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ON THE STRUCTURE OF THE 2-IWASAWA MODULE OF SOME NUMBER FIELDS OF DEGREE 16

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Abstract. Let K be an imaginary cyclic quartic number field whose 2-class group is of type $(2,2,2)$, i.e., isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. The aim of this paper is to determine the structure of the Iwasawa module of the genus field $K^{(*)}$ of K.

Keywords: cyclic quartic field; cyclotomic $\mathbb{Z}_2$-extension; 2-Iwasawa module; 2-class group; 2-rank

MSC 2020: 11R16, 11R18, 11R20, 11R23, 11R29

1. Introduction

Let k be an algebraic number field and p be a prime number. A $\mathbb{Z}_p$-extension of k is an extension k_∞/k with $\mathrm{Gal}(k_\infty/k)\simeq\mathbb{Z}_p$, the additive group of p-adic integers. It is also possible to regard a $\mathbb{Z}_p$-extension as a sequence of fields

$$k=k_0\subset k_1\subset\ldots\subset k_\infty=\bigcup_{n\geqslant 0}k_n\quad\text{with }\mathrm{Gal}(k_n/k)\simeq\mathbb{Z}/p^n\mathbb{Z}.$$

Note that the field k_n is called the nth *layer* of $\mathbb{Z}_p$-extension of k. Let A_n be the p-part of the class group of k_n. From the beautiful theorem of Iwasawa (see [12], Theorem 13.13, page 276), there exist integers $\lambda,\mu\geqslant 0$ and ν, all independent of n, and n_0 such that

$$|A_n|=p^{\lambda n+\mu p^n+\nu}\quad\text{for all }n\geqslant n_0.$$

The integers $\lambda,\mu\geqslant 0$ and ν are called the *Iwasawa invariants* of k_∞. Let A_∞ denote the projective limit of A_n. It is not easy to give the structure or an explicit description of the p-Iwasawa module A_∞ which can be finite as well as infinite. It is one of classical and difficult problems in the Iwasawa theory. However, Greenberg conjectured that A_∞ is finite if k is totally real, cf. [7], page 263.

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Let K be an imaginary cyclic quartic number field whose 2-class group is of type $(2, 2, 2)$. In the present paper, we first give explicitly the structure of the 2-Iwasawa module A_∞ of the genus field of K, as a result. Next, we give some preliminary results that will be useful in the proof. Finally, we prove our result using Kida's formula.

2. Notations

Let k be a number field and p be a prime number. The next notations are used for the rest of this article:

- ▷ n: an integer $\geqslant 0$;
- ▷ $\mathbb{Q}_n$: the maximal real subfield of $\mathbb{Q}(\zeta_{2^{n+2}})$;
- ▷ k_n: the nth layer of the $\mathbb{Z}_2$-extension of k;
- ▷ $k_\infty = \bigcup\limits_{n\geqslant 0} k_n$;
- ▷ L_n: the Hilbert 2-class field of k_n;
- ▷ $X_n = \mathrm{Gal}(L_n/k_n)$;
- ▷ $X_\infty = \varprojlim X_n$;
- ▷ A_n: the 2-part of the class group of k_n;
- ▷ $A_\infty = \varprojlim A_n$;
- ▷ τ: a topological generator of $\mathrm{Gal}(k_\infty/k)$;
- ▷ $\Lambda = \mathbb{Z}_2[\![T]\!]$ for $T = \tau - 1$;
- ▷ $\omega_n = (T+1)^{2^n} - 1$;
- ▷ $\mu(M), \lambda(M)$: the Iwasawa invariants for a Λ-torsion module M;
- ▷ $\mu(k) = \mu(A_\infty)$;
- ▷ $\lambda(k) = \lambda(A_\infty)$;
- ▷ $\lambda^-(k) = \lambda(A_\infty^-)$ (the definition of A_∞^- is given in Section 4);
- ▷ $h(k)$: the class number of k;
- ▷ $h_n = h(k_n)$;
- ▷ E_k: the unit group of k;
- ▷ W_k: the group of roots of unity contained in k;
- ▷ k^+: the maximal real subfield of a CM-field k;
- ▷ $Q_k = [E_k : W_k E_{k^+}]$: the Hasse's unit index of a CM-field k;
- ▷ $N_{L/k}$: the relative norm for an extension L/k;
- ▷ $\mathbf{C}_k(2)$: the 2-part of the class group of k;
- ▷ $\left(\frac{x}{\mathfrak{p}}\right)$: the quadratic residue symbol for k;
- ▷ $\left(\frac{x,y}{\mathfrak{p}}\right)$: the Hilbert symbol for k;
- ▷ $\left(\frac{a}{p}\right)$: the quadratic residue (Legendre) symbol;
- ▷ $\left(\frac{a}{p}\right)_4$: the biquadratic residue symbol.

3. Main theorem

Let q and l be two primes satisfying the conditions

$$(1)\qquad q\equiv 3 \pmod 4,\quad l\equiv 5 \pmod 8,\quad \left(\frac{q}{l}\right)=1,\quad \text{and}\quad \left(\frac{q}{l}\right)_4=1.$$

Denote by ε the fundamental unit of $\mathbb{Q}(\sqrt{l})$. Let $K=\mathbb{Q}\Big(\sqrt{-q\varepsilon\sqrt{l}}\Big)$ be an imaginary cyclic quartic field. From [2], Theorem 3, page 66, we have that the 2-class group $\mathbf{C}_K(2)$ of K is of type $(2,2,2)$.

Definition 3.1. The *genus field* $k^{(*)}$ of a number field k is the maximal abelian extension of k, which is a composite of an absolute abelian number field F with k and is unramified at all the finite and infinite primes of k.

Lemma 3.2. *Let $q\equiv 3 \pmod 4$ and $l\equiv 5 \pmod 8$ be two primes. Then the genus field of $K=\mathbb{Q}\Big(\sqrt{-q\varepsilon\sqrt{l}}\Big)$ is $K^{(*)}=K(\sqrt{q},\sqrt{-1})$.*

Proof. As l and q are the unique primes of $\mathbb{Q}$ different from 2, which ramify in K, of ramification indices $e_l=4$ and $e_q=2$, respectively; then, from [8], Theorem 4, pages 48–49, we have $K^{(*)}=M_lM_qK$, where M_l (or M_q) is the unique subfield of the lth (or qth) cyclotomic number field $\mathbb{Q}(\zeta_l)$ (or $\mathbb{Q}(\zeta_q)$) of degree $e_l=4$ (or $e_q=2$, respectively). Moreover, it is known that $M_l=\mathbb{Q}\Big(\sqrt{-\varepsilon\sqrt{l}}\Big)$ (cf. [10], Proposition 5.9, page 160) and $M_q=\mathbb{Q}(\sqrt{-q})$. Thus, $K^{(*)}=K(\sqrt{q},\sqrt{-1})$.

□

The main result of this paper is the following theorem.

Theorem 3.3. *Let q and l be two primes satisfying the conditions* (1). *Let A_n denote the 2-class group of the nth layer of the cyclotomic $\mathbb{Z}_2$-extension of the genus field $K^{(*)}=K(\sqrt{q},\sqrt{-1})$. Then:*

(1) *The structure of the Iwasawa module A_∞ is given by*

$$A_\infty\simeq\begin{cases}\mathbb{Z}_2^3 & \text{if } q\equiv 3 \pmod 8,\\ \mathbb{Z}_2^7 & \text{if } q\equiv 7 \pmod{16}.\end{cases}$$

(2) *The* 2-rank *of A_n is given by*

$$\operatorname{rank}_2(A_n)=\begin{cases}3 \text{ for all } n\geqslant 3 & \text{if } q\equiv 3 \pmod 8,\\ 7 \text{ for all } n\geqslant 7 & \text{if } q\equiv 7 \pmod{16}.\end{cases}$$

4. Preliminary results

Let us first collect some results that will be useful in the sequel.

Proposition 4.1 ([4], page 3). *Let $n \geqslant 2$ be a positive integer. Then we have:*

(1) *If p is a prime such that $p \equiv 3 \pmod 8$, then p decomposes into the product of 2 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$ while it is inert in $\mathbb{Q}_n$.*

(2) *If p is a prime such that $p \equiv 7 \pmod{16}$, then p decomposes into the product of 4 prime ideals of $\mathbb{Q}(\zeta_{2^{n+2}})$ while it decomposes into the product of 2 prime ideals of $\mathbb{Q}_n$.*

Definition 4.2. Let K/k be a cyclic extension of number fields of prime degree p and $\mathrm{Gal}(K/k) = \langle\sigma\rangle$.

(1) An *ideal* $\mathfrak{a}$ of K is called *ambiguous* (with respect to k), if it is fixed by σ: $\mathfrak{a}^\sigma = \mathfrak{a}$.

(2) An *ideal class* $[\mathfrak{a}]$ of K is called *ambiguous* (with respect to k), if it is fixed by σ: $[\mathfrak{a}]^\sigma = [\mathfrak{a}]$.

(3) An *ideal class* $[\mathfrak{a}]$ of K is called *strongly ambiguous* (with respect to k), if it contains an ambiguous ideal.

Let us define A_n^+ as the group of strongly ambiguous classes with respect to the extension k_n/k_n^+, where k_n^+ is the totally real subfield of k_n and $A_n^- = A_n/A_n^+$. Let A_∞^- denote the projective limit of A_n^-. We have:

Theorem 4.3 ([11], Theorem 2.5, page 374). *Let k be a CM-field containing the fourth roots of unity. Then there is no finite Λ-submodule in A_∞^-.*

Lemma 4.4. *If the extension k_n/k_n^+ is unramified and $h(k_n^+)$ is odd for all $n \geqslant 0$, then $A_\infty^- = A_\infty$.*

Proof. By the definition of the part plus A_n^+, it is clear that A_n^+ is generated by the ramified primes and the inert primes in k_n/k_n^+. Since the extension k_n/k_n^+ is unramified and $h(k_n^+)$ is odd, then A_n^+ is trivial. Therefore, $A_n^- = A_n$. In the projective limit we obtain $A_\infty^- = A_\infty$. □

Theorem 4.5 ([9], Theorem 3, page 341). *Let L/F be a finite 2-extension of abelian CM-fields. Then we have*

$$\lambda^-(L) - \delta(L) = [L_\infty : F_\infty] \cdot (\lambda^-(F) - \delta(F)) + \sum_{\beta \nmid 2}(e_\beta - 1) - \sum_{\beta^+ \nmid 2}(e_{\beta^+} - 1), \tag{2}$$

where $\delta(k)$ takes the values 1 or 0 according to whether k_∞ contains the fourth roots of unity or not, and e_β (or e_β^+) is the ramification index in L_∞/F_∞ (or L_∞^+/F_∞^+) of a finite prime β of L_∞ (or β^+ of L_∞^+, respectively).

Theorem 4.6 ([3], Theorem 3.3, page 8)**.** *Let* k_∞ *be a* $\mathbb{Z}_2$*-extension of a number field* k *and assume that any prime of* k *lying above 2 is totally ramified in* k_∞/k*. If* $\mu(k)=0$ *and* A_∞ *is an elementary* Λ*-module, then* $\mathrm{rank}_2(A_n)=\lambda(k)$ *for all* $n \geqslant \lambda(k)$*.*

Proposition 4.7 ([12], Proposition 13.22, page 284)**.** *Let* k_∞ *be a* $\mathbb{Z}_2$*-extension of a number field* k *and assume that there exists only one prime of* k *lying above 2 and that this prime is totally ramified in* k_∞/k*. Then*

$$A_n \simeq X_\infty/\omega_n X_\infty \quad \text{and} \quad 2 \nmid h_0 \Leftrightarrow 2 \nmid h_n \text{ for all } n \geqslant 0.$$

Proposition 4.8 ([1], pages 270–271)**.** *Let* q *and* l *be two primes satisfying the conditions* (1)*, and consider* $L=\mathbb{Q}\big(\sqrt{\varepsilon\sqrt{l}},\sqrt{-1}\big)$ *and* $F=\mathbb{Q}\big(\sqrt{\varepsilon\sqrt{l}},\sqrt{q}\big)$*. Then we have:*

(1) *The class number* $h(L^+)$ *of* L^+ *is odd. Moreover,* Q_L *the Hasse's unit index of* L *equals 2 and* $h(L)$ *is odd too.*

(2) *The class number* $h(F)$ *of* F *is odd.*

4.1. Quadratic residue symbol and Hilbert symbol. Let k be a number field. The *quadratic residue symbol* is defined as follows: let $\mathfrak{p}$ be a prime ideal of k. For all $x \in k^*$,

$$\left(\frac{x}{\mathfrak{p}}\right)=\begin{cases} 1 & \text{if } x \text{ is a square in } k \text{ or if } \mathfrak{p} \text{ splits in } k\big(\sqrt{x}\big),\\ -1 & \text{if } x \text{ is not a square in } k \text{ and } \mathfrak{p} \text{ remains inert in } k\big(\sqrt{x}\big),\\ 0 & \text{if } x \text{ is not a square in } k \text{ and } \mathfrak{p} \text{ ramifies in } k\big(\sqrt{x}\big).\end{cases}$$

Lemma 4.9 ([6], page 205)**.** *If the prime ideal* $\mathfrak{p}$ *is unramified in the extension* $k\big(\sqrt{x}\big)/k$*, the quadratic residue symbol can be written in terms of Artin symbols as*

$$\left(\frac{x}{\mathfrak{p}}\right)=\left(\frac{k\big(\sqrt{x}\big)/k}{\mathfrak{p}}\right)\big(\sqrt{x}\big)/\sqrt{x}.$$

Proposition 4.10 ([10], Proposition 4.2, page 112)**.** *Let* K *be a finite normal extension of* k*,* $\mathfrak{p}$ *be a prime ideal of* k *and* $\mathfrak{P}$ *be a prime ideal of* K *dividing* $\mathfrak{p}$*.*

(1) *If the inertia degree* $\mathfrak{f}(\mathfrak{P}/\mathfrak{p})=1$*, then for all* $x \in k^*$

$$\left(\frac{x}{\mathfrak{P}}\right)=\left(\frac{x}{\mathfrak{p}}\right).$$

(2) *If* K/k *is abelian and* $\mathfrak{f}(\mathfrak{P}/\mathfrak{p})=[K:k]$*, then for all* $y \in K^*$

$$\left(\frac{y}{\mathfrak{P}}\right)=\left(\frac{N_{K/k}(y)}{\mathfrak{p}}\right).$$

Remark 4.11. For $k = \mathbb{Q}$, the quadratic residue symbol defines the Legendre symbol.

We now define the *Hilbert symbol* of number field k in terms of Hasse symbols by

$$\left(\frac{x, y}{\mathfrak{p}}\right) = \left(\frac{y, k(\sqrt{x})/k}{\mathfrak{p}}\right)(\sqrt{x})/\sqrt{x},$$

where $x, y \in k^*$ and $\mathfrak{p}$ is a prime ideal of k.

Proposition 4.12 ([6], page 106). *Let K/k be a finite extension, $x \in k^*$ and $y \in K^*$. Let $\mathfrak{p}$ denote a prime ideal of k and $\mathfrak{P}$ denote a prime ideal of K. Then*
(1) $\left(\frac{x,y}{\mathfrak{P}}\right) = \left(\frac{x}{\mathfrak{P}}\right)^{v_{\mathfrak{P}}(y)}$ *if $\mathfrak{P}$ is unramified in $K(\sqrt{x})$,*
(2) $\prod\limits_{\mathfrak{P}|\mathfrak{p}} \left(\frac{x,y}{\mathfrak{P}}\right) = \left(\frac{x, N_{K/k}(y)}{\mathfrak{p}}\right)$.

For more details, see [6], [10].

5. Proof of the main theorem

In this section, we prove the main result of this paper. Recall that ε is the fundamental unit of $\mathbb{Q}(\sqrt{l})$. Let $L = \mathbb{Q}\left(\sqrt{m\varepsilon\sqrt{l}}\right)$ be a real cyclic quartic field with m being a square free integer. We need the following results.

Theorem 5.1. *The class number of L is odd if and only if m takes one of the following forms:*
(1) *m is a prime p congruent to* 3 (mod 4),
(2) *m is an even prime,*
(3) *m is equal to* 1.

P r o o f. Let us look at the forms of m such that the class number $h(L)$ of L is odd; to this end, assume that $h(L)$ is odd. Then, from [5], page 25, the 2-rank of the class group $\mathbf{C}_L$ of L is given by the formula

$$\operatorname{rank}_2(\mathbf{C}_L) = t - 1 - e = 0, \tag{3}$$

where t is the number of primes of $\mathbb{Q}(\sqrt{l})$ which ramify in L and

$$2^e = [E_{\mathbb{Q}(\sqrt{l})} \colon E_{\mathbb{Q}(\sqrt{l})} \cap N_{L/\mathbb{Q}(\sqrt{l})}(L^\times)].$$

In the following, we compute the value of e. Recall that an element x of $\mathbb{Q}(\sqrt{l})^\times$ is a norm in L if $x \in N_{L/\mathbb{Q}(\sqrt{l})}(L^\times)$. So, by [6], Hasse's norm theorem, page 179, x is a norm in L if and only if $\left(\frac{x, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) = 1$ for all prime ideals $\mathfrak{p}$ of $\mathbb{Q}(\sqrt{l})$. So we have:

(1) Let r be a positive integer and $\alpha \in \{1,2\}$. Then -1 is a norm in L if and only if m takes one of the following forms:

(a) $m = \alpha \prod_{i=1}^{r} p_i$ such that $\left(\frac{p_i}{l}\right) = -1$, where p_i is a prime;

(b) $m = \alpha \prod_{i=1}^{r} p_i$ such that $p_i \equiv 1 \pmod 4$ and $\left(\frac{p_i}{l}\right) = 1$, where p_i is a prime;

(c) $m = \alpha \prod_{i=1}^{s} q_i \cdot \prod_{i=s+1}^{r} p_i$ such that $p_i \equiv 1 \pmod 4$ and $\left(\frac{p_i}{l}\right) = -\left(\frac{q_i}{l}\right) = 1$, where q_i and p_i are two primes;

(d) $m = \alpha$.

In fact:

(i) If $\mathfrak{p} \nmid l$ and $\mathfrak{p} \nmid m$, then $v_{\mathfrak{p}}\left(m\varepsilon\sqrt{l}\right) = 0$, so

$$\begin{aligned}\left(\frac{-1, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) &= \left(\frac{-1}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(m\varepsilon\sqrt{l})} \quad \text{(by Proposition 4.12 (1))}\\ &= 1.\end{aligned}$$

(ii) If $\mathfrak{p} \mid l$, then $\mathfrak{p} = \left(\sqrt{l}\right)$ and $v_{\mathfrak{p}}(m\varepsilon\sqrt{l}) = 1$. So

$$\begin{aligned}\left(\frac{-1, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) &= \left(\frac{-1}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(m\varepsilon\sqrt{l})} \quad \text{(by Proposition 4.12 (1))}\\ &= \left(\frac{-1}{\sqrt{l}}\right)\\ &= \left(\frac{-1}{l}\right) \quad \text{(by Proposition 4.10 (1))}\\ &= (-1)^{(l-1)/2} = 1,\end{aligned}$$

because $l \equiv 1 \pmod 4$.

(iii) If $\mathfrak{p} \mid m$ and $\left(\frac{l}{p}\right) = 1$, where $\mathfrak{p} \cap \mathbb{Z} = (p \neq 2)$, then $v_{\mathfrak{p}}\left(m\varepsilon\sqrt{l}\right) = 1$. So

$$\begin{aligned}\left(\frac{-1, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) &= \left(\frac{-1}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(m\varepsilon\sqrt{l})} \quad \text{(by Proposition 4.12 (1))}\\ &= \left(\frac{-1}{\mathfrak{p}}\right)\\ &= \left(\frac{-1}{p}\right) \quad \text{(by Proposition 4.10 (1))}\\ &= (-1)^{(p-1)/2}.\end{aligned}$$

(iv) If $\mathfrak{p} \mid m$ and $\left(\frac{l}{p}\right) = -1$, where $\mathfrak{p} \cap \mathbb{Z} = (p \neq 2)$, then $v_{\mathfrak{p}}\big(m\varepsilon\sqrt{l}\big) = 1$. So

$$\left(\frac{-1, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{-1}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}\left(m\varepsilon\sqrt{l}\right)} \qquad \text{(by Proposition 4.12 (1))}$$
$$= \left(\frac{-1}{\mathfrak{p}}\right)$$
$$= \left(\frac{N_{\mathbb{Q}(\sqrt{l})/\mathbb{Q}}(-1)}{p}\right) \qquad \text{(by Proposition 4.10 (2))}$$
$$= 1.$$

(v) If $\mathfrak{p} \mid 2$, then

$$\left(\frac{-1, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{-1, N_{\mathbb{Q}(\sqrt{l})/\mathbb{Q}}\big(m\varepsilon\sqrt{l}\big)}{2}\right) \qquad \text{(by Proposition 4.12 (2))}$$
$$= \left(\frac{-1, m^2 l}{2}\right)$$
$$= \left(\frac{-1, l}{2}\right)$$
$$= \left(\frac{-1}{l}\right) \qquad \text{(cf. [10], Lemma 2.27, page 63)}$$
$$= 1,$$

because $l \equiv 1 \pmod 4$.

(2) ε is not a norm in L. In fact:

(a) If $\mathfrak{p} \nmid l$ and $\mathfrak{p} \nmid m$, then $v_{\mathfrak{p}}(m\varepsilon\sqrt{l}) = 0$, so

$$\left(\frac{\varepsilon, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{\varepsilon}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(m\varepsilon\sqrt{l})} \qquad \text{(by Proposition 4.12 (1))}$$
$$= 1$$

(b) If $\mathfrak{p} \mid l$, then $\mathfrak{p} = \big(\sqrt{l}\big)$ and $v_{\mathfrak{p}}\big(m\varepsilon\sqrt{l}\big) = 1$. So

$$\left(\frac{\varepsilon, m\varepsilon\sqrt{l}}{\mathfrak{p}}\right) = \left(\frac{\varepsilon}{\mathfrak{p}}\right)^{v_{\mathfrak{p}}(m\varepsilon\sqrt{l})} \qquad \text{(by Proposition 4.12 (1))}$$
$$= \left(\frac{\varepsilon}{\sqrt{l}}\right)$$
$$= \left(\frac{2}{l}\right) \qquad \text{(cf. [1], Proof of Proposition 4.1)}$$
$$= -1,$$

because $l \equiv 5 \pmod 8$.

Thus,

$$E_{\mathbb{Q}(\sqrt{l})}/E_{\mathbb{Q}(\sqrt{l})}\cap N_{L/\mathbb{Q}(\sqrt{l})}(L^{\times})$$
$$= \begin{cases} \{\overline{1},\overline{\varepsilon}\} & \text{if and only if } m \text{ takes one of the forms (1) (a)–(1) (d);} \\ \{\overline{1},\overline{\varepsilon},\overline{-1},\overline{-\varepsilon}\} & \text{elsewhere.} \end{cases}$$

Therefore,

$$e = \begin{cases} 1 & \text{if and only if } m \text{ takes one of the forms (1) (a)–(1) (d);} \\ 2 & \text{elsewhere;} \end{cases}$$

because $2^e = [E_{\mathbb{Q}(\sqrt{l})}\colon\ E_{\mathbb{Q}(\sqrt{l})}\cap N_{L/\mathbb{Q}(\sqrt{l})}(L^{\times})]$.

From the equality (3), we have two cases to discuss:

(1) If $e=1$, then we have $t=2$.

(2) If $e=2$, then we have $t=3$.

From [2], Paragraph 2, page 63, we get

(1) $t=2$ if and only if m takes one of the following forms:

(a) m is a prime p congruent to $3 \pmod 4$ and $\left(\frac{p}{l}\right)=-1$ ($t=\#\{\sqrt{l},\mathfrak{p}\}$, where $\mathfrak{p}\mid p$),

(b) $m\in\{1,2\}$ ($t=\#\{\sqrt{l},\mathfrak{z}\}$, where $\mathfrak{z}\mid 2$),

(2) $t=3$ if and only if m is a prime p congruent to $3 \pmod 4$ and $\left(\frac{p}{l}\right)=1$ (in this case, $t=\#\{\sqrt{l},\mathfrak{p}_1,\mathfrak{p}_2\}$, where $\mathfrak{p}_i\mid p$).

Therefore, $h(L)$ is odd if and only if m takes one of the following forms:

(1) m is a prime p congruent to $3 \pmod 4$,

(2) $m\in\{1,2\}$. □

Proposition 5.2. *Let L_n be the nth layer of the cyclotomic $\mathbb{Z}_2$-extension of L. Then, the class number of L_n is odd if and only if m takes one of the following forms:*

(1) *m is a prime p congruent to $3 \pmod 4$,*

(2) *m is an even prime,*

(3) *m is equal to 1.*

P r o o f. In order to use Proposition 4.7, we need to count the number of primes of L above 2. For this, let $\mathfrak{z}$ be a unique prime ideal of $\mathbb{Q}(\sqrt{l})$ lying above 2.

(1) If $m\in\{1,2\}$, it is clear that $\mathfrak{z}$ ramifies in L, then there is only one prime of L lying above 2.

(2) If m is a prime $p \equiv 3 \pmod 4$, then there is only one prime of L lying above 2. In fact,

$$\begin{aligned}
\left(\frac{m\varepsilon\sqrt{l}}{\mathfrak{2}}\right) &= \left(\frac{m\varepsilon\sqrt{l},2}{\mathfrak{2}}\right) && \text{(by Proposition 4.12 (1))}\\
&= \left(\frac{N_{\mathbb{Q}(\sqrt{l})/\mathbb{Q}}(m\varepsilon\sqrt{l}),2}{2}\right) && \text{(by Proposition 4.12 (2))}\\
&= \left(\frac{m^2 l,2}{2}\right) = \left(\frac{l,2}{2}\right)\\
&= \left(\frac{2}{l}\right) && \text{(cf. [10], Lemma 2.27, page 63)}\\
&= -1.
\end{aligned}$$

Let us now come back to the proof of Proposition 5.2 using Theorem 5.1 and Proposition 4.7. If $h(L_n)$ is odd for all $n \geqslant 0$, then $h(L)$ is odd (in particular, $n = 0$), hence m takes one of the forms: (1), (2) and (3). Conversely, if m takes one of the forms of Proposition 5.2, then $h(L)$ is odd, hence $A_0 \simeq X_\infty/TX_\infty = 0$, where $T = \omega_0$, this implies that $X_\infty/(2,T)X_\infty = 0$, thus $X_\infty = 0$ by Nakayama's lemma, therefore the class number of L_n is odd. $\square$

Remark 5.3. If m takes one of the forms of Theorem 5.1, then Greenberg's conjecture holds for L. Moreover, $\nu = 0$.

Now, we can prove the main theorem.

Proof of the main theorem. We begin by computing the value of $\lambda^-(K^{(*)})$ using Kida's formula (2). For this, consider Figure 1, where $L = K^{(*)}$. By Proposition 4.8, the class number of F is odd. Moreover, there is only one prime of F lying above 2. In fact, let $\mathfrak{2}$ be a unique prime ideal of F^+ lying above 2, so we have

$$\left(\frac{-\varepsilon\sqrt{l}}{\mathfrak{2}}\right) = \left(\frac{-\varepsilon\sqrt{l}}{\mathfrak{2}_{\mathbb{Q}(\sqrt{l})}}\right) = -1,$$

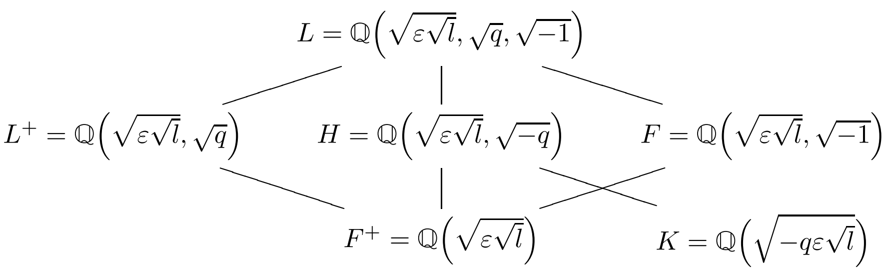

Figure 1.

then $\mathfrak{2}$ stays inert in F. Thus, the class numbers of the layers of the cyclotomic $\mathbb{Z}_2$-extension of F are odd by Proposition 4.7. Therefore $\lambda^-(F) = 0$, because

$\lambda^+(F) = \lambda(F^+) = 0$ by Proposition 5.2. On the other hand, we have q splits into 4 prime ideals of F. In fact, let $\mathfrak{q}$ be one of the two prime ideals of F^+ lying above q, so we have

$$\left(\frac{-1}{\mathfrak{q}}\right) = \left(\frac{N_{F^+/\mathbb{Q}(\sqrt{l})}(-1)}{\mathfrak{q}_{\mathbb{Q}(\sqrt{l})}}\right) = 1.$$

- ▷ If $q \equiv 3 \pmod 8$, by Proposition 4.1 we have q splits into 2 primes of $\mathbb{Q}(\zeta_{2^{n+2}})$ and it is inert in $\mathbb{Q}_n$ with $n \geqslant 2$, then q splits into the product of 8 primes in $F_n = F\mathbb{Q}_n = \mathbb{Q}\left(\sqrt{\varepsilon\sqrt{l}}, \zeta_{2^{n+2}}\right)$ while it decomposes into 4 primes in $F_n^+ = \mathbb{Q}_n\left(\sqrt{\varepsilon\sqrt{l}}\right)$.
- ▷ If $q \equiv 7 \pmod{16}$, proceeding as above, then q splits into the product of 16 primes in F_n while it decomposes into 8 primes in F_n^+.

Note that $[L_\infty : F_\infty] = [L_\infty^+ : F_\infty^+] = 2$ and $e_\beta = e_\beta^+ = 2$, then by Theorem 4.5 we have:

$$\lambda^-(L) - 1 = \begin{cases} 2\cdot(0-1)+8-4 & \text{if } q \equiv 3 \pmod 8, \\ 2\cdot(0-1)+16-8 & \text{if } q \equiv 7 \pmod{16}. \end{cases}$$

Thus,

$$\lambda^-(L) = \begin{cases} 3 & \text{if } q \equiv 3 \pmod 8, \\ 7 & \text{if } q \equiv 7 \pmod{16}. \end{cases}$$

By definition, we recall that $\lambda^+(L) = \lambda(L^+)$. One can show that $\lambda^+(L) = 0$ using Proposition 4.7. Therefore,

$$\lambda(L) = \lambda^+(L) + \lambda^-(L) = \begin{cases} 3 & \text{if } q \equiv 3 \pmod 8, \\ 7 & \text{if } q \equiv 7 \pmod{16}. \end{cases}$$

Since the extension L/L^+ is unramified, then L_n/L_n^+ is unramified too. Thus, by Lemma 4.4, $A_\infty^- = A_\infty$ because $h(L_n^+)$ is odd for all $n \geqslant 0$. By Theorem 4.3 there is no finite Λ-submodule in A_∞^-. Hence, A_∞ is a Λ-module without finite part. So,

$$A_\infty \simeq \begin{cases} \mathbb{Z}_2^3 & \text{if } q \equiv 3 \pmod 8, \\ \mathbb{Z}_2^7 & \text{if } q \equiv 7 \pmod{16}. \end{cases}$$

Finally, we have

$$A_\infty \simeq \mathbb{Z}_2^{\lambda(L)} \simeq \bigoplus_j \Lambda/(g_j(T)),$$

where each g_j is distinguished and $\sum_j \deg g_j = \lambda(L)$, and we have that $L/\mathbb{Q}$ is an abelian extension. Then, by Theorem 4.6,

$$\operatorname{rank}_2(A_n) = \begin{cases} 3 \text{ for all } n \geqslant 3 & \text{if } q \equiv 3 \pmod 8, \\ 7 \text{ for all } n \geqslant 7 & \text{if } q \equiv 7 \pmod{16}. \end{cases}$$

This completes the proof of the theorem. □

Example 5.4. Let $K = \mathbb{Q}\Big(\sqrt{-11\varepsilon\sqrt{5}}\Big)$, where $\varepsilon = \frac{1}{2}(1+\sqrt{5})$. Since $5 \equiv 5 \pmod 8$, $11 \equiv 3 \pmod 8$ and $(\frac{11}{5})_4 = 1$, we have $A_\infty \simeq \mathbb{Z}_2^3$, where A_∞ is attached to $K^{(*)}$.

Example 5.5. Let $K = \mathbb{Q}\Big(\sqrt{-7\varepsilon\sqrt{37}}\Big)$, where $\varepsilon = 6+\sqrt{37}$. Since $37 \equiv 5 \pmod 8$, $7 \equiv 7 \pmod{16}$ and $(\frac{7}{37})_4 = 1$, we have $A_\infty \simeq \mathbb{Z}_2^7$, where A_∞ is attached to $K^{(*)}$.

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