_t = 1}^{m} a_{i_1 j_1} \ldots a_{i_t j_t} \\ &\sum_{i_{t+1}, \ldots, i_n = 1}^{m} \sum_{j_{t+1}, \ldots, j_n = 1}^{m} s_{j_1 j_2 \ldots j_n} a_{i_{t+1} j_{t+1}} \ldots a_{i_n j_n} y_{i_{t+1}} \otimes \ldots \otimes y_{i_n}, \end{aligned}$$

then

$$
\begin{aligned}
D_{t-n} &= \operatorname{span}\{\alpha_{i_1 i_2 \ldots i_t} \colon\ 1 \leqslant i_1, \ldots, i_t \leqslant m\} \\
&= \operatorname{span}\Bigg\{ \sum_{i_{t+1},\ldots,i_n=1}^{m} \sum_{j_{t+1},\ldots,j_n=1}^{m} s_{j_1 j_2 \ldots j_n} a_{i_{t+1} j_{t+1}} \ldots a_{i_n j_n} y_{i_{t+1}} \otimes \ldots \otimes y_{i_n}, \\
&\qquad\qquad 1 \leqslant i_1, \ldots, i_t \leqslant m \Bigg\}.
\end{aligned}
$$

Then for each $1 \leqslant j_1, \ldots, j_t \leqslant m$,

$$
\begin{aligned}
&(s^*)^{\otimes n-t}\Bigg(\sum_{1 \leqslant j_{t+1}, j_{t+2}, \ldots, j_n \leqslant m} s_{j_1 j_2 \ldots j_n} y_{j_{t+1}} \otimes y_{j_{t+2}} \otimes \ldots \otimes y_{j_n}) \\
&\quad = \sum_{1 \leqslant i_{t+1}, i_{t+2}, \ldots, i_n \leqslant m} \Bigg(\sum_{1 \leqslant j_{t+1}, j_{t+2}, \ldots, j_n \leqslant m} s_{j_1 j_2 \ldots j_n} a_{i_{t+1} j_{t+1}} \ldots a_{i_n j_n} \Bigg) y_{i_{t+1}} \otimes \ldots \otimes y_{i_n}.
\end{aligned}
$$

So $(s^*)^{\otimes n-t}$ is the bijection from C_{t-n} to D_{t-n}. So

$$
\begin{pmatrix}
(s^*)^{\otimes n} & & 0 \\
& (s^*)^{\otimes n-1} & \\
& & \ddots \\
& 0 & & 1
\end{pmatrix}
$$

is the graded K-coalgebra morphism from C to D.

Conversely, suppose that we are given a graded K-coalgebra isomorphism $f\colon C \to D$, where f_{-i} is from C_{-i} to D_{-i}. We obtain the commutative diagrams

$$
\begin{tikzcd}
C_{-i} \arrow[r, "f_{-i}"] \arrow[d] & D_{-i} \arrow[d] \\
U^{\otimes i} \arrow[r, "(f_{-1})^{\otimes i}"] & U^{\otimes i}
\end{tikzcd}
$$

for all i, where the vertical maps are the natural injection.

Naturally, we have $K \cong C_{-n}$, $D_{-n} \cong K$, then we can obtain an isomorphism from K to K, denoted by φ. Let $c = \varphi(1)$, then we have $\varphi = c(\mathrm{id}_K)$. We have the commutative diagram

$$
\begin{tikzcd}
K \arrow[r, "c(\mathrm{id}_K)"] \arrow[d, "\cong"'] & K \arrow[d, "\cong"] \\
C_{-n} \arrow[r, "f_{-n}"] \arrow[d] & D_{-n} \arrow[d] \\
U^{\otimes n} \arrow[r, "(f_{-1})^{\otimes n}"] & U^{\otimes n}.
\end{tikzcd}
$$

Let us put $s' = t^{-1}f_{-1} \in \mathrm{GL}(U)$, $t \in K$, $t^n = c$. After that, composing vertical maps in the above diagram, we have the commutative diagram

$$\begin{tikzcd} K \arrow[r, "\mathrm{id}"] \arrow[d, "w_1^*"'] & K \arrow[d, "w_2^*"] \\ U^{\otimes n} \arrow[r, "s'^{\otimes n}"] & U^{\otimes n}. \end{tikzcd}$$

Let $s = (s')^*$. Then $(w_2)^* = (s^*)^{\otimes n} \circ (w_1)^*$. As a result, we have $w_2 = w_1 \circ (s)^{\otimes n}$. □

Remark 3.5. For two given twisted superpotentials w_1, w_2, if $\langle w_1 \rangle$ and $\langle w_2 \rangle$ are isomorphic as graded K-coalgebras, we say w_1 and w_2 are equivalent and denote it by $w_1 \sim w_2$.

Corollary 3.6. *Retain the notation as above. Let $A = C^*$, $B = D^*$, which are the corresponding Frobenius algebras of w_1, w_2, respectively. Then A and B are isomorphic as graded K-algebras if and only if $w_2 = w_1 \circ s^{\otimes n}$ for some $s \in \mathrm{GL}(V)$.*

Corollary 3.7. *Assume that w_1, w_2 are nondegenerate ν-twisted superpotentials of degree n. Then $w_1 \sim w_2$ if and only if $w_2 = w_1 \circ s^{\otimes n}$ and hence $\nu = s^*\nu(s^*)^{-1}$ for some $s \in \mathrm{GL}(V)$.*

4. Connected graded Frobenius algebra of length 3

According to Theorem 3.4, we can classify the connected graded Frobenius algebras through computing the twisted superpotentials under different matrices which are not similar. For a given matrix, it is possible that the corresponding superpotential is not the only one, so it is necessary to discuss the graded Frobenius algebras corresponding to different twisted superpotentials under the circumstances. In this section, we focus on the connected graded Frobenius algebra of length 3, whose dimension of degree 1 is 2. It can be easily obtained that the dimension of the graded Frobenius algebra in this case is 6 by Lemma 2.1.

Assume that U is a vector space over K with a basis $\{y_1, y_2\}$, and let ν be a linear automorphism of U. Let $V = U^*$, $x_1 = (y_1)^*$, $x_2 = (y_2)^*$. Then $\{x_1, x_2\}$ is the dual basis of V. To simplify notation, we omit the symbol "$\otimes$" in the tensor coalgebra $T(U)$ and simply write y_1y_2 for an element $y_1 \otimes y_2 \in U^{\otimes 2}$. The field K considered in this section is the complex number field $\mathbb{C}$. Let $K\{x_1, x_2\}$ denote the binary free algebra. And by a ν-twisted superpotential, we always mean a ν-twisted superpotentials of degree 3.

Let

$$w = \sum_{1\leqslant i_1,i_2,i_3\leqslant 2} k_{i_1i_2i_3} y_{i_1} y_{i_2} y_{i_3} \in U\otimes U\otimes U.$$

Then w is a ν-twisted superpotential if and only if

$$(\nu\otimes 1\otimes 1)(w) = \mathfrak{c}_3(w). \tag{4.1}$$

Let M be the matrix of ν associated to the basis $\{y_1, y_2\}$. We may assume that M is a Jordan-type matrix (by choosing a suitable basis if necessary). Moreover, we can discuss the twisted superpotential when M has the form $\left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\lambda & 0\\ 1 & \lambda\end{smallmatrix}\right)$.

Case 1: M is a diagonal matrix. Set

$$M = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

Naturally, the matrix $\left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$ is similar to $\left(\begin{smallmatrix}\lambda_2 & 0\\ 0 & \lambda_1\end{smallmatrix}\right)$.

By calculating equation (4.1), we obtain that if M satisfies the following situation, w is equal to 0:

(1) $\lambda_1 = 1$, $\lambda_2 \neq \pm 1$,
(2) $\lambda_2 = 1$, $\lambda_1 \neq \pm 1$,
(3) $\lambda_1, \lambda_2 \neq 1$, $\lambda_1^2\lambda_2 \neq 1$, $\lambda_2^2\lambda_1 \neq 1$.

Then we conclude that the nondegenerate ν-twisted superpotential exists in four situations as follows.

Subcase 1.1: $M = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. In this case, for each $s \in \mathrm{GL}(V)$ the equation $\nu = s^*\nu(s^*)^{-1}$ holds. According to equation (4.1), we have

$$w = \mathfrak{c}_3(w). \tag{4.2}$$

Assuming that

$$w = \sum_{1\leqslant i_1,i_2,i_3\leqslant 2} k_{i_1i_2i_3} y_{i_1} y_{i_2} y_{i_3} \in U\otimes U\otimes U,$$

we can obtain that

$$\sum_{1\leqslant i_1,i_2,i_3\leqslant 2} k_{i_1i_2i_3} y_{i_1} y_{i_2} y_{i_3} = \sum_{1\leqslant i_1,i_2,i_3\leqslant 2} k_{i_2i_3i_1} y_{i_1} y_{i_2} y_{i_3}. \tag{4.3}$$

Then the nondegenerate ν-twisted superpotential has the form

$$w = a_1y_1y_1y_1 + a_2y_2y_2y_2 + a_3(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) + a_4(y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1),$$

where a_3 and a_4 are not simultaneously zero. If $a_3a_4 \neq 0$, at least two of the following equations are not valid:

$$a_2a_3 = (a_4)^2, \quad a_1a_4 = (a_3)^2, \quad a_1a_2 = a_3a_4.$$

Otherwise, w is not nondegenerate. We want to discuss the corresponding algebra in the following four situations.

Subcase 1.1.1: Assume that at least two of a_1, a_2, a_3, a_4 are 0. There are three kinds of nondegenerate ν-twisted superpotentials which are not equivalent for each $0 \neq k_1 \in K$,

$$\begin{aligned} w_1 &= y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1, \\ w_2 &= k_1y_2y_2y_2 + y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1, \\ w_3 &= y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1 + k_1(y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1). \end{aligned}$$

Let $C = \langle w_1 \rangle$ be the graded subcoalgebras of $T(U)$ generated by w_1. According to [2], Lemma 4.8, we can obtain that

$$C = C_{-3} \oplus C_{-2} \oplus C_{-1} \oplus C_0,$$

where $C_{-3} = \mathrm{span}\{w_1\}$, $C_{-2} = \mathrm{span}\{y_1y_2 + y_2y_1, y_1y_1\}$, $C_{-1} = U$, $C_0 = K$. Consequentially, the corresponding Frobenius algebra

$$A = C^* = K \oplus V \oplus A_2 \oplus A_3,$$
$$A_2 = (C_{-2})^* = V^{\otimes 2} \,/\, R_2, \quad R_2 = L(x_1x_2 - x_2x_1, x_2x_2),$$

where $L(x_1x_2 - x_2x_1, x_2x_2)$ is the subspace generated by $\{x_1x_2 - x_2x_1, x_2x_2\}$,

$$A_3 = (C_{-3})^* = V^{\otimes 3} \,/\, R_3, \quad R_3 = R_2 \otimes V + V \otimes R_2 + L(x_1x_1x_1).$$

And we have $A \cong K\{x_1, x_2\} \,/\, (x_1x_2 - x_2x_1, x_2x_2, x_1x_1x_1)$.

Let $s_1 \in \mathrm{GL}(V)$ be defined by

$$s_1(x_1) = x_1, \quad s_1(x_2) = \frac{1}{a}x_2,$$

where $a^2 = k_1$. Then

$$w_2 \sim w_2 \circ (s_1)^{\otimes 3} = \frac{1}{a}(y_2y_2y_2 + y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1).$$

Naturally,

$$\frac{1}{a}(y_2y_2y_2 + y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) \sim y_2y_2y_2 + y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1.$$

It shows that the corresponding Frobenius algebras of w_2 are independent of the coefficient k_1.

The corresponding Frobenius algebra of w_2 is

$$A \cong K\{x_1, x_2\} \,/\, (x_1x_2 - x_2x_1, x_2x_2 - x_1x_1, x_1x_1x_1).$$

Let $s_2 \in \mathrm{GL}(V)$ be defined by

$$s_2(x_1) = bx_2, \quad s_2(x_2) = \frac{b}{k_1}x_1,$$

where $b^3 = k_1$. Then

$$w_3 \sim w_3 \circ (s_2)^{\otimes 3} = y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1 + y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1.$$

It shows that the corresponding Frobenius algebras of w_3 are independent of the coefficient k_1 as well.

The corresponding Frobenius algebra of w_3 is

$$A \cong K\{x_1, x_2\} \,/\, (x_1x_2 - x_2x_1, x_2x_2 + x_1x_1 - x_1x_2, x_2x_2x_2).$$

Subcase 1.1.2: Assume that $a_1a_2 \neq 0$, $a_3a_4 = 0$. In the sense of equivalence, there is only one kind of nondegenerate ν-twisted superpotentials as follows: for each $0 \neq k_2, k_3 \in K$,

$$w_4 = k_2y_1y_1y_1 + k_3y_2y_2y_2 + y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1.$$

The corresponding Frobenius algebra of w_4 is

$$A \cong K\{x_1, x_2\} \,/\, \left(x_1x_2 - x_2x_1, x_1x_1 - k_2x_1x_2 - \frac{1}{k_3}x_2x_2, x_1x_2x_2\right).$$

Subcase 1.1.3: Assume that $a_3a_4 \neq 0$, $a_1a_2 = 0$, $a_1 + a_2 \neq 0$. In the sense of equivalence, there is only one kind of nondegenerate ν-twisted superpotentials as follows: for each $0 \neq k_4, k_5 \in K$,

$$w_5 = y_1y_1y_1 + k_4(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) + k_5(y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1).$$

The corresponding Frobenius algebra of w_5 is

$$A \cong K\{x_1, x_2\} \,/\, \left(x_1x_2 - x_2x_1, x_1x_1 - \left(\frac{1}{k_5} - \left(\frac{k_4}{k_5}\right)^2\right)x_2x_2 - \frac{k_4}{k_5}x_1x_2, x_2x_2x_2\right).$$

Subcase 1.1.4: Assume that $a_1a_2 \neq 0$, $a_3a_4 \neq 0$, there are three kinds of nondegenerate ν-twisted superpotentials as follows: for each $0 \neq k_7, k_8, k_9, k_{10}, k_{11} \in K$, $k_7(k_8)^2 \neq 1$, $k_{10}k_{11} \neq 1$, $k_9 \neq (k_{11})^2$,

$$\begin{aligned}
w_6 &= k\Big(k_7y_1y_1y_1 + k_8y_2y_2y_2 + \frac{1}{k_8}(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) \\
&\quad + (y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1)\Big), \\
w_7 &= k((k_8)^2y_1y_1y_1 + k_7k_8y_2y_2y_2 + k_8(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) \\
&\quad + (y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1), \\
w_8 &= k(k_9y_1y_1y_1 + k_{10}y_2y_2y_2 + k_{11}(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) \\
&\quad + (y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1)).
\end{aligned}$$

Let $s_7 \in \mathrm{GL}(V)$ be defined by

$$s_3(x_1) = ax_2, \quad s_3(x_2) = ax_1,$$

where $a^3 = k_8$. Then we can obtain that $w_7 = w_6 \circ (s_3)^{\otimes 3}$, $w_7 \sim w_6$.

The corresponding Frobenius algebra of w_6 is

$$A \cong K\{x_1, x_2\} \diagup (x_1x_2 - x_2x_1, x_2x_2 - k_8x_1x_2, x_1x_1x_1 - k_7x_1x_2x_2).$$

The corresponding Frobenius algebra of w_8 is

$$\begin{aligned}
A \cong K\{x_1, x_2\} \diagup \Big(&x_1x_2 - x_2x_1, x_2x_2 - \frac{1 - k_{10}k_{11}}{k_9 - (k_{11})^2}x_1x_1 \\
&+ \frac{k_{11} - k_9k_{10}}{k_9 - (k_{11})^2}x_1x_2, x_1x_1x_2 - k_{11}x_1x_2x_2\Big).
\end{aligned}$$

Remark 4.1. Assume that

$$\begin{aligned}
w &= a_1y_1y_1y_1 + a_2y_2y_2y_2 + a_3(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) \\
&\quad + a_4(y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1), \\
w' &= a'_1y_1y_1y_1 + a'_2y_2y_2y_2 + a'_3(y_1y_2y_1 + y_1y_1y_2 + y_2y_1y_1) \\
&\quad + a'_4(y_1y_2y_2 + y_2y_1y_2 + y_2y_2y_1)
\end{aligned}$$

are different ν-twisted superpotentials. Then $w \sim w'$ if and only if the following system of equations has a solution:

$$\begin{aligned}
&a_1a^3 + a_2c^3 + 3a_3a^2c + 3a_4c^2a = a'_1, \\
&a_1b^3 + a_2d^3 + 3a_3b^2d + 3a_4d^2b = a'_2, \\
&a_1a^2b + a_2c^2d + a_3(2abc + a^2d) + a_4(2acd + c^2b) = a'_3, \\
&a_1b^2a + a_2d^2c + a_3(2abd + b^2c) + a_4(2bcd + d^2a) = a'_4.
\end{aligned} \tag{4.4}$$

The system of equations is obtained by the equation $w' = w \circ s^{\otimes 3}$, where $s \in \mathrm{GL}(V)$ is defined by

$$s(x_1) = ax_1 + cx_2, \quad s(x_2) = bx_1 + dx_2.$$

As we can see, this system of equations is difficult to solve, so we can not conclude whether any two nondegenerate ν-twisted superpotentials in subcases 1.1.2, 1.1.3, 1.1.4 are equivalent or not.

Subcase 1.2: $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. In this case, the equation $\nu = s^* \nu (s^*)^{-1}$ holds if and only if s is defined by

$$s(x_1) = ax_1, \quad s(x_2) = bx_2,$$

where $a, b, \in K$, $ab \neq 0$.

There are two kinds of nondegenerate ν-twisted superpotentials which are not equivalent as follows: for each $0 \neq k_1, k_2 \in K$,

$$\begin{aligned} w_1 &= k_1 y_1 y_1 y_1 + k_2 (y_2 y_1 y_2 - y_1 y_2 y_2 - y_2 y_2 y_1), \\ w_2 &= y_2 y_1 y_2 - y_1 y_2 y_2 - y_2 y_2 y_1. \end{aligned}$$

Let $s \in \mathrm{GL}(V)$ be defined by

$$s(x_1) = \frac{1}{a} x_1, \quad s(x_2) = \frac{1}{b} x_2,$$

where a, b are the solution of the following system of equations:

$$a^3 = k_1, \quad ab^2 = k_2. \tag{4.5}$$

Then

$$w_1 \sim w_1 \circ (s)^{\otimes 3} = y_1 y_1 y_1 + y_2 y_1 y_2 - y_1 y_2 y_2 - y_2 y_2 y_1.$$

It turns out that the corresponding Frobenius algebras of w_1 are independent of the coefficients k_1 and k_2.

The corresponding Frobenius algebra of w_1 is

$$A \cong K\{x_1, x_2\} \,/\, (x_1 x_2 + x_2 x_1, x_1 x_1 + x_2 x_2, x_2 x_2 x_2).$$

The corresponding Frobenius algebra of w_2 is

$$A \cong K\{x_1, x_2\} \,/\, (x_1 x_2 + x_2 x_1, x_1 x_1, x_2 x_2 x_2).$$

Subcase 1.3: $M = c\left(\begin{smallmatrix} 1/\lambda^2 & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, where $\lambda^3 \neq 1$, $\lambda \neq -1$. In the sense of equivalence, there is only one kind of nondegenerate ν-twisted superpotential as follows:

$$w = y_2y_1y_2 + \lambda y_1y_2y_2 + \frac{1}{\lambda}y_2y_2y_1.$$

The corresponding Frobenius algebra of w is

$$A \cong K\{x_1, x_2\} \,/\, \left(x_2x_1 - \frac{1}{\lambda}x_1x_2, x_1x_1, x_2x_2x_2\right).$$

Subcase 1.4: $M = \left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi \end{smallmatrix}\right)$, where $\xi \neq 1$, $\xi^3 = 1$. In the sense of equivalence, there is only one kind of nondegenerate ν-twisted superpotentials as follows:

$$w = y_1y_1y_2 + \xi y_1y_2y_1 + \xi^2 y_2y_1y_1.$$

The corresponding Frobenius algebra of w is

$$A \cong K\{x_1, x_2\} \,/\, (x_1x_2 - \xi x_2x_1, x_1x_1, x_2x_2x_2).$$

Case 2: M is not a diagonal matrix. Set

$$M = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}. \tag{4.6}$$

By calculating equation (4.1), we obtain that if M satisfies the following situations, w is either equal to 0 or is not nondegenerate.

(1) $\lambda = 1$, the ν-twisted superpotential $w = ky_2y_2y_2$ is not nondegenerate.

(2) $\lambda^3 \neq 1$, the ν-twisted superpotential $w = 0$.

Accordingly, the ν-twisted superpotential w is nondegenerate only in the following case:

$$M = \begin{pmatrix} \xi & 0 \\ 1 & \xi \end{pmatrix},$$

where $\xi^3 = 1$, $\xi \neq 1$. In the sense of equivalence, there is only one kind of nondegenerate ν-twisted superpotential as follows:

$$w = y_1y_2y_2 + \xi y_2y_2y_1 + \xi^2 y_2y_1y_2 + \frac{1}{1-\xi}y_2y_2y_2.$$

The corresponding Frobenius algebra of w is

$$A \cong K\{x_1, x_2\} \,/\, \left(x_1x_2 - \xi x_2x_1, x_1x_1, x_2x_2x_2 - \frac{1}{1-\xi}x_1x_2x_2\right).$$

On the whole, we can obtain the following theorem.

Theorem 4.2. *Let A be a connected graded Frobenius algebra of length 3 over the complex number field $\mathbb{C}$, whose dimension of degree 1 is 2. Then A is isomorphic to one of the Frobenius algebras in Case 1 and Case 2.*

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