Basma Ammous; Nour Ben Mahmoud; Mohamed Hbaib
On the quasi-periodic p-adic Ruban continued fractions

Czechoslovak Mathematical Journal, Vol. 72 (2022), No. 4, 1157–1166

Persistent URL: http://dml.cz/dmlcz/151137

Czechoslovak Mathematical Journal, 72 (147) (2022), 1157–1166

ON THE QUASI-PERIODIC p-ADIC RUBAN CONTINUED FRACTIONS

Basma Ammous, Nour Ben Mahmoud, Mohamed Hbaib, Sfax

Received October 28, 2021. Published online July 28, 2022.

Abstract. We study a family of quasi periodic p-adic Ruban continued fractions in the p-adic field $\mathbb{Q}_p$ and we give a criterion of a quadratic or transcendental p-adic number which based on the p-adic version of the subspace theorem due to Schlickewei.

Keywords: continued fraction; p-adic number; transcendence; subspace theorem

MSC 2020: 11A55, 11D88, 11J81

1. Introduction

In 1906, Maillet (see [8]) was the first person to give explicit transcendental continued fractions with bounded partial quotients. Later, in 1962, Baker (see [4]) extended Maillet's results with LeVeque, see [6].

Recently, Bugeaud and Adamczewski in [3] have improved Baker's results, which in particular show the transcendence of irrationals, where the development of a continued fraction has a specific irregularity.

Let $(n_k)_{k\geqslant 0}$ be an increasing sequence of positive integers. Let $(\lambda_k)_{k\geqslant 0}$ and $(r_k)_{\geqslant 0}$ be sequences of positive integers. Assume that the propriety $(*)$ verifies for all k

$$(*)\qquad \begin{gathered} n_{k+1} \geqslant n_k + \lambda_k r_k, \\ a_{m+r_k} = a_m \quad \text{for } n_k \leqslant m \leqslant n_k + (\lambda_k - 1)r_k - 1. \end{gathered}$$

Theorem 1.1 ([3]). *Let $a = (a_n)_{n\geqslant 0}$ be a sequence of positive integers which satisfies $(*)$ and is not ultimately periodic. Let $(p_n/q_n)_{n\geqslant 0}$ denote the sequence of convergents to the real number $\xi = [a_0, a_1, \ldots, a_n, \ldots]$.*

DOI: 10.21136/CMJ.2022.0409-21

Assume that the sequence $(q_n^{1/n})_{n\geqslant 1}$ is bounded (which is in particular the case when the sequence a is bounded), and that

$$\limsup_{k\to\infty} \frac{\lambda_k}{n_k} > 0.$$

Then, the real number ξ is transcendental.

Moreover, Bugeaud and Adamczewski gave in [1], [2] several transcendence criteria both in the decimal as well as in the continued fraction. The proofs are mainly based on the Schmidt Subspace Theorem, see [13].

Besides, a continued fraction expansion exists in the p-adic number field $\mathbb{Q}_p$. The p-adic analogue of real continued fractions was first studied by Mahler in 1934, see [7]. After that, in 1970, Ruban in [11] developed the algorithm given by Mahler and showed that these p-adic continued fractions enjoy nice elementary properties similar to the real case. However, the p-adic continued fractions of Ruban have many important differences with respect to the real case. For example, the p-adic continued fraction expansion of a rational number might not be finite. Thus, in 1985, Laohakosol in [5] proved that a number $\alpha \in \mathbb{Q}_p$ is rational if and only if the continued fraction of α is finite or ultimately periodic with all partial quotients in the period equal to $(p-p^{-1})$. Moreover, in $\mathbb{Q}_p$, we have not found a characterization of quadratic p-adic numbers until now. In this context, recently, Ooto in [10] proved that the analogue of Lagrange's theorem about the periodicity of real continued fractions does not hold for Ruban's continued fractions in $\mathbb{Q}_p$.

In this work, we study the family of quasi periodic p-adic Ruban continued fractions and give a criterion of a quadratic or transcendental p-adic number which based on the p-adic version of subspace theorem, see [12]. More precisely, we investigate the analogous of Theorem 1.1 in $\mathbb{Q}_p$.

The present paper is organized as follows: In Section 2, we introduce the p-adic field $\mathbb{Q}_p$ and the expression of a Ruban continued fraction over this field. In Section 3, we state our transcendence criterion of a p-adic number. After that, we review some properties, we present some lemmas needed to prove our result, and finally we give the proof of our main theorem (see Theorem 3.1) and an example to illustrate our criterion. By the end, we conclude with open questions.

2. Background

The letter p denotes a prime number. The p-adic absolute value in $\mathbb{Q}$ noted $|\cdot|_p$ is defined as the unique absolute value satisfying $|p|_p = p^{-1}$ and $|p'|_p = 1$ for all prime number $p' \neq p$ (it extends completely multiplicatively to all integers and hence to rationals). The field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ for this absolute p-adic value.

Consequently, every $x \in \mathbb{Q}_p$ has a unique representation in the form:

$$x = \sum_{i \geqslant d} c_i p^i, \quad d \in \mathbb{Z} \quad \text{and} \quad c_i \in \{0, 1, \ldots, p-1\} \quad \forall\, i \geqslant d.$$

The p-adic absolute value in $\mathbb{Q}$ is extended to $\mathbb{Q}_p$ as follows:

$$|x|_p = p^{-\inf\{i / c_i \neq 0\}} \quad \text{for } x \neq 0.$$

Recall that this absolute value is non-archimedean.

Definition 2.1. An element α is called *algebraic* over $\mathbb{Q}$ if there is a polynomial

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{Q}[x] \quad \text{with } P(\alpha) = 0.$$

It turns out that algebraic elements over $\mathbb{Q}$ are not necessarily contained in $\mathbb{Q}_p$. In our context, we will only need that $|\cdot|_p$ can be extended uniquely from $\mathbb{Q}_p$ to all of its algebraic extensions. This follows from the next theorem, which holds generally in non-archimedean fields.

Theorem 2.1 ([9], Chapter II, Theorem 4.8)**.** *Let K be a field which is complete with respect to $|\cdot|$ has a unique extension to L defined by*

$$|\alpha| = \sqrt[n]{|N_{L/K}(\alpha)|},$$

and L is complete with respect to this extension.

Now, we expand elements in $\mathbb{Q}_p$ in a Ruban continued fraction and we define the p-adic integral and fractional parts of x by

$$[x]_p = c_d p^d + \ldots + c_{-1} p^{-1} + c_0 \quad \text{and} \quad \{x\}_p = c_1 p^1 + c_2 p^2 + \ldots$$

According to x, the integral and fractional parts of x are uniquely determined, and so we can uniquely write $x = [x]_p + \{x\}_p$. The algorithm proceeds as follows:

If $\{x\}_p = 0$, then $x = [x]_p = [a_0]_p$. Or else, we write $x = a_0 + 1/x_1$, where $a_0 = [x]_p$, $x_1 = 1/\{x\}_p$.

Otherwise, since $|x_1|_p > p$, by repeating the previous steps, we can uniquely write $x_1 = a_1 + 1/x_2$, where $a_1 = [x_1]_p$, $x_2 = 1/\{x_1\}_p$. If $\{x_1\}_p = 0$ we stop, otherwise we proceed in the same manner. Since the $(a_i)_{i>0}$ obtained are unique, each $x \in \mathbb{Q}_p$ has a unique RCF (p-adic Ruban continued fraction) of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots}}}} = [a_0, a_1, \ldots, a_n, \ldots]_p.$$

For an infinite Ruban continued fraction $x = [a_0; a_1, a_2, \ldots]_p$, we define nonnegative rational numbers p_n, q_n by using recurrence equations:

$$p_{-1} = 1, \quad p_0 = a_0, \quad q_{-1} = 0, \quad q_0 = 1$$

and

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for any } n \geqslant 0.$$

As we know, we call p_n/q_n the nth convergent of the RCF of x. Then the Ruban continued fraction has the following properties which are the same properties as the continued fraction expansions for real numbers: For all $n \geqslant 0$,

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]_p \quad \forall\, n \geqslant 0, \tag{2.1}$$

$$[a_0; a_1, \ldots, a_{n-1}, x_n]_p = \frac{x_n p_n + p_{n-1}}{x_n q_n + q_{n-1}}, \tag{2.2}$$

$$p_{n-1} q_n - p_n q_{n-1} = (-1)^n. \tag{2.3}$$

3. Main results

Throughout the present paper, let $(n_k)_{k\geqslant 0}$, $(\lambda_k)_{k\geqslant 0}$ and $(r_k)_{\geqslant 0}$ be sequences defined in propriety $(*)$ mentioned above and consider the p-adic number ξ defined by

$$\xi = [a_0; a_1, \ldots, a_n, \ldots]_p.$$

Then ξ has a quasi-periodic continued fraction expansion in the following sense: for any positive integer k, a block of r_k consecutive partial quotients is repeated λ_k times, such repetition occurs just after the $(n_k - 1)$th partial quotient.

For positive reals $a_1, \ldots, a_m$, denote by $K_m(a_1, \ldots, a_m)$ the denominator of the p-adic number $[0; a_1, a_2, \ldots, a_m]_p$. It is commonly called a *continuant.*

The main theorem is the following:

Theorem 3.1. *Let* $\mathbf{a} = (a_k)_{k\geqslant 1}$ *be as in the above and not ultimately periodic. Let* $(p_k/q_k)_{k\geqslant 1}$ *denote the sequence of convergents to the p-adic number* $\xi = [0; a_1, a_2, \ldots, a_k, \ldots]_p$ *with* $K_{n_k}(a_{n_k}, \ldots, a_{n_k+r_k-1})$ *is bounded, and that*

$$\limsup_{k\to\infty} \frac{\lambda_k r_k}{n_k} > 2.$$

Then ξ is either quadratic or transcendental.

Remark 3.1. If the sequence $(r_k)_{k\geqslant 0}$ is bounded, then $K_{n_k}(a_{n_k}, \ldots, a_{n_k+r_k-1})$ is bounded.

Corollary 3.1. *Under the same hypothesis of the previous theorem and if the sequence $(r_k)_{k\geqslant 0}$ is increasing and*

$$\limsup_{k\to\infty} \frac{\lambda_k}{n_k} > 0,$$

then ξ is either quadratic or transcendental.

The proof of the main theorem of this paper is based on the p-adic version of the Schmidt subspace theorem, established by Schlickewei (see [12]), which is recalled below:

Let $k \geqslant 2$ be an integer, $X = (X_1, \dots, X_k)$ a k-tuple of rational numbers. Put $|X|_\infty = \{\max |X_i| \colon 1 \leqslant i \leqslant k\}$ and $|X|_p = \{\max |X_i|_p \colon 1 \leqslant i \leqslant k\}$.

Theorem 3.2 ([12])**.** *Let p be a prime number, $L_{1,\infty}, \dots, L_{k,\infty}$ be k linearly independent forms with variable X and algebraic real coefficients, $L_{1,p}, \dots, L_{k,p}$ be k linearly independent forms with algebraic p-adic coefficients and same variables and $\varepsilon > 0$ a real number. Then, the set of solutions $X \in \mathbb{Z}^k$ of the inequality:*

$$\prod_{i=1}^{k} (|L_{i,\infty}(X)|_\infty |L_{i,p}(X)|_p) \leqslant |X|_\infty^{-\varepsilon}$$

is contained in the union of a finite number of proper subspaces of $\mathbb{Q}^k$.

Of equal importance, the proofs of our theorem lean on the following four lemmas:

Lemma 3.1 ([14])**.**

$$|q_n|_p = |a_1 \dots a_n|_p \quad \forall\, n \geqslant 1, \tag{3.1}$$

$$\begin{cases} |p_n|_p = |a_0 \dots a_n|_p & \forall\, n \geqslant 1 \textit{ if } a_0 \neq 0, \\ |p_1|_p = 1, \quad |p_n|_p = |a_2 \dots a_n|_p & \forall\, n \geqslant 2 \textit{ if } a_0 = 0, \end{cases} \tag{3.2}$$

$$|q_n|_p < |q_{n+1}|_p \quad \textit{and} \quad |p_n|_p < |p_{n+1}|_p, \tag{3.3}$$

$$\left| x - \frac{p_n}{q_n} \right|_p = \frac{1}{|a_{n+1}|_p |q_n|_p^2} \quad \forall\, n \geqslant 0. \tag{3.4}$$

In what follows, equality (3.5) will be referred to as the mirror formula.

Lemma 3.2 ([10])**.** *Let $x = [0; a_1, a_2, \dots]_p$ be a p-adic number with nth convergent p_n/q_n. Then*

$$\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \dots, a_1]_p. \tag{3.5}$$

Lemma 3.3. *Let $x = [a_0; a_1, a_2, \ldots]_p$ and $y = [a'_0; a'_1, a'_2, \ldots]_p$ be two p-adic numbers having the same first $(n+1)$ partial quotients. Then*

$$|x-y|_p \leqslant \frac{1}{|q_n|_p^2}.$$

Proof. Since p_n/q_n is a nth convergent to both x and y, and (7), the lemma follows. □

Lemma 3.4. *Let $x = [0; a_1, a_2, \ldots]_p$ be a p-adic number with nth convergent p_n/q_n. Then*

$$\max(p_n, q_n) < (p+1)^{n+1}.$$

Proof. Using the fact $a_n < p$ for all $n \geqslant 0$, and the induction on n, we find that $p_0 = a_0 < p < (p+1)^{0+1}$, $p_1 = a_1 a_0 + 1 < p^2 + 1 < (p+1)^{1+1}$.

We suppose that the statement holds true for all $k < n$. Then, we find

$$\begin{aligned} p_n = a_n p_{n-1} - p_{n-2} &< p(p+1)^n + (p+1)^{n-1} < (p+1)^{n-1}(p(p+1)+1) \\ &< (p+1)^{n-1}(p+1)^2 < (p+1)^{n+1}. \end{aligned}$$

The same holds true for q_n. □

Notations 3.1. Let $a_k \in \mathbb{Z}[1/p] \cap (0,p)$. Set $a_k = b_k/c_k$, where $b_k \in \mathbb{N}^*$ and $c_k = p^{-\nu_p(a_k)} \in \mathbb{N}^*$. We take

$$P_k = \left(\prod_{j=0}^{k} c_j\right) p_k, \quad Q_k = \left(\prod_{j=0}^{k} c_j\right) q_k, \quad P'_k = \left(\prod_{j=0}^{k-1} c_j\right) p'_k \quad \text{and} \quad Q'_k = \left(\prod_{j=0}^{k-1} c_j\right) q'_k.$$

It is clear that P_k, P'_k, Q_k and Q'_k are integers.

Proof of Theorem 3.1. For any $k \geqslant 0$, set

$$K_k = K_{n_k}(a_{n_k}, \ldots, a_{n_k + r_k - 1}).$$

By assumption, there exists an infinite set of integers $\mathcal{K}_1$, ranged in increasing order, such that $(K_k)_{k \in \mathcal{K}_1}$ is bounded. Since the K_k's are nonnegative reals, it follows that infinitely of them take the same value. Then, Lemma 3.4 implies that there exists a positive integer r, positive integers $c_0, \ldots, c_{r-1}$ and an infinite set $\mathcal{K}_2$ of positive integers such that

$$r_k = r, \quad a_{n_k + j} = c_j, \quad 0 \leqslant j \leqslant r-1,$$

for any k in $\mathcal{K}_2$. Let α denote the p-adic number having the purely periodic Ruban continued fraction expansion with period $C = (c_{r-1}, \ldots, c_0)$, with $c_j \neq p - p^{-1}$ for all $0 \leqslant j \leqslant r-1$, that is,

$$\alpha = [c_{r-1}, c_{r-2}, \ldots, c_0, c_{r-1}, \ldots, c_0, c_{r-1}, \ldots]_p = [C, C, \ldots, C, \ldots]_p.$$

Then, α is a quadratic p-adic number. Since we need to introduce some more notation, we denote p_n/q_n (or by r_n/s_n) the nth convergent to ξ (or to α, respectively).

Hence, for any i in $\mathcal{K}_2$, set $p_k = p_{n_k+\lambda_k r_k - 1}$, $q_k = q_{n_k+\lambda_k r_k - 1}$, $p'_k = p_{n_k+\lambda_k r_k - 2}$, $q'_k = q_{n_k+\lambda_k r_k - 2}$ and $s_k = s_{\lambda_k r_k - 1}$.

By assumption, we already know that ξ is an irrational and not a quadratic p-adic number. Therefore, we assume that ξ is an algebraic p-adic number of degree $d > 2$ and we aim at deriving a contradiction.

Let k be in $\mathcal{K}_2$. By the theory of continued fractions, we have

$$|q_k\xi - p_k|_p \leqslant \frac{1}{|q_k|_p} \quad \text{and} \quad |q'_k\xi - p'_k|_p \leqslant \frac{1}{|q'_k|_p}. \tag{3.6}$$

Furthermore, we obtain

$$\frac{p_k}{q_k} = [0; a_1, \ldots, a_{n_k-1}, \underbrace{C, C, \ldots, C}_{\lambda_k}]_p.$$

We get from the mirror formula (see Lemma 3.2)

$$\frac{q_k}{q'_k} = [\underbrace{C, C, \ldots, C}_{\lambda_k}, a_{n_k-1}, \ldots, a_1]_p.$$

Since α and q_k/q'_k have the same first $(\lambda_k r_k)$ partial quotients, then through Lemma 3.3, we have

$$|\alpha q'_k - q_k|_p \leqslant \frac{|q'_k|_p}{|s_k|_p^2}. \tag{3.7}$$

When $k \to \infty$ and $|q_k|_p \to \infty$, we obtain

$$\lim_{\mathcal{K}_2 \ni k \to \infty} \frac{q_k}{q'_k} = \alpha \quad (\text{in } \mathbb{Q}_p). \tag{3.8}$$

Let us consider the eight following independent linear forms with algebraic coefficients in variable $X = (X_1, X_2, X_3, X_4)$.

$$\begin{aligned}
&L_{i,\infty}(X) = X_i \quad \text{for } 1 \leqslant i \leqslant 4,\\
&L_{1,p}(X) = \xi X_1 - X_3, \quad L_{2,p}(X) = \xi X_2 - X_4,\\
&L_{3,p}(X) = \alpha X_2 - X_1, \quad L_{4,p}(X) = X_1.
\end{aligned}$$

Keeping Notations 3.1, we evaluate the product of these linear forms at the integer points $X = (Q_k, c_kQ'_k, P_k, c_kP'_k)$, and it follows from (3.6) and (3.7) that

$$\prod_{i=1}^{4} |L_{i,p}(X)|_p \leqslant \frac{|\prod_{j=0}^{k} c_j|_p^4}{|s_k|_p^2},$$

we also have

$$\prod_{i=1}^{4} |L_{i,\infty}(X)|_\infty = \left|\prod_{j=1}^{k} c_j\right|_\infty^4 |q_k p_k q'_k p'_k|_\infty \leqslant |X|_\infty^4.$$

Therefore, we obtain the following inequality:

$$\prod_{i=1}^{4} |L_{i,\infty}(X)|_\infty |L_{i,p}(X)|_p \leqslant \frac{|X|_\infty^4}{|s_k|_p^2}. \tag{3.9}$$

For any k in $\mathcal{K}_2$, Lemma 3.1 implies

$$|s_k|_p \geqslant p^k = p^{\lambda_k r_k - 1}.$$

Then,

$$|s_k|_p \geqslant (p+1)^{(\lambda_k r_k - 1)\log p/\log(p+1)}.$$

Moreover, we infer from Lemma 3.4 that $(p+1) \geqslant |X|_\infty^{1/(k+1)}$ for any k in $\mathcal{K}_2$. So

$$|s_k|_p \geqslant |X|_\infty^\delta,$$

where

$$\delta = \frac{\log p}{\log(p+1)} \frac{\lambda_k r_k - 1}{n_k + \lambda_k r_k}.$$

Returing to inequality (3.9), we conclude that

$$\prod_{i=1}^{4} |L_{i,\infty}(X)|_\infty |L_{i,p}(X)|_p \leqslant \frac{1}{|X|_\infty^\varepsilon} = |X|_\infty^{-\varepsilon},$$

where

$$\varepsilon = 2\frac{\log p}{\log(p+1)} \frac{\lambda_k r_k - 1}{n_k + \lambda_k r_k} - 4.$$

By the Theorem hypothesis, we get $\varepsilon > 0$.

Schlickewei's theorem confirms the existence of nonzero integer quadruplet (x_1, x_2, x_3, x_4) and an infinite set of distinct positif integers $\mathcal{K}_3 \subset \mathcal{K}_2$, which gives

$$x_1 Q_k + x_2 c_k Q'_k + x_3 P_k + x_4 P'_k = 0 \quad \forall\, k \in \mathcal{K}_3,$$

that is

$$x_1 q_k + x_2 q'_k + x_3 p_k + x_4 p'_k = 0. \tag{3.10}$$

Dividing (3.10) by q'_k, we obtain

$$x_1 \frac{q_k}{q'_k} + x_2 + x_3 \frac{p_k}{q_k} \frac{q_k}{q'_k} + x_4 \frac{p'_k}{q'_k} = 0. \tag{3.11}$$

By letting k tend to infinity along $\mathcal{K}_3$ in (3.11) and through the use of (3.8), we obtain

$$x_1\alpha + x_2 + (x_3\alpha + x_4)\xi = 0.$$

Since ξ is not a quadratic irrational, then $x_3\alpha + x_4 = 0$ and since α is irrational, then $x_3 = x_4 = 0$. Consequently, again by using that α is irrational, we reach that $x_1 = x_2 = x_3 = x_4 = 0$, which is a contradiction. $\square$

Example 3.1. Let p a prime number, a, b two distinct p-adic numbers in $\mathbb{Z}[1/p] \cap]0, p[$. Let

$$\xi = [0; \overline{bb}^{5^0}, \underbrace{a, a}_{2\,\text{times}}, \overline{bb}^{5^1}, \underbrace{a, a, a}_{3\,\text{times}}, \overline{bb}^{5^2}, \ldots, \underbrace{a, a, \ldots, a}_{(n+1)\,\text{times}}, \overline{bb}^{5^n}, \ldots]_p.$$

In this case we take $r_i = 2$, $\lambda_i = 5^i$ for all $i \geqslant 0$ and $n_i = 2 \sum\limits_{j=0}^{i-1} 5^j + \frac{1}{2}(i+1)(i+2)$ for all $i \geqslant 1$ with $n_0 = 1$. Then ξ is either quadratic or transcendental because:

$$\begin{aligned}
\limsup_{i\to\infty} \frac{r_i\lambda_i}{n_i} &= \limsup_{i\to\infty} \frac{2.5^i}{2\sum_{j=0}^{i-1} 5^j + \frac{1}{2}(i+1)(i+2)} \\
&= \limsup_{i\to\infty} \frac{2.5^i}{\frac{1}{2}(5^i - 1) + \frac{1}{2}(i+1)(i+2)} = 4 > 2.
\end{aligned}$$

4. Concluding remarks

Under the assumptions of Theorem 3.1, we get that the p-adic number α is quadratic or transcendental. It is natural to ask the following question: is it possible to give some refined conditions to our theorem for obtain a transcendence criterion in $\mathbb{Q}_p$? The problem seems to be difficult because until now we have not had any characterization of quadratic p-adic numbers.

Bugeaud and Adamczewski gave several transcendence criteria of irrational numbers both in the decimal and in an integer base as well as continued fraction, see [1], [2], [3]. Can we find transcendence criteria of p-adic numbers by using the p-adic expansion as we get with the Ruban continued fraction?

Acknowledgements. The authors would like to thank the referee for many useful comments.

References

[1] *B. Adamczewski, Y. Bugeaud*: On the decimal expansion of algebraic numbers. Fiz. Mat. Fak. Moksl. Semin. Darb. *8* (2005), 5–13. zbl MR

[2] *B. Adamczewski, Y. Bugeaud*: On the complexity of algebraic numbers. I. Expansions in integer bases. Ann. Math. (2) *165* (2007), 547–565. zbl MR doi

[3] *B. Adamczewski, Y. Bugeaud*: On the Maillet-Baker continued fractions. J. Reine Angew. Math. *606* (2007), 105–121. zbl MR doi

[4] *A. Baker*: Continued fractions of transcendental numbers. Mathematika, Lond. *9* (1962), 1–8. zbl MR doi

[5] *V. Laohakosol*: A characterization of rational numbers by p-adic Ruban continued fractions. J. Aust. Math. Soc., Ser. A *39* (1985), 300–305. zbl MR

[6] *W. J. LeVeque*: Topics in Number Theory. II. Addison-Wesley, Reading, 1956. zbl MR

[7] *K. Mahler*: Zur Approximation p-adischer Irrationalzahlen. Nieuw Arch. Wiskd. *18* (1934), 22–34. (In German.) zbl

[8] *E. Maillet*: Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions. Gauthier-Villars, Paris, 1906. (In French.) zbl

[9] *J. Neukirch*: Algebraic Number Theory. Grundlehren der Mathematischen Wissenschaften 322. Springer, Berlin, 1999. zbl MR doi

[10] *T. Ooto*: Transcendental p-adic continued fractions. Math. Z. *287* (2017), 1053–1064. zbl MR doi

[11] *A. A. Ruban*: Some metric properties of p-adic numbers. Sib. Math. J. *11* (1970), 176–180. zbl MR doi

[12] *H. P. Schlickewei*: The p-adic Thue-Siegel-Roth-Schmidt theorem. Arch. Math. *29* (1977), 267–270. zbl MR doi

[13] *W. M. Schmidt*: Diophantine Approximation. Lecture Notes in Mathematics 785. Springer, Berlin, 1980. zbl MR doi

[14] *L. Wang*: P-adic continued fractions. I. Sci. Sin., Ser. A *28* (1985), 1009–1017. zbl MR

Authors' address: Basma Ammous, Nour Ben Mahmoud, Mohamed Hbaib (corresponding author), University of Sfax, Faculty of Sciences, Departement of Mathematics, Route de l'Aéroport Km 0.5, BP 1169.3029 Sfax, Tunisia, e-mail: `basma.ammous@fss.usf.tn`, `nour.benmahmoud23@gmail.com`, `mohamed.hbaieb@fss.usf.tn`.